Depths of codimension one foliations

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1 Introduction

Theory of foliations is originated from studies of global behavior of solutions of differential equations. The development of the theory was provoked by the following question; "Does there exist a two dimensional foliation on the three sphere S^3?" which had been proposed by H.Hopf in a different form in about 1935. This question was answered affirmatively by G.Reeb, and this is regarded as the initiation of the research field "foliation". In 1970's two new concepts on foliations were introduced. In [16], D.Sullivan called a foliation on a Riemannian manifold geometrically taut if every leaf is a minimal surface with respect to the Riemannian metric, and showed that codimension one foliation is geometrically taut if and only if each compact leaf admits a closed transversal curve. This geometric property is simply called taut, and turned out to be very useful in three dimensional topology via the theory of Thurston norm defined by W.P.Thurston [17]. The other notion, depth of codimension one foliation was introduced by Nishimori in 1977 [12]. The original definition of the depth of $\mathcal{F}$, denoted by $d(\mathcal{F})$, is as follows.

> For leaves F_1, F_2 of $\mathcal{F}$, we say $F_1 \leq F_2$ if and only if $F_1 \subset \overline{F_2}$. $F_1 < F_2$ if and only if $F_1 \leq F_2$ and $F_1 \neq F_2$. For a leaf F of $\mathcal{F}$, $d(F)$ is the supremum of k such that there are k leaves $F_1, \ldots, F_k$ such that $F_1 < \cdots < F_k = F$. The depth of $\mathcal{F}$, $d(\mathcal{F})$ is the supremum of $d(F)$ for the leaves F of $\mathcal{F}$.

(Note that the value $d(\mathcal{F})$ given by this definition differs by one from the value of the depth given in Definition 2.1.8 of this paper.) It can be regarded that depth is a quantity which describes how far from a fiber bundle structure the foliation is. Nishimori described some fundamental properties of the depth and several authors studied the invariant afterwards.

In 1980's D.Gabai developed the theory of codimension one foliations on three manifolds. He gave a powerful method which is called sutured manifold theory, for constructing taut foliations on three dimensional manifolds [5]. Particularly in [6], he showed that for any knot K in S^3, there exists a codimension one, transversely oriented, taut C^0 foliation $\mathcal{F}$ of finite depth on the knot exterior $E(K)$ such that $\mathcal{F}|_{\partial E(K)}$ is a foliation by circles. (As a consequence of this theorem, Property R Conjecture, which was one of the most important problem in knot theory, follows immediately.) Inspired by this result, Cantwell-Conlon [2] introduced an invariant for knots called depth of knots which is the minimal depth of the foliations on the knot exterior and studied the invariant in a sequence of papers ([2, 3], etc.). (For example, in [2], they showed that for each n (≥ 0), there exists a knot at depth n.) The purpose of this paper is, as a sequal of these researches, to propose more delicate treatments of depth. In the first result, we pay our attension to the number of depth 0 leaves of the foliation under consideration. Here, we note that Cantwell-Conlon often assumed that each of the foliations under consideration has exactly one depth 0 leaf. The motivation of the first research is the following question.

Question. For the finite depth foliations on a given 3-manifold M, is there a difference between the minimal value of the depths of the foliations on M each of which admits exactly one depth 0 leaf and the minimal value of the depths of the foliations on M without such an assumption?

In this paper, we discuss this question for $\Sigma^{(n)}(K,0)$, the n-fold cyclic covering space of $S^3(K,0)$, where $S^3(K,0)$ denotes the manifold obtained from S^3 by performing 0-surgery on a knot K. The main result is as follows.

Theorem 1.0.1 *Let $\Sigma^{(n)}(K,0)$ be as above. Suppose that K is a 0-twisted double of a non-cable knot. Then for each n, we have*

$$\mathrm{depth}^0_{1,\alpha}(\Sigma^{(n)}(K,0)) \geq 1 + \left[\frac{n}{2}\right] ,$$

where $\mathit{depth}^0_{1,\alpha}(\Sigma^{(n)}(K,0))$ denotes the minimal depth of codimension one, transversely oriented, taut C^0 foliations on $\Sigma^{(n)}(K,0)$ each of which admits exactly one depth 0 leaf representing the homology class corresponding to a generator α of $H_1(S^3(K,0))$ and $[x]$ denotes the greatest integer among the integers which are not greater than x.

Let k be the minimal depth of the codimension one, transversely oriented, taut C^0 foliation on $S^3(K,0)$ (note that by Gabai [6], k is always finite). Suppose $n > 1$. Then, by lifting the depth k foliation on $S^3(K,0)$, we see that $\Sigma^{(n)}(K,0)$ admits a codimension one, transversely oriented, taut C^0 foliation of depth k with more than one (in fact, at least n) depth 0 leaves. This together with Theorem 1.0.1 gives an affirmative answer to Question.

In the second research, we introduce a quantity called "gap" of the foliation to deal with behaviors of depths of leaves of foliations of finite depth. We know by the definition of depth of leaves (see Section 2.1.2) that each depth $k(\geq 1)$

leaf of $\mathcal{F}$ is adjacent to a depth $k-1$ leaf. Note that if k does not represent the maximal depth in $\mathcal{F}$, it is not necessary the case that there exists a depth $k+1$ leaf which is adjacent to the leaf. However even if there does not exist such a leaf, there could be a leaf with depth more than $k+1$ which is adjacent to the depth k leaf. We phrase this situation "There is a gap between the depths of the leaves." More precisely, for a leaf L, we consider the minimum value of the differences between the depth of L and the depths of leaves which are adjacent to L and of depth greater than that of L. Then the gap of the foliation is the maximum of such values among the leaves of the foliation. For the formal definition of gap, see Section 2.1.2. By using this invariant, we give an estimation of depth of foliations on $\Sigma^{(n)}(K, 0)$ above.

Theorem 1.0.2 *Let $\mathcal{F}$ be a codimension one, transversely oriented, taut, C^0 foliation with C^∞ leaves on $\Sigma^{(n)}(K, 0)$ with exactly one depth 0 leaf representing $[\alpha]$, where α is corresponding to a generator of $H_1(S^3(K, 0)) \cong \mathbb{Z}$. Suppose $\hat{G}(\hat{\mathcal{F}})$ is a tree. Then for each n, we have:*

$$depth(\mathcal{F}) \geq \frac{n + gap(\widetilde{\mathcal{F}})}{2}.$$

For the notations $\hat{G}(\hat{\mathcal{F}})$ and $\text{gap}(\widetilde{\mathcal{F}})$, see Section 2.1.2.

This paper is organized as follows. In Section 2.1, we give definitions concerning about foliations and describe some facts related to the concepts. We also introduce Semistability Theorem given by Dippolito [4,] ☆ (Theorem 2.1.1). In Section 2.1.1, we give definition of depth of foliations and show some facts related to the concepts. In Section 2.1.2, we give the definition of gap of foliations. For the definition, we define eauivalence relation on leaves and introduce graph of foliations. In Section 2.2, we give some definitions concerning topology of three dimensional manifolds (Thurston norm, knots and links, sutured manifolds, etc.). We also introduce a theorem given by Gabai [6,] (Theorem 2.2.1) and give definition of depth of knots. Let $\mathcal{F}$ be a codimension one, transversely oriented C^0 foliation of finite depth with C^∞ leaves on a compact, orientable manifold. In Section 3, we give a means of modifying $\mathcal{F}$ to obtain a "good" foliation to which we can define the gap in Section 2.1.2. In Section 4, we give the proofs of Theorems 1.0.1 and 1.0.2. In Section 4.1, for a q-twisted doubled knot K_q with companion K', and standard Seifert surface S_q (for the definition of these terms, see Section 2.2), we show that the manifold M_{S_q} obtained from the complementary sutured manifold of S_q by attaching a 2-handle along its suture is homeomorphic to the exterior of a 2-component link obtained from K' by taking parallel copies with linking number q (Proposition 4.1.1). This implies that M_{S_q} admits a decomposition $M_{S_q} = E(K') \cup Q$, where $E(K')$ is the exterior of K', and Q is the manifold homeomorphic to (disk with two holes)$\times S^1$. Note that $\Sigma^{(n)}(K, 0)$ is a union of n copies of M_{S_q}, say $\Sigma^{(n)}(K, 0) = M_1 \cup \cdots \cup M_n$, where each M_i is homeomorphic to M_{S_q} (moreover, according to the above decomposition of M_{S_q}, each M_i admits a decomposition $M_i = E(K')_i \cup Q_i$). Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K, 0)$ by cutting along a torus obtained from a lift of S_q by capping off the boundary by a meridian disk of the solid torus (for the 0-sugered manifold $S^3(K, 0)$). Let $\mathcal{F}$ be a codimension one, transversely oriented, taut C^0 foliation of finite depth on $M^{(n)}$ such that the union of the compact leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$. In Section 4.2, for the proof of Theorem 1.0.1, we study the depth of $\mathcal{F}$. In Section 4.2.1, for a doubled knot K, we show that there exists a submanifold $M^{(n)\prime}$ which is obtained from $M^{(n)}$ by removing a regular neighbourhood of $\partial M^{(n)}$ such that $\mathcal{F}$ is transverse to $\partial M^{(n)\prime}$, hence $\mathcal{F} |_{M^{(n)\prime}}$ (: $\mathcal{F}'$, say) is a foliation on $M^{(n)\prime}$ transverse to $\partial M^{(n)\prime}$. Then we show that $\mathrm{depth}(\mathcal{F}') \leq \mathrm{depth}(\mathcal{F}) - 1$. These imply that our research can be reduced to the study of the depth of $\mathcal{F}'$. Since $M^{(n)\prime}$ is homeomorphic to $M^{(n)}$, we abuse notation by denoting $M^{(n)\prime} = M_1 \cup \cdots \cup M_n = (E(K')_1 \cup Q_1) \cup \cdots \cup (E(K')_n \cup Q_n)$. In Section 4.2.2, we mimic the arguments of Cantwell-Conlon [2, Section 2] to show that if K' is a non-cable knot, then via an ambient isotopy, we can put $\mathcal{F}'$ in a position which is nice with respect to the submanifolds $E(K')_i$. Let d_i be the minimal value of the depths of the leaves of $\mathcal{F}'$ which meet $M(K)_i$. In Section 4.2.3, we show with adding condition $q = 0$ that via an amibient isotopy, we can put $\mathcal{F}'$ in a position which is nice with respect to Q_i. In Section 4.2.4, we analyze the behaviors of d_i's by using arguments introduced in Section 4.2.3 (for example, we give the inequalities $d_1 < \cdots < d_k \leq d_{k+1} > \cdots > d_n$), which imply a lower bound of $\mathcal{F}'$ (Corollary 4.2.1). In Section 4.3, we give the proof of Theorem 1.0.1 by showing that the depth 0 leaf of each foliation treated in Theorem 1.0.1 is isotopic to a lift of a torus which is obtained from S_q by capping off the boundary by a meridian disk. In Section 4.4, we give the proof of Theorem 1.0.2. For a given foliation with exactly one depth 0 leaf on $\Sigma^{(n)}(K, 0)$ above, we can apply the modification in Section 3 to obtain the foliation to which we can define the gap. By applying the arguments as in Section 4.3, we can obtain a foliation treated in Section 4.2. For such a foliation, we show some properties about the graph $\hat{G}(\hat{\mathcal{F}})$ (Lemmas 4.4.1$\sim$4.4.3). Then by considering positions of leaves representing vertices on the unique path joining the depth 0 vertices of $\hat{G}(\hat{\mathcal{F}})$ when it is a tree, we give estimations of depths of $\mathcal{F}$. By using these arguments, we obtain Theorem 1.0.2.

2 Preliminaries

2.1 Codimension one foliations

Let M be a Riemannian manifold of dimension n. In this subsection, we suppose that M is compact and orientable.

Definition 2.1.1 A *codimension* q *(or dimension* $n-q$*)* $C^r(0 \leq r \leq \infty)$ *foliation* on M is a C^r atlas $\mathcal{F}$ on M with the following properties.

1. If $(U, \varphi) \in \mathcal{F}$, then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^{n-q} \times \mathbb{R}^q$ where U_1 (U_2 resp.) is an open disk in $\mathbb{R}^{n-q}$ ($\mathbb{R}^q$ resp.).

2. If (U, φ) and $(V, \psi) \in \mathcal{F}$ are such that $U \cap V \neq \emptyset$, then the change of coordinates map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is of the form $\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y))$.

The charts $(U, \varphi) \in \mathcal{F}$ will be called *foliation charts*. We call the pair $(M, \mathcal{F})$ a *foliated manifold*.

For the basic terminologies concerning foliations (holonomy, etc.), see [1]. Let $\mathcal{F}$ be a codimension q $C^r(0 \leq r \leq \infty)$ foliation on M. Let (U, φ) be a foliation chart. The sets of the form $\varphi^{-1}(U_1 \times \{c\})$, $c \in U_2$ are called *plaques* of U, or else plaques of $\mathcal{F}$.

Definition 2.1.2 A *path of plaques* of $\mathcal{F}$ is a sequence of plaques $P_1, \ldots, P_k$ of $\mathcal{F}$ such that $P_j \cap P_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$. Since M is covered by plaques of $\mathcal{F}$, we can define on M the following equivalence relation: $p_1 \cong p_2$ if there exists a path of plaques $P_1, \ldots, P_k$ with $p_1 \in P_1, p_2 \in P_k$. The equivalence classes of the relation are called *leaves* of $\mathcal{F}$.

In the remainder of this section, suppose $q = 1$, i.e., $\mathcal{F}$ is a codimension one foliation.

Definition 2.1.3 We say that a leaf of $\mathcal{F}$ is *proper* if its topology as a manifold coincides with the topology induced from that of M. A foliation $\mathcal{F}$ is called *proper* if every leaf of $\mathcal{F}$ is proper.

Definition 2.1.4 We say that $\mathcal{F}$ is *taut* if for any leaf L of $\mathcal{F}$, there is a properly embedded (possibly, closed) transverse curve which meets L.

Definition 2.1.5 Let $\mathcal{F}$ be a C^r foliation. We say that $\mathcal{F}$ is a C^r foliation with C^∞ leaves on M if each leaf is a C^∞ immersed manifold.

By using a partition of unity argument, we can show that any codimension one, transversely oriented foliation with C^∞ leaves has a one dimensional C^∞ foliation which is transverse to $\mathcal{F}$. Let $\mathcal{F}^\perp$ be a one dimensional C^∞ foliation which is transverse to $\mathcal{F}$.

Let $(M, \mathcal{F})$ be a foliated manifold. A subset U of M is called *saturated* if U is a union of leaves of $\mathcal{F}$. It is clear that closed leaves are always proper, and it is easy to see that each proper leaf L has an open saturated neighbourhood U in which it is relatively closed ($\overline{L} \cap U = L$).

Notation 2.1.1 Let U be an open saturated set, and $\iota : U \longrightarrow M$ be the inclusion. There is an induced Riemannian metric on $U \subset M$. Then $\hat{U}$ denotes the metric completion of U, and $\hat{\iota} : \hat{U} \longrightarrow M$ denotes the extended isometric immersion. Let $\hat{\mathcal{F}} = \hat{\iota}^{-1}(\mathcal{F})$, and $\hat{\mathcal{F}}^\perp = \hat{\iota}^{-1}(\mathcal{F}^\perp)$ be the induced foliations on $\hat{U}$. The set $\delta U = \hat{\iota}(\partial \hat{U})$ is called the *border* of U.

In the remainder of this section, we suppose $\mathcal{F}$ is transversely oriented.

Definition 2.1.6 An $(\mathcal{F}, \mathcal{F}^\perp)$ *coordinate atlas* is a locally finite collection of C^r embeddings $\varphi_i : D^{n-1} \times [0,1] \to M$ such that the interior of the images cover M, and the restriction of φ_i to each $D^{n-1} \times \{t\}$ (to each $\{x\} \times [0,1]$ resp.) is a C^∞(C^r resp.) embedding into a leaf of $\mathcal{F}$($\mathcal{F}^\perp$ resp.).

The unit tangent bundle $q : \widetilde{M} \longrightarrow M$ of $\mathcal{F}^\perp$ is a C^∞ double covering of M. Since $\mathcal{F}$ is transversely oriented, for each leaf L of $\mathcal{F}$, $q^{-1}(L)$ consists of two components. Each component of $q^{-1}(L)$ is called a *side* of L.

Definition 2.1.7 A side $\widetilde{L}$ of $q(\widetilde{L}) = L$ is *proper* if there are a transverse curve $\tau : [0,1] \longrightarrow M$ starting from L in the direction of $\widetilde{L}$ and $\varepsilon(> 0)$ such that $\tau(t) \notin L$ for $0 < t < \varepsilon$. Let $\widetilde{L}$ be a proper side of L. The leaf L has *unbounded holonomy* on the side $\widetilde{L}$ if there are a transverse curve $\gamma : [0,1] \longrightarrow M$ starting from L in the direction of $\widetilde{L}$ and a sequence $h_1, h_2, \dots$ of holonomy pseudogroup elements with domain containing $\mathrm{im}(\gamma)$ such that

$$h_i(\mathrm{im}(\gamma)) = \gamma([0, \varepsilon_i]), \ \varepsilon_i \searrow 0.$$

The leaf L is *semistable* on the side $\widetilde{L}$ if there is a sequence $e_1, e_2, \dots$ of C^∞ immersions of $\widetilde{L} \times [0,1]$ (with its manifold structure) into M such that $e_i(x, 0) =$

$q(x)$ for all x and i, $e_{i*}(\frac{\partial}{\partial t})|_{t=0}$ points in the direction $\widetilde{L}$, $e_{i*}(\frac{\partial}{\partial t})$ is always tangent to $\mathcal{F}^{\perp}$, each $e_i(\widetilde{L} \times \{1\})$ is a leaf of $\mathcal{F}$, and $\bigcap_i e_i(\widetilde{L} \times [0,1]) = L$.

In [4], Dippolito showed the following.

Theorem 2.1.1 (Semistability Theorem [4, ☆]) *Let $\mathcal{F}$ be a codimension one foliation with C^∞ leaves on a closed manifold. If $\widetilde{L}$ is a proper side of a leaf L of $\mathcal{F}$, then L either is semistable or has unbounded holonomy on the side $\widetilde{L}$.*

Remark 2.1.1 Any side of a proper leaf is proper.

2.1.1 Depth of foliations

Definition 2.1.8 A leaf L of $\mathcal{F}$ is at *depth 0* if it is compact. Inductively, when leaves of at depth less than k are defined, L is at *depth* $k \geq 1$ if $\overline{L} \setminus L$ consists of leaves at strictly less than k, and at least one of which is at depth $k-1$. If L is at depth k, we use the notation $\operatorname{depth}(L) = k$, and call L a *depth k leaf.* The foliation $\mathcal{F}$ is of depth $k < \infty$ if every leaf of $\mathcal{F}$ is at depth at most k and k is the least integer for which this is true. If $\mathcal{F}$ is of depth k, we use the notation $\operatorname{depth}(\mathcal{F}) = k$. If there is no integer $k < \infty$ which satisfies the above condition, the foliation $\mathcal{F}$ is of infinite depth.

In the remainder of this section, we suppose $\mathcal{F}$ is of finite depth.

Remark 2.1.2 Let L, L' be leaves of $\mathcal{F}$. By Definition 2.1.8, we see that if $\overline{L} \setminus L \subset \overline{L'} \setminus L'$, then $\operatorname{depth}(L) \leq \operatorname{depth}(L')$.

Remark 2.1.3 It is known that any leaf of $\mathcal{F}$ is proper.

Lemma 2.1.1 *For any leaf L of $\mathcal{F}$, there exists a depth 0 leaf of $\mathcal{F}$ in $\overline{L}$.*

Proof. By the definition of depth of leaves (Definition 2.1.8), there exists a leaf L_1 of $\mathcal{F}$ such that $L_1 \subset \overline{L} \setminus L$, and $\operatorname{depth}(L_1) < \operatorname{depth}(L)$. If $\operatorname{depth}(L_1) = 0$, then the lemma holds. Suppose $\operatorname{depth}(L_1) > 0$. We claim that $\overline{L} \supset \overline{L_1}$. Let x be a point in $\overline{L_1}$. Since M is a complete metric space, there is a Cauchy sequence $\{x_i\}_{i=1,2,\ldots}$ ($x_i \in L_1$) such that x_i converges to x. Since $x_i \in \overline{L}$, there exists y_i in L such that $d(x_i, y_i) \leq \frac{1}{i}$. Clearly, y_i converges to x, i.e., $x \in \overline{L}$. By the definition of depth of leaves (Definition 2.1.8), there exists a leaf L_2 in $\overline{L_1}$ such that $\operatorname{depth}(L_2) < \operatorname{depth}(L_1)$. If $\operatorname{depth}(L_2) = 0$, then the lemma holds. Suppose that $\operatorname{depth}(L_2) > 0$. Since $\overline{L} \supset \overline{L_1} \supset L_2$, we apply the above argument to L_2 to show that $\overline{L} \supset \overline{L_2}$. Then by applying the above arguments repeatedly, we see that the lemma holds. $\square$

Lemma 2.1.2 *Let L be a leaf of $\mathcal{F}$. Suppose L has unbounded holonomy on the side $\widetilde{L}$ and let γ be as in Definition 2.1.7. Then for any leaf L' of $\mathcal{F}$ such that $L' \cap \gamma \neq \emptyset$, we have depth$(L) <$ depth(L').*

Proof. Let $h_1, h_2, \dots$ be as in Definition 2.1.7. Fix a point $x_0 \in L' \cap \gamma$. Let $x_i = h_i(x_0)$ $(i = 1, 2, \dots)$. Then $x_i \in L'$, and x_i converges to the point $\gamma(0) \in L$. This shows that $L \subset \overline{L'}$. Since $L \neq L'$, this implies depth$(L) <$ depth(L'). $\square$

Lemma 2.1.3 *Let L, L' be leaves of $\mathcal{F}$. Suppose L is semistable on the side $\widetilde{L}$ and let $e_1, e_2, \dots$ be as in Definition 2.1.7. Suppose there exists i such that $e_i(\widetilde{L} \times [0,1]) \supset L'$. Then we have depth$(L) \leq$ depth(L').*

Proof. If L is compact, then obviously the lemma holds. Suppose L is non-compact. Then $\overline{L} \setminus L \neq \emptyset$. Let L^* be a leaf contained in $\overline{L} \setminus L$. Fix a point x^* in L^*. Let P be a plaque of $\mathcal{F}^{\perp}$ through x^*. Let P' be the closure of a component of $P \setminus x^*$ such that $x^* \in \overline{P' \cap L}$. Then we can take points $x_1, x_2, \dots$ in $P' \cap L$ such that x_i monotonously converges to x^*. Let $\widetilde{x}_j$ be the points in $\widetilde{L}$ such that $e_i(\widetilde{x}_j \times \{0\}) = x_j$. Let $P_j = e_i(\widetilde{x}_j \times [0,1])$. Then $P_2, P_3, \dots$ are mutually disjoint arcs embedded in P'. Since $L' \subset e_i(\widetilde{L} \times [0,1])$, $L' \cap P_j \neq \emptyset$ $(j = 2, 3, \dots)$ Fix a point $x'_j \in L' \cap P_j$. Then $\{x'_j\}_{j=2,3,\dots}$ converges to x^*. Hence $L^* \subset \overline{L'}$. Since $L^* \neq L'$, this implies that $\overline{L} \setminus L \subset \overline{L'} \setminus L'$. By Remark 2.1.2, we have depth$(L) \leq$ depth(L'). $\square$

Lemma 2.1.4 *Let $\{L_i^{(d)}\}$ be a set of depth d leaves of $\mathcal{F}$, U a component of $M \setminus \overline{\cup L_i^{(d)}}$, and F a component of $\partial \hat{U}$. Then depth$(L) \leq d$, where L denotes the leaf $\hat{\imath}(F)$ of $\mathcal{F}$.*

Proof. Fix a point x in L. Let P be a plaque of $\mathcal{F}^{\perp}$ through x. Let P_1, P_2 be the closures of the components of $P \setminus x$. We may suppose P_2 is contained in $\hat{\imath}(\hat{U})$. If there exists a subarc P_1' of P_1 with $x \in \partial P_1'$ such that $P_1' \subset \hat{\imath}(\hat{U})$, then obviously $L \in \{L_i^{(d)}\}$. Hence we may suppose for any subarc P_1' of P_1 with $x \in \partial P_1'$, we have $P_1' \not\subset \hat{\imath}(\hat{U})$. If there exists a subarc P' of P_1 with $x \in \partial P'$ such that P' does not intersect $\cup L_i^{(d)}$, then $L \in \{L_i^{(d)}\}$, hence depth$(L) = d$.

Suppose for any subarc P' of P_1 with $x \in \partial P'$, there exists $L' \in \{L_i^{(d)}\}$ such that $L' \cap P_1' \neq \emptyset$. Then the situation is divided into the following two cases.

Case 1 L is semistable on the side $\widetilde{L}$ which contains P_1.

Let e_1, e_2, $\ldots$ be as in Definition 2.1.7. Take a subarc P'' of P_1 with $x \in \partial P''$ and $P'' \subset \mathrm{im}(e_1)$. Let L'' be an element of $\{L_i^{(d)}\}$ such that $L'' \cap P'' \neq \emptyset$. By Lemma 2.1.3, we have $\mathrm{depth}(L) \leq \mathrm{depth}(L'') = d$.

Case 2 L has unbounded holonomy on the side $\widetilde{L}$ which contains P_1.

Take a subarc γ'' of P_1 with $x \in \partial\gamma''$ and satisfies the condition of γ in Definition 2.1.7. By Lemma 2.1.2, we have $\mathrm{depth}(L) < \mathrm{depth}(L') = d$. □

Lemma 2.1.5 *Let $\{L_i^{(d)}\}$, U be as in Lemma 2.1.4. Suppose there exists a pair of components of $\partial\hat{U}$ representing the same leaf L of $\mathcal{F}$. Then L is an element of $\{L_i^{(d)}\}$.*

Proof. Let x, P_1 be as in the proof of Lemma 2.1.4. It is clear that there exists a subarc P_1' of P_1 with $x \in \partial P_1'$ such that $P_1' \subset \hat{\imath}(\hat{U})$. This implies $L \in \{L_i^{(d)}\}$. □

2.1.2 Gap of foliations

Definition 2.1.9 For leaves L_1 and L_2 of $\mathcal{F}$, we say that L_1 is *equivalent* to L_2 if $L_1 = L_2$ or there exisits an embedding $\phi : L_1 \times [0,1] \longrightarrow M$ such that the image of $L_1 \times \{0\}$ ($L_1 \times \{1\}$ resp.) coincides with L_1 (L_2 resp.), and the image of $\{x\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^{\perp}$ for each $x \in L_1$. Moreover, if $\widetilde{L}$ is the side of L such that $\phi_*(\frac{\partial}{\partial t})|_{t=0}/||\phi_*(\frac{\partial}{\partial t})|_{t=0}||$ is contained in $\widetilde{L}$, then we say that L is equivalent to L' *through the side* $\widetilde{L}$.

Remark 2.1.4 Let L be a leaf of $\mathcal{F}$. Suppose that L is semistable on the proper side $\widetilde{L}$ and let $e_i : \widetilde{L} \times [0,1] \longrightarrow M$ be as in Definition 2.1.7. It is clear that for each i, L is equivalent to $e_i(\widetilde{L} \times \{1\})$ through the side $\widetilde{L}$.

Suppose further M is closed. Let $\widetilde{\mathcal{F}}$ be a codimension one, transversely oriented C^r foliation of finite depth with C^∞ leaves on M which satisfies the following conditions:

1. the number of equivalence classes of the leaves of $\widetilde{\mathcal{F}}$ is finite;
2. let L_1, L_2 be leaves of $\widetilde{\mathcal{F}}$. If L_1 is equivalent to L_2 through the side $\widetilde{L}_1$, then the restriction of $\widetilde{\mathcal{F}}$ to the region between L_1 and L_2 containing the side $\widetilde{L}_1$ is a product foliation with each leaf is homeomorphic to L_1.

Let $[L_i^0]$ be the equivalence classes of the depth 0 leaves of $\widetilde{\mathcal{F}}$. Let $\hat{M}$ be the union of the metric completions of the components of $M \setminus (\bigcup_i L_i^0)$. Let $\hat{\mathcal{F}}$ be the foliation on $\hat{M}$ induced from $\widetilde{\mathcal{F}}$. By the definition of depth, we immediately have the following.

Lemma 2.1.6 *Under the above notations, we have depth*$(\hat{\mathcal{F}}) =$ *depth*$(\widetilde{\mathcal{F}})$.

Let $[L_j]$ be the equivalence classes of the leaves of $\hat{\mathcal{F}}$.

Definition 2.1.10 The graph of $\hat{\mathcal{F}}$ denoted by $\hat{G}(\hat{\mathcal{F}}) = \{V, E\}$ is the directed graph with the vertex set $V = \{v_j\}$ and the edge set $E = \{e_{k\ell}\}$ such that each v_j corresponds to the equivalence class $[L_j]$ of the leaves of $\hat{\mathcal{F}}$ and there is an edge $e_{k\ell}$ from v_k to v_ℓ if $L_\ell \subset \overline{L_k} \setminus L_k$, and there does not exist a leaf L such that $L \subset \overline{L_k} \setminus L_k$ and $L_\ell \subset \overline{L} \setminus L$.

By the construction, the foliated manifold $(M, \widetilde{\mathcal{F}})$ is recovered from $(\hat{M}, \hat{\mathcal{F}})$ by identifying pairs of depth 0 leaves L_i^{0+} and L_i^{0-}, each corresponding to L_i^0. Then we define the graph of $\widetilde{\mathcal{F}}$ as follows.

Definition 2.1.11 The graph of $\widetilde{\mathcal{F}}$ denoted by $G(\widetilde{\mathcal{F}})$ is the graph obtained from $\hat{G}(\hat{\mathcal{F}})$ by identifying pairs of vertices corresponding to L_i^{0+} and L_i^{0-} for each depth 0 leaf L_i^0.

By the definition, we immediately have the following.

Lemma 2.1.7 *The following three conditions are equivalent to each other.*

1. *$\hat{G}(\hat{\mathcal{F}})$ is the graph consisting of exactly one vertex.*
2. *$\widetilde{\mathcal{F}}$ is a foliation given by a fiber bundle structure over S^1.*
3. *There exists a leaf L_i^0 of $\widetilde{\mathcal{F}}$ such that L_i^{0+} and L_i^{0-} corresponding to the same vertex of $\hat{G}(\hat{\mathcal{F}})$.*

Definition 2.1.12 Let v be a vertex of $\hat{G}(\hat{\mathcal{F}})$ or $G(\widetilde{\mathcal{F}})$. We say that v is at *depth k* if v represents a leaf at depth k. If v is at depth k, we use the notation $\mathrm{depth}(v) = k$, and call v a *depth k vertex*.

Definition 2.1.13 Let e be an edge of $\hat{G}(\hat{\mathcal{F}})$ or $G(\widetilde{\mathcal{F}})$. Let v be the initial point and v' the terminal point of e. Then, we define the *length* of e as follows:

$$\mathrm{length}(e) = \mathrm{depth}(v) - \mathrm{depth}(v').$$

Remark 2.1.5 Let v be a vertex of $\hat{G}(\hat{\mathcal{F}})$ or $G(\widetilde{\mathcal{F}})$. If $\mathrm{depth}(v) \neq 0$, then by the definition of the depth, we see that there exists a directed path $\Gamma = e_1 \cup \cdots \cup e_n$ from v to a depth 0 vertex such that $\mathrm{length}(e_i) = 1$ $(i = 1, \ldots, n)$

Definition 2.1.14 We define the *gap* of the foliation $\widetilde{\mathcal{F}}$ as follows:

$$\mathrm{gap}(\widetilde{\mathcal{F}}) = \begin{cases} 0 & \text{if } G(\widetilde{\mathcal{F}}) \text{ has no edges,} \\ \max_{e:\text{edges of } G(\widetilde{\mathcal{F}})}\{\mathrm{length}(e)\} & \text{if } G(\widetilde{\mathcal{F}}) \text{ has an edge.} \end{cases}$$

2.2 Topology of three dimensional manifolds

In this subsection, we introduce some basic terminologies concerning three dimensional manifolds. Throughout this subsection, we suppose submanifolds are differentiable, hence each submanifolds admits a regular neighbourhood. For a submanifold G of a manifold M, $N(G, M)$ denotes a regular neighbourhood of G in M. When M is clear from the context, we often abbreviate $N(G, M)$ by denoting $N(G)$. Let β be a simple closed curve embedded in a surface. We say that β is *inessential* if there exists a disk D in the surface such that $\partial D = \beta$. The simple closed curve β is *essential* if it is not inessential.

In the remainder of this subsection, M denotes a three dimensional manifold. We say that M is *irreducible* if for any two-sphere S^2 embedded in M, there exists a three-ball B^3 in M such that $\partial B^3 = S^2$. Let F be a surface properly embedded in M. We say that F is *compressible* if there is a disk D in M such that $D \cap F = \partial D$ and ∂D is essential in F. The disk D is called a *compression disk*. The surface F is *incompressible* if it is not compressible. Let F_1, F_2 be surfaces embedded in M such that $\partial F_1 = \partial F_2$ or $\partial F_1 \cap \partial F_2 = \emptyset$. We say that F_1 and F_2 are parallel (or F_2 is parallel to F_1) if there is a submanifold N in M such that N is homeomorphic to $F_1 \times [0, 1]$ where we have the following.

1. If $\partial F_1 = \partial F_2$, then F_1(F_2 resp.) corresponds to the closure of the component of $\partial(F_1 \times [0, 1]) \setminus (\partial F_1 \times \{1/2\})$ that contains $F_1 \times \{0\}$($F_1 \times \{1\}$ resp.).

2. If $\partial F_1 \cap \partial F_2 = \emptyset$, then F_1(F_2 resp.) corresponds to $F_1 \times \{0\}$($F_1 \times \{1\}$ resp.) and $N \cap \partial M$ corresponds to $\partial F_1 \times [0, 1]$.

Suppose F is connected. We say that F is *∂-parallel* if F is parallel to a subsurface of ∂M. We say that F is *essential* if F is incompressible and not ∂-parallel.

2.2.1 Thurston norm

Let F be a surface in ∂M. Then, for a connected surface S properly embedded in (M, F), let $\chi_-(S) = \max\{0, -\chi(S)\}$. In general, for a surface $\mathscr{S}$, let $\chi_-(\mathscr{S}) = \Sigma_{i=1}^n \chi_-(S_i)$ ($S_1, \dots, S_n$ are the components of $\mathscr{S}$). For a nontrivial homology class $a \in H_2(M, F; \mathbb{Q})$, we define $x(a) = \min\{\chi_-(\mathscr{S}) \mid \mathscr{S}$ is a surface properly embedded in (M, F) which represents $a \in H_2(M, F; \mathbb{Q})\}$. Let G be a surface properly embedded in (M, F). We say that G is *norm minimizing* if $\chi_-(G) = x([G])$, where $[G]$ is the element of $H_2(M, F; \mathbb{Q})$ represented by G. Let S be a surface properly embedded in M. We say that S is *taut* if S is incompressible and norm minimizing in $H_2(M, N(\partial S, \partial M))$.

Let V be a solid torus in M. A simple closed curve m in ∂V is called a *meridian* of V if m is essential in ∂V and there exists a disk D properly embedded in V such that $\partial D = m$. A simple closed curve ℓ is called a *longitude* of V if it is null-homologous in $\overline{M \setminus V}$.

Remark 2.2.1 For a taut foliation, it is known that any compact leaf is norm minimizing [17, Corollary 2]. We can show that any leaf of a taut foliation is incompressible by the arguments in the proof of Lemma 7 of [15], hence any compact leaf of a taut foliation is taut.

2.2.2 Knots and links

The union of finite number of mutually disjoint oriented simple closed curves in a 3-manifold M is called a *link*. For a link L in M, $E(L)$ denotes $\overline{M \setminus N(L, M)}$. We call $E(L)$ an *exterior* of L. A link which consists of one component is called a *knot*. Let K_1, K_2 be knots in M. We say that K_1 and K_2 are *equivalent* (or K_2 is equivalent to K_1) if there exists a homeomorphism $h : M \longrightarrow M$ such that $h(K_2) = K_1$. A *Seifert surface* for K is an oriented connected surface S embedded in M such that $\partial S = K$. Note that $N(K)$ is a solid torus. Then a meridian (longitude resp.) of $N(K)$ is called a meridian (longitude resp.) of K. It is known that for any knot K in the 3-sphere S^3, there exists a Seifert surface for K. This implies that there exists a longitude of K intersecting a meridian of K transversely in one point. If a Seifert surface for K has minimal genus among all Seifert surfaces for K, it is called a *minimal genus Seifert surface* for K and the genus is called the *genus of* K. A knot K in S^3 is called a *trivial knot* if there exists a disk D^2 embedded in S^3 such that $\partial D^2 = K$, otherwise K is called a *non-trivial knot*.

Definition 2.2.1 Let R, R_1, R_2 be oriented surfaces in S^3. We say that R is obtained by plumbing R_1 and R_2 if they satisfy the following.

1. $R = R_1 \cup R_2$, where $R_1 \cap R_2$ is a rectangle D with edges a_1, b_1, a_2, b_2 as in Figure 2.1 such that $a_1, a_2 \subset \partial R_1$ and $b_1, b_2 \subset \partial R_2$, a_1 and a_2 (b_1 and b_2 resp.) are arcs properly embedded in R_2 (R_1 resp.).

2. There exist 3-balls B_1, B_2 in S^3 which satisfy the following.

   (a) $B_1 \cup B_2 = S^3$.

   (b) $B_1 \cap B_2 = \partial B_1 = \partial B_2$.

   (c) $B_i \supset R_i$ $(i = 1, 2)$.

   (d) $\partial B_1 \cap R_1 = \partial B_2 \cap R_2 = D$.

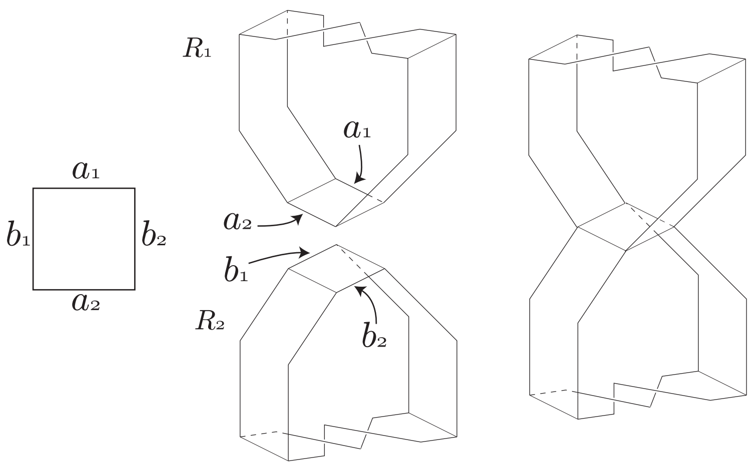


Figure 2.1

Let K' be a knot in S^3, and $V = D^2 \times S^1$ an unknotted solid torus in S^3. Let L be a link in V such that L is not contained in any 3-ball in V, and $h : V \longrightarrow N(K')$ a homeomorphism. Then the link $h(L)$ is called a *satellite* for K', and K' is called a *companion* for $h(L)$. Let ℓ (m resp.) be a longitude (a meridian resp.) of V. If L is a knot in ∂V representing a homology class $p[m] + q[\ell] (\in H_1(\partial V, \mathbb{Z}))$ (with $|q| \geq 2$) then $h(L)$ is called a *cable* of K'. We say that a knot K is a *cable knot* if there exists a knot K'' such that K is a cable of K''. Let C be the knot in V as in Figure 2.2 and ℓ' (m' resp.) $(\subset \partial N(K'))$ a longitude (a meridian resp.) of $N(K')$. For an integer q, let $h_q : V \longrightarrow N(K')$ be a homeomorphism with $(h_q)_*([m]) = [m'], (h_q)_*([\ell]) = [\ell'] + q[m']$. Then, we call the satellite $h_q(C)$ a *q-twisted double* of K' (or simply, we say that $h_q(C)$ is a q-twisted doubled knot). Let S be the genus one surface in V, as in Figure 2.3. Clearly $h_q(S)$ is a Seifert surface for $h_q(C)$. We often use the notation S_q for denoting this Seifert surface. Note that if K' is a non-trivial knot, then for any $q \in \mathbb{Z}$, $h_q(C)$ is a non-trivial knot [13, IV.10]. Since S_q is of genus one, this implies that if K' is a non-trivial knot, then S_q is a minimal genus Seifert surface for $h_q(C)$. We call S_q a *standard Seifert surface* for $h_q(C)$. An unknotted annulus in S^3 with positive or negative one full twist is called a *Hopf annulus* (see Figure 2.4). Let A be the annulus embedded in V as in Figure 2.3, and let $A_q = h_q(A)$. Then by Figure 2.3 , we see that S_q is obtained by plumbing A_q and a Hopf annulus.

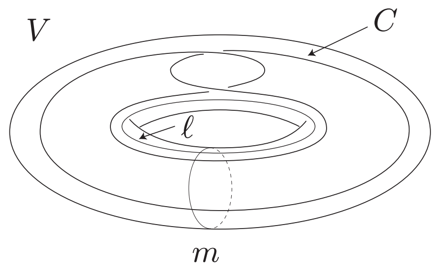


Figure 2.2

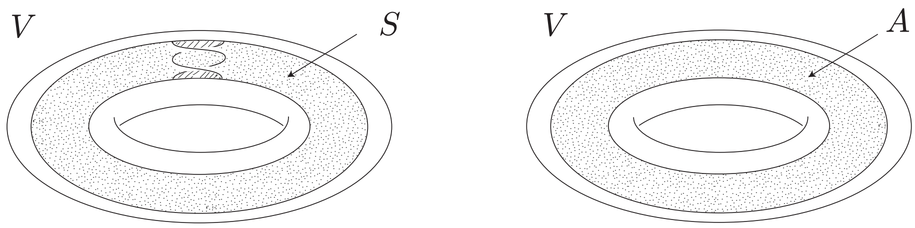


Figure 2.3

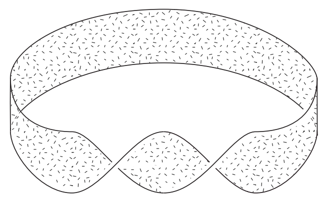


Figure 2.4

2.2.3 Depth of knots

Recall that M denotes a 3-manifold. In this subsection, we suppose that M is compact and oriented.

D. Gabai showed the following in [6].

Theorem 2.2.1 ([6, Theorem ☆]) *Let K be a knot in S^3, and S a minimal genus Seifert surface for K. Then there exists a codimension one, transversely oriented, taut C^0 foliation $\mathcal{F}$ of finite depth on $E(K)$ such that $S \cap E(K)$ is a compact leaf of $\mathcal{F}$, and that $\mathcal{F} \mid_{\partial E(K)}$ is a foliation by circles.*

Cantwell-Conlon [3] proposed the following.

Definition 2.2.2 Let K be a knot in S^3. For an integer r $(0 \leq r \leq \infty)$, K is of C^r-depth at most k if the knot exterior $E(K)$ admits a codimension one, transversely oriented, taut C^r foliation which is transverse to $\partial E(K)$ and of depth at most k. If k is the least integer for which this is true, then K *is of C^r-depth k*. If such an integer does not exist, the knot K is of infinite C^r-depth.

The following definition which gives a variation of depth is required for the statement of Theorem 1.0.1.

Definition 2.2.3 Suppose a nontrivial element α of $H_2(M, \mathbb{Q})$ is represented by a connected, taut surface F. Then, we define C^r 1-depth of M associated to α which is denoted by $\mathrm{depth}^r_{1,\alpha}(M)$ as follows.
$\mathrm{depth}^r_{1,\alpha}(M) = \min\{\mathrm{depth}(\mathcal{F}) \mid \mathcal{F}$ is a codimension one, transversely oriented, taut C^r foliation with exactly one depth 0 leaf, which represents $\alpha\}$

2.2.4 Sutured manifolds

In this subsection, we quickly recall the definition of sutured manifold introduced by Gabai [6] and some related concepts.

Definition 2.2.4 Let M be a compact, oriented 3-manifold. The pair (M, γ) is *a sutured manifold* if γ is a union of mutually disjoint annuli in ∂M satisfying the following conditions.

1. Each component of γ contains an oriented simple closed curve which is homologically non-trivial in γ. The simple closed curve is called a *suture*, and $s(\gamma)$ denotes the union of the sutures of γ.

2. Let $R(\gamma) = \partial M \setminus \mathrm{int}\gamma$. Then $R(\gamma)$ is oriented as follows. For each component δ of $\partial R(\gamma)$ and each component A_1 of γ such that $A_1 \cap \delta \neq \emptyset$, we have the following. If s_1 is the suture in A_1, then δ and s_1 are homologous in A_1.

Then $R_+(\gamma)$ ($R_-(\gamma)$ resp.) denotes the union of the components of $R(\gamma)$ with its normal vectors point outward (inward resp.).

Note that in the original definition of sutured manifold, γ may have torus components. However in our setting this situation does not occur. Hence we gave a definition for the restricted case.

Let S be a compact surface with $\partial S \neq \emptyset$, $M = S \times [0,1]$, and $\gamma = \partial S \times [0,1]$. It is easy to see that (M, γ) admits a sutured manifold structure with $R_+(\gamma) = S \times \{0\}$ (see Figure 2.5). We call the sutured manifold a *product sutured manifold.*

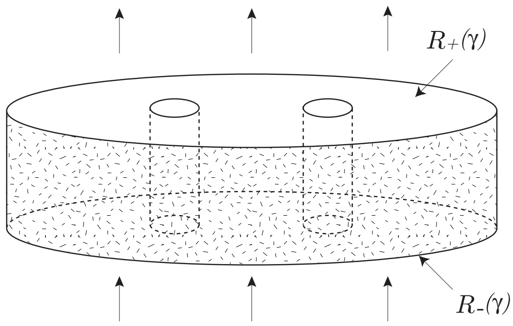


Figure 2.5

Let (M, γ) be a sutured manifold and D a disk properly embedded in M. We call D a *product disk* for (M, γ), if $D \cap R_+(\gamma)$ consists of an arc properly embedded in $R_+(\gamma)$, and $D \cap R_-(\gamma)$ consists of an arc properly embedded in $R_-(\gamma)$. Let M' be the manifold obtained from M by cutting along D, $\hat{\gamma}$ the image of γ in M', D^+ and D^- the copies of D in M'. Then let $\gamma' = \hat{\gamma} \cup D^+ \cup D^-$. It is easy to see that (M', γ') inherits a sutured manifold structure from (M, γ). We say that the sutured manifold (M', γ') is obtained from (M, γ) by the *product decomposition* along the product disk D (Figure 2.6).

Let K be a knot in an oriented 3-manifold and S a Seifert surface for K. We may suppose that $S \cap E(K) = \overline{S \setminus N(\partial S, S)}$ and $S \cap E(K)$ is properly embedded in $E(K)$. Then S_E denotes the surface $S \cap E(K)$. Let M_1 be a regular neighbourhood of S_E in $E(K)$ and $\gamma_1 = M_1 \cap \partial E(K)$. Clearly, (M_1, γ_1)

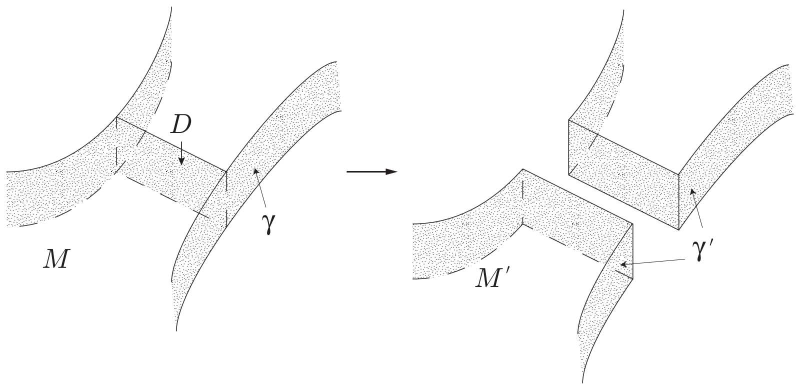


Figure 2.6

is a product sutured manifold. Let $X_S = \overline{E(K) \setminus M_1}$, $\gamma_S = \overline{\partial E(K) \setminus \gamma_1}$ $(= X_S \cap \partial E(K))$. It is easy to see that (X_S, γ_S) admits a sutured manifold structure with $R_{\pm}(\gamma_S) = R_{\mp}(\gamma_1)$. We call the sutured manifold (X_S, γ_S) the *complementary sutured manifold* of S.

3 Modifying foliations

Let M be a compact, oriented n dimensional manifold and $\mathcal{F}$ a codimension one, transversely oriented C^r foliation of finite depth with C^∞ leaves on M. We further suppose $\mathrm{depth}(\mathcal{F}) \neq 0$. In this section, we show that for any foliation as above, we can modify $\mathcal{F}$ to obtain a foliation with the condition soon after Remark 2.1.4 in Section 2.1.2. Let $\mathcal{F}^\perp$ be a one dimensional C^∞ foliation on M which is transverse to $\mathcal{F}$. Since M is compact, we can take an $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas $\{\varphi_i\}$ of $m(< \infty)$ components.

3.1 First step of Modification

In this subsection, we describe a procedure for modifying $\mathcal{F}$ by using depth 0 leaves of $\mathcal{F}$.

Lemma 3.1.1 *Under the equivalence relation of Definition 2.1.9, the number of the equivalence classes represented by the depth 0 leaves is at most* $2m +$ *rank* $H_1(M, \mathbb{R}) - 1$.

Proof. Let $\{L_j^{(0)}\}$ be representatives of the equivalence classes of the depth 0 leaves of $\mathcal{F}$. We assume that $\{L_j^{(0)}\}$ has $2m+\mathrm{rank}\ \mathrm{H}_1(M, \mathbb{R})$ elements. By slightly modifying the $(\mathcal{F}, \mathcal{F}^\perp)$-coodinate atlas $\{\varphi_i\}$ if necessary, we may suppose that $(\cup L_j^{(0)}) \cap (\cup \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Note that if we take any subset of $\{L_j^{(0)}\}$ consisting of at least $\mathrm{rank}\ \mathrm{H}_1(M, \mathbb{R}) + 1$ elements, then the union of them separates M. Hence the number of the components of $M \setminus \cup L_j^{(0)}$ is at least $2m+1$. Hence we can find U, a component of $M \setminus \cup L_j^{(0)}$ such that $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Note that U is a saturated set. Hence we use notations in Notation 2.1.1 in Section 2.1. For any point x in $\partial \hat{U}$, let $\hat{\tau}_x$ be the leaf of $\hat{\mathcal{F}}^\perp$ which meets x. Since $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$, $\hat{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some i and $c \in D^{n-1}$. Hence $\hat{\tau}_x$ is an arc properly embedded in $\hat{U}$ with endpoints x and y, say. Since $\mathcal{F}$ is transversely oriented, x and y are contained in different components of $\partial \hat{U}$. If $\hat{\imath}(x)$ and $\hat{\imath}(y)$ are contained in the same leaf of $\mathcal{F}$, this implies that $\{L_j^{(0)}\}$ consists of one element, contradicting the assumption that $\{L_j^{(0)}\}$ has $2m + \mathrm{rank}\ \mathrm{H}_1(M, \mathbb{R})$ elements. Thus $\hat{\imath}(x)$ and $\hat{\imath}(y)$ are contained in different leaves, say F_x and F_y of $\mathcal{F}$. Obviously, we can take an embedding $\phi : F_x \times [0,1] \to M$ which gives equivalence relation between F_x and F_y, this contradicts the assumption that each pair of elements of $\{L_j^{(0)}\}$ is not mutually equivalent. $\square$

For an equivalence class $[L]$ represented by a depth 0 leaf L, $\cup L_\alpha$ denotes the union of the leaves of $\mathcal{F}$ representing $[L]$.

Claim Under the above notations, $\cup L_\alpha$ is closed.

Proof of Claim. Let $\{x_i\}_{i=1,2,\dots}$ be a Cauchy sequence in M such that each x_i is contained in $\cup L_\alpha$ and converges to x_∞. We show that $x_\infty \in \cup L_\alpha$. Let L_∞ be the leaf of $\mathcal{F}$ which contains x_∞. Let P be a plaque of $\mathcal{F}^\perp$ through x_∞. We may suppose each x_i is contained in P. Let P_+, P_- be the components of $P \setminus x_\infty$. Then, by retaking x_i if necessary, we may suppose that each x_i is contained in P_+. Let L_i be the leaf of $\mathcal{F}$ which contains x_i. Since L_i is compact, L_i intersects P finitely many times. Thus we may suppose that x_i is the nearest to x_∞ among all the points of $L_i \cap P$. Suppose L_∞ has unbounded holonomy. Let $\gamma, h_1, h_2, \dots$ be as in Definition 2.1.7. Since x_i converges to x_∞, we may suppose $x_n \in \mathrm{im}(\gamma)$, for $n \gg 0$. Fix such n. Take δ_n such that $\gamma([0,\delta_n])$ is the subarc of $\mathrm{im}(\gamma)$ with endpoints x_∞, x_n. Since x_n is the nearest to x_∞, $h_i(\mathrm{im}(\gamma)) \not\subset \gamma([0,\delta_n])$ for any n, a contradiction. Hence L_∞ does not have unbounded holonomy on the side which contains x_i. By Theorem 2.1.1, L_∞ is semistable on the side which contains x_i. Hence L_∞ is equivalent to L_i (Remark 2.1.4). Thus x_∞ is contained in $\cup L_\alpha$. $\square$

Now, we describe how to modify $\mathcal{F}$ near L to obtain a new finite depth foliation $\mathcal{F}^1$. The situation is divided into the following two cases.

Case 1 There exists more than one leaves of $\mathcal{F}$ representing $[L]$.

Case 2 There exists exactly one leaf of $\mathcal{F}$ representing $[L]$.

In Case 1, let $\{\phi_\beta\}$ be the set of all the embeddings which give equivalence relations between L and the leaves representing $[L]$. Let $\mathcal{U} = \cup\phi_\beta(L \times [0,1])$. The situation is divided into the following two subcases.

Case 1.1 $\mathcal{U} = M$.

Claim 1 For each side of L, there is a leaf L' to which L is equivalent through the side.

Proof of Claim 1. Let P be a plaque of $\mathcal{F}^\perp$ through $x \in L$. For a side $\widetilde{L}$ of L, let P_+ be the component of $P \setminus x$ which is contained in the side $\widetilde{L}$. Let x_i be a sequence of points in P_+ which converges to x. Since $\mathcal{U} = M$, for each i, there exists a leaf L_i to which L is equivalent via embedding $\phi_i : L \times [0,1] \to M$ such that $\phi_i(L \times [0,1]) \ni x_i$. If there exists i such that $\phi_i(L \times [0,1])$ contains the side $\widetilde{L}$, then the leaf $\phi_i(L \times \{1\})$ is equivalent to L through the side $\widetilde{L}$. Suppose for each i, $\phi_i(L \times [0,1])$ does not contain the side $\widetilde{L}$. Let $L_i = \phi_i(L \times \{1\})$. Then

L_i is a depth 0 leaf such that $L_i \cap P$ contains a point y_i such that $y_i = x$ or y_i is nearer to x than x_i in P_+. By applying the argument as in the proof of Claim given soon after the proof of Lemma 3.1.1, we see that L_i is semistable on the side $\widetilde{L}$. Hence by Remark 2.1.4, there exisits a leaf L' to which L is equivalent through the side $\widetilde{L}$, this completes the proof of the claim. $\square$

Let $\{\phi^{\pm}_{\beta}\}$ be the set of all the embeddings which give equivalence relations between L and the leaves representing $[L]$ through the side $\widetilde{L}_{\pm}$. Let $\mathcal{U}^{\pm} = \cup\phi^{\pm}_{\beta}(L \times [0,1])$.

Claim 2 Both $\mathcal{U}^+$ and $\mathcal{U}^-$ are closed.

Proof of Claim 2. Since the situation is symmetric, we give the proof for $\mathcal{U}^+$. Let $\{x_i\}_{i=1,2,\ldots}$ be a Cauchy sequence in M such that each x_i is contained in $\mathcal{U}^+$ and converges to x_∞. We show that $x_\infty \in \mathcal{U}^+$. Assume $x_\infty \notin \mathcal{U}^+$. This implies that there does not exisit j such that $x_\infty \in \phi^+_j(L \times [0,1])$. For each i, we fix an embedding ϕ^+_i giving an equivalence relation through the side $\widetilde{L}_+$ such that $x_i \in \phi^+_i(L \times [0,1])$. Let L_∞ be the leaf of $\mathcal{F}$ which contains x_∞. Let $L_i = \phi^+_i(L \times \{1\})$. Let P be a plaque of $\mathcal{F}^\perp$ through x_∞. We may suppose each x_i is contained in P. Let P_+, P_- be the components of $P \setminus x_\infty$. Since $x_\infty \notin \phi^+_i(L\times[0,1])$, all of the x_i's are contained in P_+ or P_-, say P_+. Let y_i be the point of $L_i \cap P_+$ which is the nearest to x_∞. By applying the argument in the proof of Claim given soon after the proof of Lemma 3.1.1, we can show that L_∞ does not have unbounded holonomy on the side which contains x_i. By Theorem 2.1.1, L_∞ is semistable on the side which contains x_i. Hence for large n, L_∞ is equivalent to L_n through the side which contains x_i. Let $\phi' : L_n \times [0,1] \to M$ be an embedding which gives the equivalence relation. Then, by composing ϕ^+_n and ϕ', we can obtain an embedding which gives equivalence relation between L and L_∞ through the side $\widetilde{L}_+$. Hence $x_\infty \in \mathcal{U}^+$, a contradiction. $\square$

Claim 3 There exists a leaf L_* to which L is equivalent through both sides of L.

Proof of Claim 3. Since $\mathcal{U} = M$ and M is connected, the above Claim 2 implies that $\mathcal{U}^+ \cap \mathcal{U}^- \neq \emptyset$. For a point y in $\mathcal{U}^+ \cap \mathcal{U}^-$, let $\phi^{\pm}$ be an embedding from $L \times [0,1]$ to M which gives equivalence relation through $\widetilde{L}_{\pm}$ such that $\phi^{\pm}(L \times [0,1])$ contains y. Let $L^{\pm} = \phi^{\pm}(L \times \{1\})$. If $L^+ = L^-$, then L is equivalent to $L^+ = L^-$ through both sides of L. Suppose $L^+ \subset \mathrm{int}\phi^-(L \times [0,1])$. Since L^+ is transverse to $\mathcal{F}^\perp$, L is equivalent to L^+ through the side $\widetilde{L}^-$. Thus

L^+ satisfies the condition of L_*. The case of $L^- \subset \mathrm{int}\phi^+(L \times [0,1])$ is treated in the same manner, these complete the proof of the claim. $\square$

By joining the embeddings giving equivalence relation between L and L_* in the above Claim 3, we see that there is an immersion $\phi' : L \times [0,1] \to M$ such that the image of $\{x\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$, $\phi'(L \times \{0\}) = \phi'(L \times \{1\}) = L$. Hence M admits a fiber bundle structure over S^1 with each fiber homeomorphic to L and transverse to $\mathcal{F}^\perp$, and L is a fiber. In this case, we let $\mathcal{F}^1$ be the foliation given by this bundle structure, i.e., each leaf of $\mathcal{F}^1$ is a fiber of the fibration.

Case 1.2 $\mathcal{U} \neq M$.

In this case, we first show the following claims (Claims 1$\sim$3).

By applying the argument as in the proof of Claim 2 of Case 1.1, we can show the following.

Claim 1 $\mathcal{U}$ is closed.

Let τ be a leaf of $\mathcal{F}^\perp|_\mathcal{U}$ which meets a component of $\partial\mathcal{U}$, say L_0.

Claim 2 The leaf τ is an arc properly embedded in $\mathcal{U}$.

Proof of Claim 2. Assume not, i.e., τ meets $\partial\mathcal{U}$ in one point. Let $\{x_i\}_{i=1,2,\ldots}$ be a sequence of points on τ such that $d_\tau(x_0, x_i) > i$, where d_τ is the path metric on τ induced from M. By the above Claim 1, $\mathcal{U}$ is compact. Hence there exists an accumulating point of $\cup x_i$. By taking a subsequence of $\{x_i\}_{i=1,2,\ldots}$ if necessary, we may suppose that x_i converges to x_∞. Since $\mathcal{U}$ is closed, $x_\infty \in \mathcal{U}$. Then $x_\infty \in L$ or there exists $\phi_\infty \in \{\phi_\beta\}$ such that $x_\infty \in \phi_\infty(L \times [0,1])$. Since τ meets $\partial\mathcal{U}$ in one point, $x_\infty \notin \partial\mathcal{U}$.

Case 1.2.1 $x_\infty \in L$.

In this case, $L_0 \neq L$. Let ϕ_0 be an embedding which gives equivalence relation between L and L_0. Since x_i converges to x_∞, and $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_0(L \times [0,1])$ is a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$ contained in τ, contradicting that τ meets $\partial\mathcal{U}$ in one point.

Case 1.2.2 $x_\infty \notin L$.

In this case, there exists $\phi_\infty \in \{\phi_\beta\}$ such that $x_\infty \in \phi_\infty(L \times [0,1])$. Let $L_\infty = \phi_\infty(L \times \{1\})$. On the other hand, since $\mathcal{U}$ is closed, $L_0 \subset \mathcal{U}$. This implies that L_0 is equivalent to L. Suppose $L_0 = L$. Since $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$ those are contained in τ, contradicting that τ meets $\partial\mathcal{U}$ in one point.

Suppose $L_0 \neq L$. Let ϕ_0 be an embeddeing which gives equivalence relation between L and L_0. The situation is divided into the following two cases.

Case 1.2.2.1 $\phi_0(L \times [0,1]) \cap \phi_\infty(L \times [0,1]) = \phi_\infty(L \times [0,1])$.

In this case, since the length of each fiber of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$ is finite, and $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$. Hence $\tau \cap \phi_0(L \times [0,1])$ is also a union of infinitely many subarcs of τ which are properly embedded in $\phi_0(L \times [0,1])$, contradicting that τ meets $\partial\mathcal{U}$ in one point.

Case 1.2.2.2 $\phi_0(L \times [0,1]) \cap \phi_\infty(L \times [0,1]) = L$.

In this case, by applying the argument as in Case 1.2.2.1, we can show that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$. Since each fiber of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$ is adjacent to a fiber of $\mathcal{F}^\perp|_{\phi_\infty(L\times[0,1])}$, $\tau \cap \phi_0(L \times [0,1])$ is also a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_0(L\times[0,1])}$. This contradicts the assumption that τ meets $\partial\mathcal{U}$ in one point. $\square$

Claim 3 The boundary of $\mathcal{U}$ consists of two components.

Proof of Claim 3. Since $\mathcal{F}$ is transversely oriented, we see by Claim 2 of Case 1.2 that $\partial\mathcal{U}$ consists of at least two components. Suppose $L \subset \partial\mathcal{U}$. Let L'' be another component of $\partial\mathcal{U}$. Since $\mathcal{U}$ is closed, $L'' \subset \mathcal{U}$. Hence L'' is equivalent to L. Let ϕ'' be an embedding which gives equivalence relation between L and L''. Then, it is obvious that $\mathcal{U} = \phi''(L \times [0,1])$, hence $\partial\mathcal{U} = L \cup L''$. Suppose $L \subset \operatorname{int}\mathcal{U}$. Let L_1, L_2 be different components of $\partial\mathcal{U}$. Let ϕ_1 (ϕ_2 resp.) be an embedding which gives equivalence relation between L and L_1 (L_2 resp.). Then it is obvious that $\mathcal{U} = \phi_1(L \times [0,1]) \cup \phi_2(L \times [0,1])$. Hence $\partial\mathcal{U} = L_1 \cup L_2$. This completes the proof of the claim. $\square$

Let $\partial\mathcal{U} = L_\infty \cup L_{-\infty}$. Obviously, L_∞ is equivalent to $L_{-\infty}$, i.e., there exists $\phi^* : L_\infty \times [0,1] \to M$ such that $\phi^*(L_\infty \times [0,1]) = \mathcal{U}$. Now, we modify $\mathcal{F}$ by replacing $\mathcal{F}|_\mathcal{U}$ with the image of the product foliation on $L_\infty \times [0,1]$. The modification near the depth 0 leaf L is completed.

In Case 2 (the case that there exists exactly one leaf of $\mathcal{F}$ representing $[L]$), let $U = M \setminus L$. Then $\partial\hat{U} = L_+ \cup L_-$, where L_+ (L_- resp.) is homeomoprhic to L. The situation is divided into the following two subcases.

Case 2.1 There exists a homeomorphism $h : L \times [0,1] \to \hat{U}$ such that the image of each $x \times [0,1]$ is a leaf of $\hat{\mathcal{F}}^\perp$.

In this case M admits a fiber bundle structure over S^1 with each fiber homeomorphic to L and transverse to $\mathcal{F}^\perp$, and L is a fiber. Then, $\mathcal{F}^1$ is the foliation given by this bundle structure,

Case 2.2 There does not exist a homeomorphism from $L \times [0,1]$ to $\hat{U}$ as in Case 2.1.

In this case $\mathcal{F}$ is unchanged by the modification.

In Cases 1.2 and 2.2, we further modify the foliation by using another equivalence class of the depth 0 leaves. By Lemma 3.1.1, this terminates in finitely many steps. Let $\mathcal{F}^1$ be the foliation which is obtained by repeating the procedure for all equivalence classes of the depth 0 leaves. Note that this modification does not change the transverse foliation $\mathcal{F}^\perp$, i.e., $\mathcal{F}^{1\perp} = \mathcal{F}^\perp$.

3.2 Second step of Modification

In this subsection, we describe a procedure for modifying $\mathcal{F}^1$ obtained in Section 3.1 by using depth 1 leaves of $\mathcal{F}^1$.

Lemma 3.2.1 *For the modified foliation $\mathcal{F}^1$, the number of the equivalence classes represented by the depth 1 leaves is finite.*

Proof. Let $\{L_k^{(1)}\}$ be a set of depth 1 leaves of $\mathcal{F}^1$ such that each pair of elements is not mutually equivalent. We assume that $\{L_k^{(1)}\}$ has infinitely many elements. By Lemma 2.1.2 and Theorem 2.1.1, we can show that each leaf of $\{L_k^{(1)}\}$ is isolated in $\cup L_k^{(1)}$. By slightly modifying the $(\mathcal{F}, \mathcal{F}^\perp)$-coodinate atlas $\{\varphi_i\}$ if necessary, we may suppose that $\overline{(\cup L_k^{(1)})} \cap (\cup \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Hence we can find U, a component of $M \setminus \overline{\cup L_k^{(1)}}$ such that $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. For any point x in $\partial\hat{U}$, let $\hat{\tau}_x$ be the leaf of $\hat{\mathcal{F}}^{1\perp}$ which meets x. Since $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$, $\hat{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some i and $c \in D^{n-1}$. Hence $\hat{\tau}_x$ is an arc properly embedded in $\hat{U}$ with endpoints x and y, say. Since $\mathcal{F}$ is transversely oriented, x and y are contained in different components of $\partial\hat{U}$. Let F_x (F_y resp.) be the leaf of $\mathcal{F}^1$ which meets $\hat{\imath}(x)$ ($\hat{\imath}(y)$ resp.). Then we immediately have the following.

Claim 1 $\hat{U}$ is homeomorphic to $F_x \times [0,1]$, where each $\{p\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$.

Moreover we have the following.

Claim 2 If $F_x \neq F_y$, then $\mathcal{F}^1|_U$ is a product foliation with each leaf is at depth 0.

Proof of Claim 2. If F_x or F_y is a depth 1 leaf, then obviously F_x is equivalent to F_y, this contradicts the assumption that each pair of elements of $\{L_k^{(1)}\}$ is not mutually equivalent. This together with Lemma 2.1.4, ∂U consists of depth 0 leaves. Since $\mathcal{F}^1$ is modified for all equivalence classes of the depth 0 leaves, $\mathcal{F}^1|_U$ is a product foliation with each leaf is at depth 0. $\square$

Suppose $F_x = F_y$. Let $\widetilde{U} = U \cup F_x$.

Claim 3 Under the above conditions, we have the following:

1. F_x is a depth 1 leaf;
2. $\partial\widetilde{U} = \overline{F_x} \setminus F_x$; and
3. F_x is the only element of $\{L_k^{(1)}\}$ which meets $\widetilde{U}$.

Remark 3.2.1 By 1 and 2 of the above Claim 3, we see that $\partial\widetilde{U}$ consists of depth 0 leaves.

Proof of Claim 3. By Lemma 2.1.5, it is obvious that 1 of the claim holds. We show that $\partial\widetilde{U} \supset \overline{F_x} \setminus F_x$. Note that since F_x is a depth 1 leaf, $\overline{F_x} \setminus F_x$ is a union of depth 0 leaves. Let L_0 be a leaf contained in $\overline{F_x} \setminus F_x$. Since $U \cong F_x \times (0,1)$, F_x is noncompact, and each leaf of $\hat{\mathcal{F}}^1$ is transverse to $\hat{\mathcal{F}}^\perp$, every leaf of $\mathcal{F}^1|_U$ is noncompact. Since L_0 is compact, this shows that $L_0 \cap U = \emptyset$. Hence $L_0 \cap \widetilde{U} = \emptyset$. On the other hand, since $L_0 \subset \overline{F_x}$, we have $L_0 \subset \overline{\widetilde{U}}$. These imply $L_0 \subset \partial\widetilde{U}$. Then, we show that $\partial\widetilde{U} \subset \overline{F_x} \setminus F_x$. Let a be a point in $\partial\widetilde{U}$ and N_a a neighbourhood of a. Then there exist points b_1 and b_2 of N_a such that $b_1 \in \widetilde{U}$ and $b_2 \notin \widetilde{U}$. Suppose $b_1 \notin F_x$. Then take an arc $\widehat{b_1 b_2}$ in N_a connecting b_1 and b_2. Then there is a point $b \in \widehat{b_1 b_2}$ such that $b \in F_x$. This shows that $a \in \overline{F_x}$. Since $\delta U = F_x$, both sides of F_x are contained in U. Hence $F_x \subset \operatorname{int} \widetilde{U}$, and this shows that $a \notin F_x$. These show $\partial\widetilde{U} \subset \overline{F_x} \setminus F_x$, and 2 of the claim holds. We can show 3 of the claim immediately by the above Claim 1. $\square$

We know that the number of the equivalence classes represented by the depth 0 leaves of $\mathcal{F}$ is finite (Lemma 3.1.1). Since the modification does not change the number of the equivalence classes represented by the depth 0 leaves, the number of the equivalence classes represented by the depth 0 leaves of $\mathcal{F}^1$ is also finite. This fact together with the above Claim 2, 3 of the above Claim 3 and Remark 3.2.1 imply that $\{L_k^{(1)}\}$ consists of finitely many elements, a contradiction. $\square$

Now, we modify the foliation $\mathcal{F}^1$. Since the number of the equivalence classes represented by the depth 0 leaves is finite (Lemma 3.1.1), $M \setminus \cup$(depth 0 leaves) consists of finite number of components, say $U_1, U_2, \ldots, U_k$. Note that there is a depth 1 leaf in each U_i. Let $L(\subset U_1)$ be a depth 1 leaf. The situation is divided into the following two cases.

Case 1 There exist more than one leaves of $\mathcal{F}^1$ representing $[L]$.

Case 2 There exists exactly one leaf of $\mathcal{F}^1$ representing $[L]$.

In Case 1, let $\{\phi_\beta\}$ be the set of all the embeddings which give equivalence relations between L and the leaves representing $[L]$. Let $\mathcal{U}^{(1)} = \cup \phi_\beta(L \times [0,1])$. The situation is divided into the following two subcases.

Case 1.1 $\mathcal{U}^{(1)} = U_1$.

In this case, by applying the argument as in the proof of Claim 3 of Case 1.1 in Section 3.1, we can show that there exists a depth 1 leaf $L' \subset U$ such that for each side of L, L is equivalent to L' through the side. This implies that $U_1 \setminus L \cong L \times (0,1)$ with each $x \times (0,1)$ is contained in a leaf of $\mathcal{F}^\perp$. We modify $\mathcal{F}^1$ by replacing $\mathcal{F}^1|_{\mathcal{U}^{(1)}}$ with the image of the product foliation on $L \times [0,1]$. Note that in this case, the modification on U_1 is completed.

Case 1.2 $\mathcal{U}^{(1)} \neq U_1$.

In this case, by applying the argument as in the proof of Claim 1 of Case 1.2 in Section 3.1, we can show that $\mathcal{U}^{(1)}$ is complete with respect to the induced Riemannian metric, which implies that $\mathcal{U}^{(1)} \cong L \times [0,1]$ with each $x \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$. Let $\partial \mathcal{U}^{(1)} = L_- \cup L_+$. We modify $\mathcal{F}^1$ by replacing $\mathcal{F}^1|_{\mathcal{U}^{(1)}}$ with the image of the product foliation on $L_- \times [0,1]$.

Case 2 is divided into the following two subcases.

Case 2.1 There exists an immersion $h : L \times [0,1] \to U_1 \setminus L$ such that the image of each $x \times [0,1]$ is contained in a leaf of $\hat{\mathcal{F}}^\perp$.

In this case, we replace $\mathcal{F}^1|_{U_1 \setminus L}$ by the image of the product foliation on $L \times [0,1]$. Note that in this case, the modification on U_1 is completed.

Case 2.2 There does not exist an immersion h as in Case 2.1.

In this case $\mathcal{F}^1$ is unchanged by the modification.

In Cases 1.2 and 2.2, we further modify the foliation for another equivalence class of a depth 1 leaf in U_1, and repeat the procedure to modify the foliation in U_1. Then the desired foliation $\mathcal{F}^2$ is obtained by repeating the procedure for all U_i's.

Note that this modification does not change the transverse foliation $\mathcal{F}^{\perp}$, i.e., $\mathcal{F}^{2\perp} = \mathcal{F}^{\perp}$.

3.3 Completion of modification

In this subsection, we apply the similar modifications for leaves at higher depth to give a means of completing the modification.

Lemma 3.3.1 *For the modified foliation $\mathcal{F}^2$, the number of equivalence classes represented by the depth 2 leaves is finite.*

Proof. Let $\{L_\ell^{(2)}\}$ be a set of depth 2 leaves of $\mathcal{F}^2$ such that each pair of elements is not mutually equivalent. We assume that $\{L_\ell^{(2)}\}$ has infinitely many elements. By slightly modifying an $(\mathcal{F}, \mathcal{F}^{\perp})$-coordinate atlas if necessary, we may suppose that $(\overline{\cup L_\ell^{(2)}}) \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Hence we can find U, a component of $M \setminus \overline{\cup L_\ell^{(2)}}$ such that $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. For any point x in $\partial \hat{U}$, let $\hat{\tau}_x$ be the leaf of $\hat{\mathcal{F}}^2$ which meets x. Since $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$, $\hat{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some i and $c \in D^{n-1}$. Hence $\hat{\tau}_x$ is an arc properly embedded in $\hat{U}$ with endpoints x and y, say. Since $\mathcal{F}^2$ is transversely oriented, x and y are contained in different components of $\partial \hat{U}$. Let F_x (F_y resp.) be a leaf of $\mathcal{F}^2$ which meets x (y resp.).

We immediately have the following.

Claim 1 $\hat{U}$ is homeomorphic to $F_x \times [0,1]$, where each $\{p\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^{\perp}$.

By applying the arguments as in the proof of Claim 2 in Section 3.2, we have the following claim.

Claim 2 If $F_x \neq F_y$, then $\mathcal{F}^2|_U$ is a product foliation with depth 0 leaves or depth 1 leaves.

Suppose $F_x = F_y$. Let $\widetilde{U} = U \cup F_x$. By applying the arguments as in the proof of Claim 3 in Section 3.2, we have the following.

Claim 3 Under the above conditions, we have the following:

1. F_x is a depth 2 leaf;
2. $\partial \widetilde{U} = \overline{F_x} \setminus F_x$; and
3. F_x is the only element of $\{L_\ell^{(2)}\}$ which meets $\widetilde{U}$.

Remark 3.3.1 By 1 and 2 of the above Claim 3, we see that each component of $\partial\widetilde{U}$ is a depth 0 leaf or a depth 1 leaf.

We know that the number of the equivalence classes of the depth 0 leaves and depth 1 leaves of $\mathcal{F}^1$ are finite (Lemmas 3.1.1 and 3.2.1). Since the modification in Section 3.2 does not change the number of the equivalence classes represented by the depth 0 or depth 1 leaves, the number of the equivalence classes represented by the depth 0 or depth 1 leaves of $\mathcal{F}^2$ is also finite. This fact together with the above Claim 2, 3 of the above Claim 3, and Remark 3.3.1 imply that $\{L_\ell^{(2)}\}$ consists of finitely many elements, a contradiction. $\square$

Then we can further apply the similar modifications for higher depth leaves to obtain a modified foliation $\widetilde{\mathcal{F}}$ that cannot be modified any more. Note that the modification for depth d leaves does not affect the leaves at depth less than d. Note that if $\mathcal{F}^\perp$ is fixed, then for each i, $\mathcal{F}^i$ is unique. Hence $\widetilde{\mathcal{F}}$ is unique. And by the construction, it is easy to show the following.

Fact 3.3.1 *depth*$(\widetilde{\mathcal{F}}) \leq$ *depth*$(\mathcal{F})$.

Moreover, by applying the argument as in the proof of Lemma 3.3.1, we can show the following.

Proposition 3.3.1 *The number of the equivalence classes of the leaves of* $\widetilde{\mathcal{F}}$ *is finite.*

Here, we note that the modified foliation $\widetilde{\mathcal{F}}$ satisfies the conditions soon after Remark 2.1.4 in Section 2.1.2.

4 Proof of Theorems

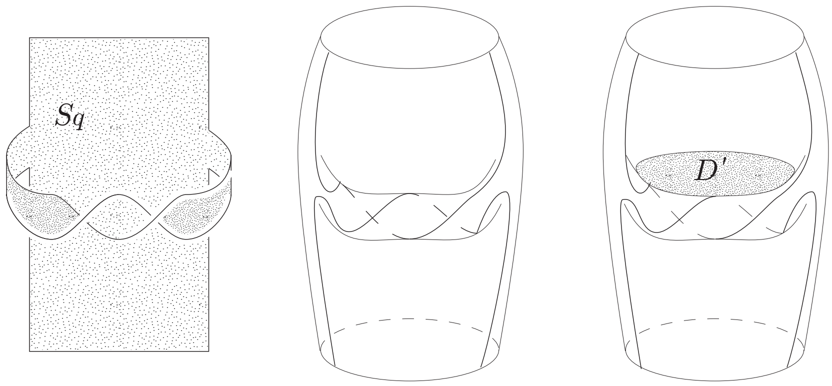
4.1 Structure of the manifold M_{S_q}

Let $K_q(= h_q(C))$ be a q-twisted double of K', and $S_q(= h_q(S))$, $A_q(= h_q(A))$ as in Section 2.2.2. Recall that (X_{S_q}, γ_{S_q}) is a complementary sutured manifold of S_q. Let M_{S_q} be the manifold obtained from X_{S_q} by attaching a 2-handle along γ_{S_q} (see Section 2.2.4) with the orientation inherited from X_{S_q}. We further suppose that each component of ∂M_{S_q} is equipped with the orientation inherited from $R(\gamma_{S_q})$. Since S_q is a genus one surface, the boundary of M_{S_q} consists of two tori. Then T^+ denotes the component of ∂M_{S_q} whose normal vectors point outward, and T^- the other.

Let L_q be the link ∂A_q equipped with the orientation such that each component of L_q is parallel to K', hence the linking number of L_q is q.

Proposition 4.1.1 *M_{S_q} is homeomorphic to $E(L_q)$.*

Proof. Recall that S_q is obtained by plumbing A_q and a Hopf annulus (see Section 2.2.2). Note that a part of S_q near the Hopf annulus looks as in Figure 4.1(a). The part of (X_{S_q}, γ_{S_q}) corresponding to the part of S_q in Figure 4.1(a),



(a) (b) (c)

Figure 4.1

looks as in Figure 4.1(b). Let D' be the product disk for (X_{S_q}, γ_{S_q}) as in Figure 4.1(c). Recall that M_{S_q} is obtained from X_{S_q} and $D^2 \times [0,1]$ by identifying γ_{S_q} and $\partial D^2 \times [0,1]$. Hence we may regard that D' is a subspace of M_{S_q}. Since D' is a product disk for (X_{S_q}, γ_{S_q}), $D' \cap \gamma_{S_q}$ $(= D' \cap (\partial D^2 \times [0,1]))$ consists of two arcs in $\partial D'$, say a_1, a_2. Let D'' be a product disk for $(D^2 \times [0,1], \partial D^2 \times [0,1])$ such that $D'' \cap (\partial D^2 \times [0,1]) = a_1 \cup a_2$. Let $A_M = D' \cup D''$. Note that A_M is an annulus proprely embedded in M_{S_q} such that a component of ∂A_M is contained in T^+ and the other in T^- (see Figure 4.2).

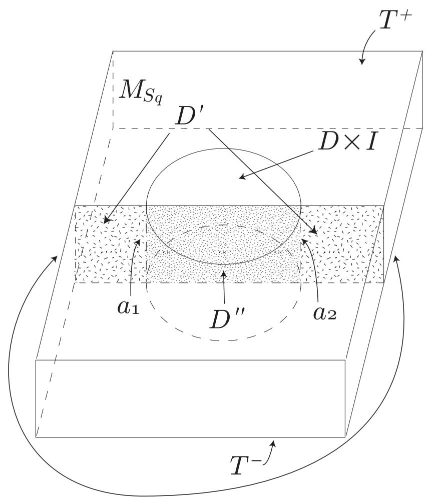
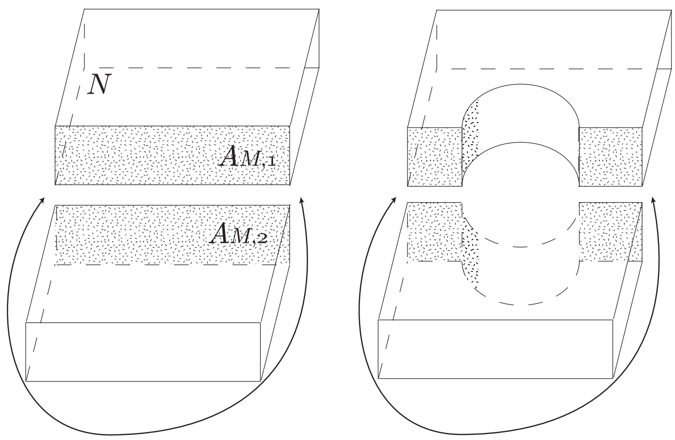


Figure 4.2

Now, let N be the manifold which is obtained from M_{S_q} by cutting along A_M and $A_{M,1}, A_{M,2}$ the copies of A_M in ∂N (see Figure 4.3(a)).



(a) (b)

Figure 4.3

Note that $(N, A_{M,1} \cup A_{M,2})$ naturally inherits a sutured manifold structure from (X_{S_q}, γ_{S_q}), i.e., $R_\pm(A_{M,1} \cup A_{M,2})$ is the image of $T^\pm$.

Let $(\widetilde{X}_{S_q}, \widetilde{\gamma}_{S_q})$ be the sutured manifold obtained from (X_{S_q}, γ_{S_q}) by the product decomposition along D' (see Figure 4.3(b)). By Figure 4.3, we have the following.

Claim The manifold pair $(N, A_{M,1} \cup A_{M,2})$ is homeomorphic to $(\widetilde{X}_{S_q}, \widetilde{\gamma}_{S_q})$.

Since M_{S_q} is retrieved from N by identifying $A_{M,1}$ and $A_{M,2}$, this claim implies that M_{S_q} is obtained from $\widetilde{X}_{S_q}$ by identifying the components of $\widetilde{\gamma}_{S_q}$ and that the image of $\widetilde{\gamma}_{S_q}$ is A_M. It is directly observed from Figure 4.4 that $\widetilde{X}_{S_q}$ is

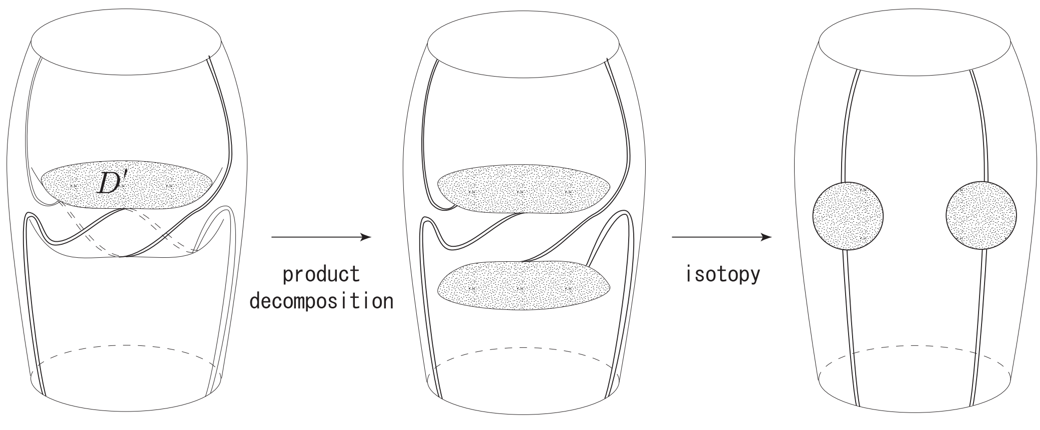


Figure 4.4

homeomorphic to $E(K')$, and the union of core curves of $\widetilde{\gamma}_{S_q}$ is equivalent to the link L_q. Here we note that one component of ∂A_M is contained in T^+ and the other is in T^-, and that M_{S_q} is orientable. Hence the identification has to be as in Figure 4.5. These show that M_{S_q} is homeomorphic to $E(L_q)$. □

We describe two facts that will be required for the proof of Theorem 1.0.1. Let A_M be as in the proof of Proposition 4.1.1.

Fact 4.1.1 *Let T' be the component of $\partial N(\partial M_{S_q} \cup A_M, M_{S_q})$ which is contained in* $\mathrm{int} M_{S_q}$ *(Figure 4.6). By Proposition 4.1.1, we may regard T' is contained in $E(L_q)$. Moreover it is directly observed that the closure of a component of $E(L_q) \setminus T'$ is homeomorphic to $E(K')$ and the closure of the other, say Q, is homeomorphic to (disk with two holes) $\times S^1$.*

For the statement of the second fact, we prepare some notations. By Proposition 4.1.1, we can consider T^+, T^- as boundary components of $E(L_q)$. Let

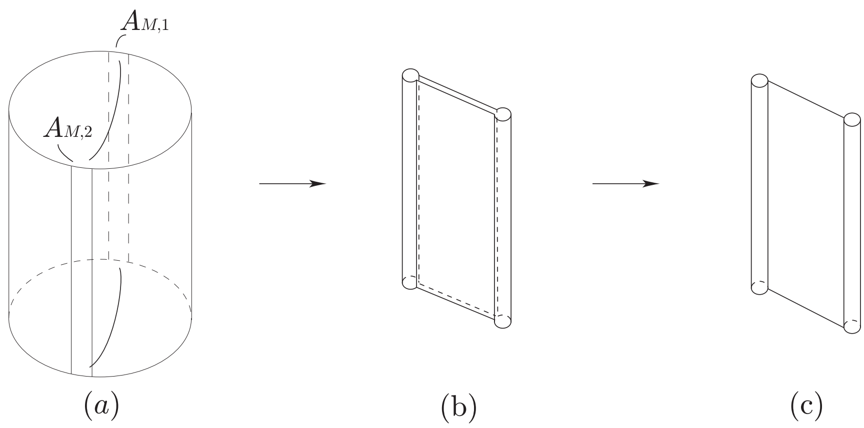


Figure 4.5

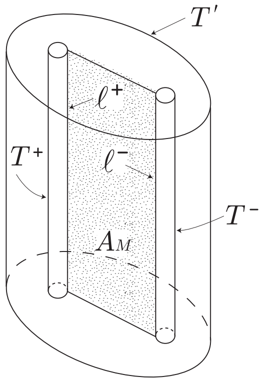


Figure 4.6

$\ell^\pm = T^\pm \cap A_M$. Note that $\ell^\pm$ is isotopic to a S^1-fiber of Q. Recall that $M_{S_q} = X_{S_q} \cup (D^2 \times [0,1])$ and $T^+ = R^+(\gamma_{S_q}) \cup (D^2 \times \{1\})$, $T^- = R^-(\gamma_{S_q}) \cup (D^2 \times \{0\})$. By deforming $\ell^\pm$ by an ambient isotopy, if necessary, we may suppose that $\ell^\pm \subset R^\pm(\gamma_{S_q})$. Recall that (X_{S_q}, γ_{S_q}) is the complementary sutured manifold of S_q, hence $R^\pm(\gamma_{S_q})$ is homeomorphic to S_q. Then,

$$p^\pm : R^\pm(\gamma_{S_q}) \longrightarrow S_q$$

denotes the natural homeomorphism. By Figure 4.5(b), $\ell^+ \cup \ell^-$ looks as in Figure 4.6. By tracing the deformations of Figure 4.4 conversely together with $\ell^+ \cup \ell^-$, we obtain Figure 4.7. This observation implies the following.

Fact 4.1.2 *Under the above notations, $p^+(\ell^+)$ and $p^-(\ell^-)$ are isotopic on S_q to loops which meet transversely in one point.*

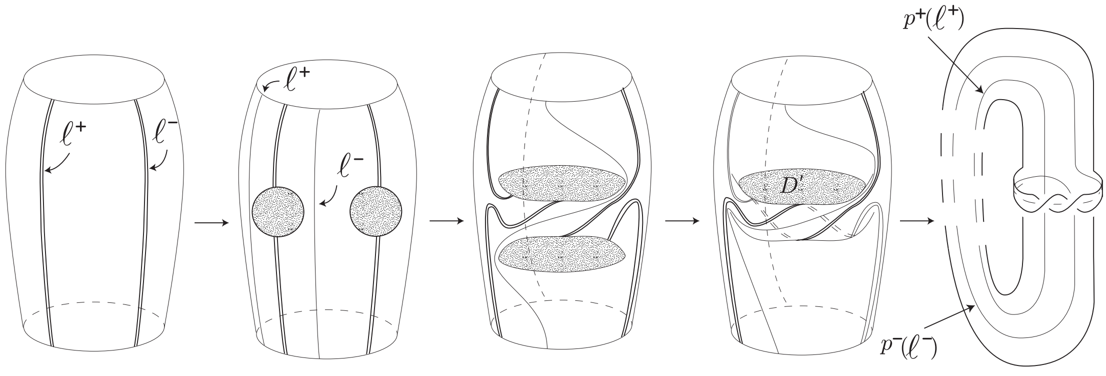


Figure 4.7

4.2 Foliations on $M^{(n)}$

Let K be a q-twisted double of K' (in this subsection, we basically follow the notations in Section 4.1, but we use K, S, L, M_S for K_q, S_q, L_q, M_{S_q} for simplicity). Let $S^3(K,0)$ be the manifold obtained from S^3 by performing 0-surgery on K. Note that $H_1(S^3(K,0);\mathbb{Z}) \cong \mathbb{Z}$. Let $\Sigma^{(n)}(K,0)$ be the n-fold cyclic covering space of $S^3(K,0)$ (see Section 1). Note that $\Sigma^{(n)}(K,0)$ admits a decomposition $\Sigma^{(n)}(K,0) = M_1 \cup \cdots \cup M_n$ where each M_i ($i = 1, \ldots, n$) is homeomorphic to M_S, and $M_1, \ldots, M_n$ are arrayed cyclically i.e., $M_i \cap M_{i+1} = \partial M_i \cap \partial M_{i+1}$ consists of a torus T_i (if $n > 2$) (subscript is taken in mod n) or $M_1 \cap M_2 = \partial M_1 = \partial M_2$ ($= T_1 \cup T_2$, say)(if $n = 2$). Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K,0)$ by cutting along T_n. In this subsection, for the proof of Theorem 1.0.1, we study depth of foliations on $M^{(n)}$ the union of the depth 0 leaves of each of which coincides with $\partial M^{(n)}$.

We may regard $M^{(n)} = M_1 \cup_{T_1} \cdots \cup_{T_{n-1}} M_n$. Then, we abuse notation T_n for denoting the component of $\partial M^{(n)}$ such that $T_n \subset M_n$ and T_0 denotes the other component of $\partial M^{(n)}$. By Proposition 4.1.1, each M_i is homeomorphic to $E(L)$. Let T_i' be the torus in M_i corresponding to T' in Fact 4.1.1. Hence the closure of a component of $M_i \setminus T_i'$, say $E(K')_i$, is homeomorphic to $E(K')$ and the closure of the other component of $M_i \setminus T_i'$, say Q_i, is homeomorphic to (disk with two holes)$\times S^1$. Let ℓ_j^+ (ℓ_j^- resp.) ($j = 1, \ldots, n-1$) be a simple closed curve in T_j which is isotopic to an S^1-fiber of Q_j (Q_{j+1} resp.). By Fact 4.1.2, we obtain the following.

Lemma 4.2.1 *Under the above notations, ℓ_j^+ and ℓ_j^- are isotopic to loops on T_j*

which meet transversely in one point.

4.2.1 Foliation $\mathcal{F}'$ on $M^{(n)'}$

Let $\mathbb{T} = (\bigcup_{j=1}^{n-1} T_j) \cup (\bigcup_{i=1}^{n} T_i')$. Let $\mathcal{F}$ be a codimension one, transversely oriented taut C^0 foliation of finite depth on $M^{(n)}$ such that the union of the depth 0 leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$.

Lemma 4.2.2 *By deforming* $\mathbb{T}$ *by an ambient isotopy in* $M^{(n)}$*, if necessary, we may suppose* $\mathbb{T}$ *is transverse to* $\mathcal{F}$*.*

Proof. Since $\mathcal{F}$ is taut, Theorem 4 of [17] implies that by ambient isotopy in $M^{(n)}$, we can deform $\mathbb{T}$ so that either $\mathbb{T}$ is transverse to $\mathcal{F}$ or there exists T, a component of $\mathbb{T}$ such that T is a leaf of $\mathcal{F}$. However by the assumption that the union of the depth 0 leaves coincides with $\partial M^{(n)}$, we see that there does not exist such T as above. Hence we may suppose $\mathbb{T}$ is transverse to $\mathcal{F}$. This completes the proof of the lemma. $\square$

Let $\overset{\circ}{T}_0$, $\overset{\circ}{T}_n$ be tori in int$M^{(n)}$ such that $\overset{\circ}{T}_0$ ($\overset{\circ}{T}_n$ resp.) is parallel to T_0 (T_n resp.), and $\overset{\circ}{T}_0 \cap \mathbb{T} = \emptyset$ ($\overset{\circ}{T}_n \cap \mathbb{T} = \emptyset$ resp.). By applying the arguments as in the proof of Lemma 4.2.2, we can assume that $\overset{\circ}{T}_0 \cup \overset{\circ}{T}_n \cup \mathbb{T}$ is transverse to $\mathcal{F}$. Let $M^{(n)'}$ be the closure of the component of $M^{(n)} \setminus (\overset{\circ}{T}_0 \cup \overset{\circ}{T}_n)$ which does not meet $\partial M^{(n)}$. Note that $M^{(n)'}$ is homeomorphic to $M^{(n)}$. Let $\mathcal{F}' = \mathcal{F}\mid_{M^{(n)'}}$.

Lemma 4.2.3 $\mathcal{F}'$ *is of finite depth and for each leaf* L' *of* $\mathcal{F}'$ *we have: if* L *is the leaf of* $\mathcal{F}$ *such that* $L \supset L'$*, then* $\mathrm{depth}(L) > \mathrm{depth}(L')$*. In particular,* $\mathrm{depth}(\mathcal{F}') \leq \mathrm{depth}(\mathcal{F}) - 1$*.*

For the proof of the lemma, for each i $(0 \leq i \leq \mathrm{depth}(\mathcal{F}))$, we show the following inductively. Note that Assertion $(\mathrm{depth}(\mathcal{F}))$ gives the lemma.

Assertion (i) Let L be a leaf of $\mathcal{F}$ such that $\mathrm{depth}(L) = i$. Then, either one of the following holds:

1. $L \cap M^{(n)'} = \emptyset$; or
2. for any component F of $L \cap M^{(n)'}$, we can define $\mathrm{depth}(F)$ (note that $\mathrm{depth}(F)$ denotes the depth of F as a leaf of $\mathcal{F}'$), and $\mathrm{depth}(F) \leq i - 1$, i.e., $\overline{F} \setminus F$ is a union of leaves F_α such that $\mathrm{depth}(F_\alpha) \leq i - 2$.

Proof of Assertion (i)**.** Suppose $i = 0$. Then, clearly $L \cap M^{(n)\prime} = \emptyset$. Suppose for $i \leq k$ ($0 \leq k \leq \text{depth}(\mathcal{F})-1$), Assertion (i) holds. Consider the case $i = k+1$. If $L \cap M^{(n)\prime} = \emptyset$, then 1 holds. Suppose $L \cap M^{(n)\prime} \neq \emptyset$. Let F be a component of $L \cap M^{(n)\prime}$. Recall that $\overline{L} \setminus L$ is a union of leaves L_β such that $\text{depth}(L_\beta) \leq k$.

Claim 1 Suppose $(\cup L_\beta) \cap M^{(n)\prime} = \emptyset$. Then $L \cap M^{(n)\prime}$ is compact.

Proof of Claim 1. Assume $L \cap M^{(n)\prime}$ is not compact. Then there exists a sequence of points in $L \cap M^{(n)\prime}$, say $\{x_i\}_{i=1,2,\dots}$ with an accumulating point x such that $x \notin L \cap M^{(n)\prime}$. Let L_x be the leaf of $\mathcal{F}'$ which contains x. Then we have $L_x \cap (\overline{L \cap M^{(n)\prime}} \setminus (L \cap M^{(n)\prime})) \neq \emptyset$, thus $L_x \subset \overline{L \cap M^{(n)\prime}} \setminus (L \cap M^{(n)\prime})$, contradicting the assumption that $(\cup L_\beta) \cap M^{(n)'} = \emptyset$. $\square$

Claim 2 Suppose $L \cap M^{(n)\prime}$ is compact. Then each component of $L \cap M^{(n)\prime}$ is compact.

Proof of Claim 2. Assume there is a component F' of $L \cap M^{(n)\prime}$ which is not compact. Then there exists a sequence of points in F', say $\{x_i\}_{i=1,2,\dots}$ with an accumulating point x such that $x \notin F'$. On the other hand, since $L \cap M^{(n)\prime}$ is compact, $x \in L \cap M^{(n)\prime}$. This contradicts the fact that L is proper (Remark 2.1.3). $\square$

By the above Claims 1 and 2, we see if for each β, $L_\beta \cap M^{(n)\prime} = \emptyset$, then $\text{depth}(F) = 0 \leq (k+1) - 1$, that is, the conclusion 2 of Assertion $(k+1)$ holds. Suppose $(\cup L_\beta) \cap M^{(n)\prime} \neq \emptyset$. Since Assertion (i) holds for $i \leq k$, for each β either one of the following holds:

(a) $L_\beta \cap M^{(n)\prime} = \emptyset$; or

(b) for any component $F_{\beta,\gamma}$ of $L_\beta \cap M^{(n)\prime}$, $\text{depth}(F_{\beta,\gamma}) \leq k - 1$.

Let $\{L_{\beta'}\}$ be the set of elements of $\{L_\beta\}$ intersecting $M^{(n)\prime}$. Recall that $\overline{L} \setminus L = \cup L_\beta$, and $L_{\beta'} \cap M^{(n)\prime} = \cup_\gamma F_{\beta',\gamma}$. This implies that $(\overline{L} \setminus L) \cap M^{(n)\prime} = \bigcup_{\beta'}(\bigcup_\gamma F_{\beta',\gamma})$.

Claim 3 $\overline{F} \setminus F \subset (\overline{L} \setminus L)$.

Proof of Claim 3. Since $\overline{F} \setminus F \subset \overline{L}$ is clear, we show that $(\overline{F} \setminus F) \cap L = \emptyset$. Asuume that $(\overline{F} \setminus F) \cap L \neq \emptyset$. Let x be a point in $(\overline{F} \setminus F) \cap L$. Then there exists a sequence of points in F, say $\{x_i\}_{i=1,2,\dots}$ with an accumulating point x such that $x \notin F$ and $x \in L$. However this contradicts the fact that L is proper (Remark 2.1.3). $\square$

Hence $\overline{F} \setminus F \subset \bigcup_{\beta'}(\bigcup_{\gamma} F_{\beta',\gamma})$. Hence by (b), depth$(F) \leq k$, this completes the proof of Assertion $(k+1)$. $\square$

Since each component of $M^{(n)\prime} \setminus \mathbb{T}$ naturally corresponds to each component of $M^{(n)} \setminus \mathbb{T}$, in the remainder of this subsection, we use the notation that $M^{(n)\prime} = M_1 \cup_{T_1} \cdots \cup_{T_{n-1}} M_n = (E(K')_1 \cup_{T_1'} Q_1) \cup_{T_1} \cdots \cup_{T_{n-1}} (E(K')_n \cup_{T_n'} Q_n)$. Let $\widetilde{\mathbb{T}}$ be a union of tori which is ambient isotopic to $\mathbb{T}$ and transverse to $\mathcal{F}'$. For such $\widetilde{\mathbb{T}}$, let $\widetilde{T_i}$ be the component of $\widetilde{\mathbb{T}}$ corresponding to T_i, and $\widetilde{M_i}$ the closure of the component of $M^{(n)} \setminus \bigcup_{i=1}^{n-1} \widetilde{T_i}$ corresponding to M_i $(1 \leq i \leq n)$. Let $d_i(\widetilde{\mathbb{T}})$ be the minimal value of the depths of the leaves of $\mathcal{F}'$ which meet $\widetilde{M_i}$.

Let L_i be a leaf of $\mathcal{F}'$ which meets M_i such that depth$(L_i) = d_i(\mathbb{T})$ and $L_i^* = L_i \cap M_i$.

Lemma 4.2.4 *L_i^* is compact and incompressible in M_i.*

Proof. Assume L_i^* is not compact. Then there exists a sequence of points in L_i^*, say $\{x_i\}_{i=1,2,\ldots}$ with an accumulating point x such that $x \notin L_i^*$. Let L_x be the leaf of $\mathcal{F}'$ which contains x. Then we have $L_x \cap (\overline{L_i^*} \setminus L_i^*) \neq \emptyset$, thus $L_x \cap M_i \subset (\overline{L_i^*} \setminus L_i^*)$. By the definition of depth of leaves, the depth of L_x is less than $d_i(\mathbb{T})$ and this contradicts the definition of $d_i(\mathbb{T})$. Thus L_i^* is compact. Since $\mathcal{F}'$ is transverse to $\mathbb{T}$, each component of ∂L_i^* is essential in $\cup T_i$. Suppose there is a compression disk D for L_i^* in M_i. Since $\mathcal{F}$ is taut, the leaf of $\mathcal{F}$ containing L_i (say, $\hat{L}_i$) is incompressible (Remark 2.2.1). Hence ∂D is inessential in $\hat{L}_i$, i.e., there is a disk D' in $\hat{L}_i$ such that $\partial D' = \partial D$. Since D is a compression disk for L_i^*, D' is not contained in M_i, hence $D \cap (\cup T_i) \neq \emptyset$. Let ℓ be a component of $D' \cap (\cup T_i)$ which is innermost in D'. Let $\Delta (\subset D')$ be the disk bounded by ℓ. Since ℓ is essential in $\cup T_i$, Δ is a compression disk for $\cup T_i$, a contradiction. $\square$

4.2.2 Putting $\mathbb{T}$ in a nice position (with assuming K' a non-cable knot)

In the remainder of this subsection, we suppose K' is a non-cable knot. Let $\mathcal{F}$ be a codimension one, transversely oriented, taut C^0 foliation of finite depth on $M^{(n)}$ such that the union of the depth 0 leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$. Let $\mathbb{T}$, $\mathring{T}_0$, $\mathring{T}_n$ be as in Section 4.2.1. We suppose that $\mathring{T}_0 \cup \mathring{T}_n \cup \mathbb{T}$ is isotoped as in the paragraph preceding Lemma 4.2.3 (hence, $M^{(n)\prime}, \mathcal{F}'$ are defined). Recall from Section 4.2.1 that we use the notation $M^{(n)\prime} = M_1 \cup_{T_1} \cdots \cup_{T_{n-1}} M_n = (E(K')_1 \cup_{T_1'} Q_1) \cup_{T_1} \cdots \cup_{T_{n-1}} (E(K')_n \cup_{T_n'} Q_n)$.

Lemma 4.2.5 *By deforming T_i' by an ambient isotopy, we may suppose that $\mathcal{F}' \mid_{E(K')_i}$ has no annular leaves for each i $(1 \le i \le n)$.*

Proof. We first note that the basic idea of the following proof is exactly the same as that of Lemma 2.3 of [2]. The key of our proof is to remove the assumption that the considered foliation is of class C^2 which is required in Lemma 2.3 of [2].

Assume there exists k $(1 \le k \le n)$ such that $\mathcal{F}' \mid_{E(K')_k}$ has an annular leaf, say A. By the proof of Lemma 2.3 of [2], we see that $\partial A \subset T_k'$ is a pair of essential simple closed curves which devides T_k' into two annuli, A', A'', where either $A \cup A'$ or $A \cup A''$ is isotopic to T_k' in $E(K')_k$. Now, assume that $T_A = A \cup A'$ is isotopic to T_k'. Let E_A be a component of $E(K')_k \setminus A$. We say that E_A is *outside* if $\partial E_A = T_A$ and that E_A is *inside* otherwise. If there exists another annular leaf B in $\mathcal{F}' \mid_{E(K')_k}$, we say $A < B$ if A is contained in the inside of B, or $A \le B$ if we include the case $A = B$. Now, suppose $A < B$. Let G be the closure of the component of $E(K')_k \setminus (A \cup B)$ which is between A and B. Then G is homeomorphic to $A \times [0,1]$, where $A \times \{0\} = A,\ A \times \{1\} = B$ and $\partial A \times [0,1]$ corresponds to $G \cap T_k'$. In this case, we can think of B as "concentric" with A and pushed out from A. Now, we consider the set of annular leaves B such that $A \le B$. By using the above observation that A and B are concentric, it is easy to show that this set is linearly ordered by $<$ and the union of these leaves is a closed set with an outermost annular leaf A^∞. (That is, A^∞ is maximal with the property that $A \le A^\infty$.) Let $A^{\infty\prime} \subset T_k'$ be the annulus such that $T^\infty = A^\infty \cup A^{\infty\prime}$ is isotopic to T_k'.

Claim There exists a torus $\mathscr{T}$ in outside of A^∞ which is transverse to $\mathcal{F}'$ and isotopic to T^∞.

Proof of Claim. Remark that A^∞ is an essential annulus in T^∞ (Figure 4.8). Fix a point $p_0 \in A^\infty$. Let Σ_0 be a transverse section to A^∞ at p_0, Σ_0^o be the closure of the component of $\Sigma_0 \setminus p_0$ such that Σ_0^o is contained in the outside of A^∞. Let $\alpha' : [0,1] \longrightarrow A^\infty$ be a simple closed curve representing a generator of $\pi_1(A^\infty, p_0) \cong \mathbb{Z}$. By applying the argument as in the proof of Lemma 2 of [15], we see that there is a map $F : [0,1] \times [0,1] \longrightarrow E(K')_k$ such that $F|_{(0,1)\times(0,1)}$ is an embedding, $F(\{0\} \times [0,1]) \subset \Sigma_0^o$, $F(\{1\} \times [0,1]) \subset \Sigma_0^o$, $F|_{[0,1]\times\{0\}} = \alpha'$, and F is transverse to $\mathcal{F}|_{E(K')_k}$. This together with Theorem 2 of [15] shows that there is a homeomorphism $\Phi : \pi_1(A^\infty, p_0) \longrightarrow G(\Sigma_0, p_0)$, where $G(\Sigma_0, p_0)$ is the group of germs of C^0-homeomorphism of Σ_0 which leaves p_0 fixed. Hence $\Phi(\pi_1(A^\infty, p_0))$ is the holonomy group of A^∞ at p_0.

If $\Phi(\pi_1(A^\infty, p_0))$ is trivial on Σ_0^o, then there exist annular leaves outside of A^∞, this contradicts the maximality of A^∞. By taking α'^{-1} instead of α', if necessary, we may assume $\Phi(\alpha')$ is contracting on Σ_0^o, i.e., for each point $x \in F(\{0\} \times (0,1])$, if x proceeds along the leaf of $F([0,1] \times [0,1]) \cap \mathcal{F}'$ in the α'-direction, then x approaches A^∞ (Figure 4.9).

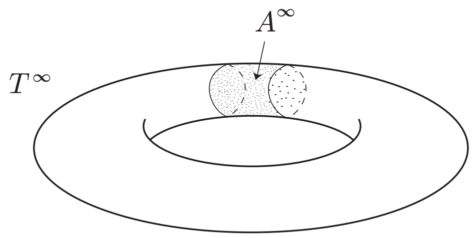


Figure 4.8

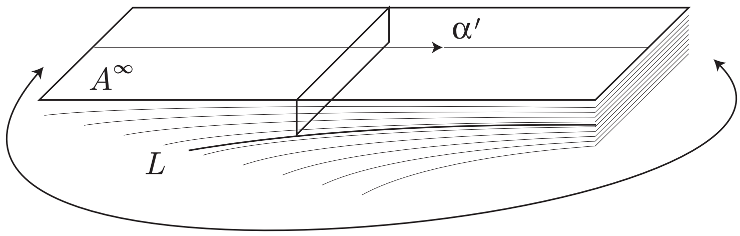


Figure 4.9

Let b_1 be an essential arc properly embedded in A^∞ such that $p_0 \in b_1$. By using the augument of the proof of Lemma 2 of [15], we can show that there is an embedding $F' : [0,1] \times [0,1] \longrightarrow E(K')_k$ such that $F'(\{0\} \times [0,1]) \subset T'_k$, $F'(\{1\} \times [0,1]) \subset T'_k$, $F'(\{\frac{1}{2}\} \times [0,1]) = \Sigma_0^o$, $F'([0,1] \times \{0\}) = b_1$, $F'([0,1] \times \{1\})$ is contained in a leaf, and F' is transverse to $\mathcal{F}'$. Let $R = F'([0,1] \times [0,1])$. Note that $\mathcal{F}'|_R$ is isomorphic to the product foliation on $[0,1] \times [0,1]$, i.e., the foliation on $[0,1] \times [0,1]$ with each leaf being $\{*\} \times [0,1]$. Let $b_2 = F'([0,1] \times \{1\})$, $p_1 = F'(\{0\} \times \{0\})$, $p_2 = F'(\{1\} \times \{0\})$, $q_1 = F'(\{0\} \times \{1\})$, $q_2 = F'(\{1\} \times \{1\})$, $a_1 = F'(\{0\} \times [0,1])$, $a_2 = F'(\{1\} \times [0,1])$. Note that p_i, q_i are points on T'_k (Figure 4.10).

Let L_{b_2} be the leaf of $\mathcal{F}'$ which contains b_2. Let ℓ_i be the arc in $T^\infty \cap L_{b_2}$ with $\partial \ell_i = \ell_i \cap a_i = q_i \cup r_i$ (where r_i is a point in int a_i), b'_2 the component of $L_{b_2} \cap R$ such that $\partial b'_2 = r_1 \cup r_2$ (Figure 4.11). Note that b'_2 is an arc properly embedded in R and by [15, Theorem 2], we see that $b_2 \cup (\ell_1 \cup \ell_2) \cup b'_2$ bounds a

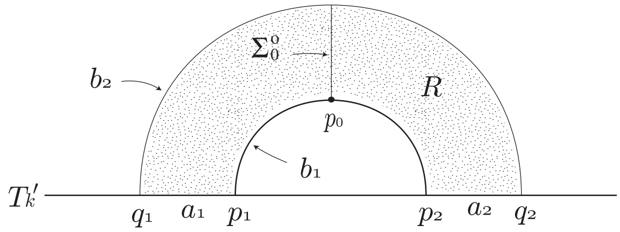


Figure 4.10

rectangle $\check{A}_1$ in L_{b_2}. Let a_i' be the subarc in a_i with $\partial a_i' = p_i \cup r_i$, a_i'' the subarc in a_i with $\partial a_i'' = r_i \cup q_i$, and R' the rectangle in R with edges a_1'', b_2', a_2'', b_2 (Figure 4.11). Then, $\ell_i \cup a_i''$ is a simple closed curve in T^∞ disjoint from ∂A^∞. Let $\ell_1^\infty, \ell_2^\infty$ be the components of ∂A^∞ such that $p_i \in \ell_i^\infty$ (Figure 4.12). Let B_1, B_2 be mutually disjoint annuli in T^∞ such that $\partial B_i = (\ell_i \cup a_i'') \cup \ell_i^\infty$ (Figure 4.12). Then $A^\infty \cup (B_1 \cup B_2) \cup \check{A}_1 \cup R'$ bounds a solid torus $\mathscr{V}$ in $E(K')_k$ (Figure 4.11 and Figure 4.12).

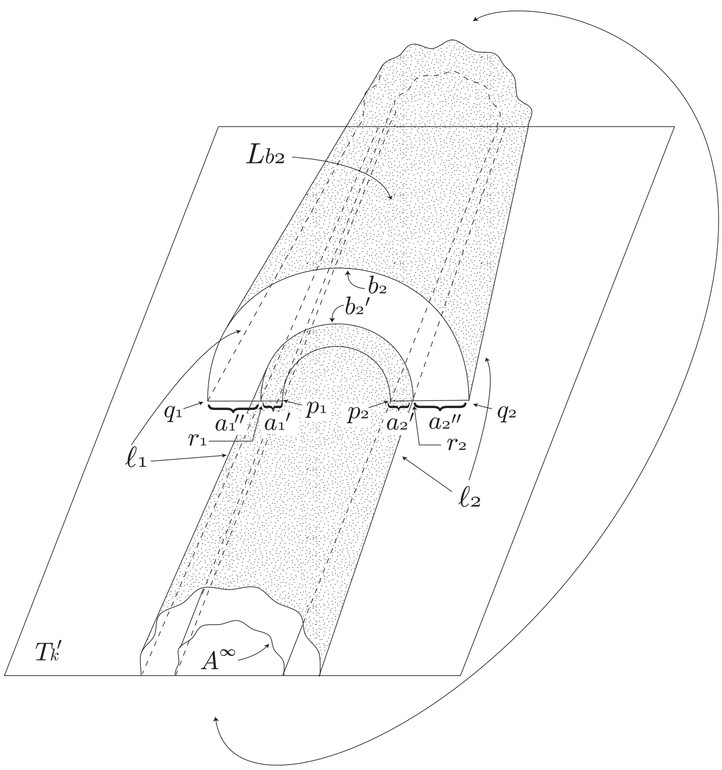


Figure 4.11

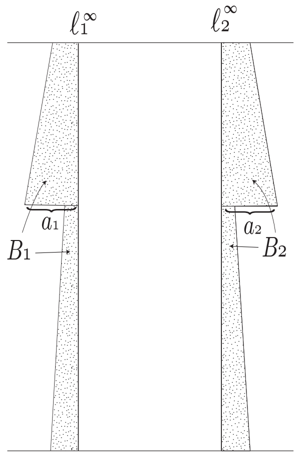


Figure 4.12

Let $\check{A^\infty} = A^\infty \setminus b_1$. Let $\overline{\check{A^\infty}}$ be the metric completion of $\check{A^\infty}$. Note that $\overline{\check{A^\infty}}$ is obtained from $\check{A^\infty}$ by adding two edges, say b_1^+, b_1^-, each corresponding to b_1, where there is a simple closed curve in A^∞ representing α' in $\pi_1(A^\infty, p_0)$ such that the image of the simple closed curve is an oriented arc in $\overline{\check{A^\infty}}$ which goes from b_1^+ to b_1^- (Figure 4.13).

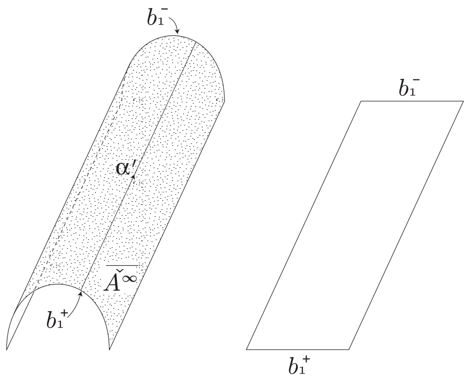


Figure 4.13

Let $f : b_1^- \longrightarrow b_1^+$ be the natural homeomorphism and $g : [0,1] \longrightarrow [0,1]$ the local homeomorphism induced from the holonomy map corresponding to $\Phi([\alpha'])|_{\Sigma_0^\circ}$. Since $\overline{\check{A^\infty}}$ is simply connected, by [15, Theorem 2], we see that the foliated manifold $(\mathscr{V}, \mathcal{F}'|_{\mathscr{V}})$ is isomorphic to $(\overline{\check{A^\infty}} \times [0,1]/\sim, \check{\mathcal{F}}/\sim)$, where $\check{\mathcal{F}}$ is a product foliation on $\overline{\check{A^\infty}} \times [0,1]$ and the equivalence relation $\sim$ is defined as follows; for $x \in b_1^-, y \in [0,1]$, $(x,y) \sim (f(x), g(y))$.

Let $\check{A}^*$ be the rectangle in $\overline{\check{A^\infty}} \times [0,1]$ corresponding to the flat rectangle properly embedded in $\overline{\check{A^\infty}} \times [0,1]$ ($\cong [0,1]^2 \times [0,1]$) such that $([0,1] \times \{0\}) \times \{g(1)\}$ ($= b_1^+ \times \{g(1)\}$) and $([0,1] \times \{1\}) \times \{1\}$ ($= b_1^- \times \{1\}$) are edges of $\check{A}^*$ (Figure 4.14). Note that $\check{A}^*$ is transverse to $\check{\mathcal{F}}/\sim$. Let A^* be the annulus in $\mathscr{V}$ corresponding to $\check{A}^*/\sim$. Note that one component of ∂A^* is contained in B_1, and the other is contained in B_2 and A^* is transverse to $\mathcal{F}'$. Let $A^{*\prime}$ be the closure of the component of $T^\infty \setminus \partial A^*$ which is contained in $A^{\infty\prime}$. Let $\mathscr{T} = A^* \cup A^{*\prime}$. Then $\mathscr{T}$ is transverse to $\mathcal{F}'$. Further, $\mathscr{T}$ is ambient isotopic to T'_k since $\mathscr{V}$ is a solid torus and T^∞ is isotopic to T'_k. This completes the proof of the claim. □

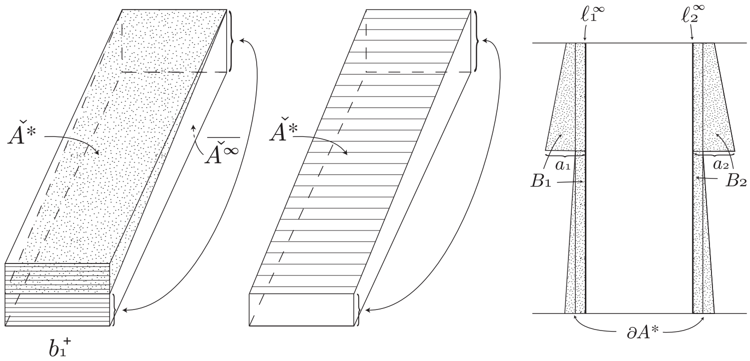


Figure 4.14

Let $\mathscr{T}$ be as in the proof of Claim. By taking $\mathscr{T}$ instead of T^∞, we can remove such annuli B ($\geq A$).

Assume there exist infinitely many pairs of annular leaves $A^{\infty,1}$, $A^{\infty,2}$, ... which are maximal. This implies that there exist infinitely many simple closed curves $\partial A^{\infty,1}$, $\partial A^{\infty,2}$, ... Then, the distances of each pair of simple closed curves will be close to 0, this contradicts the condition that leaves are transversely oriented. Thus, by repeating the operation finitely many times, we can remove all annular leaves. This completes the proof of the lemma. □

In what follows, we assume that the considerd foliation satisfies the condition of Lemma 4.2.5.

Lemma 4.2.6 *For each i, $\mathcal{F}' |_{E(K')_i}$ is taut.*

Proof. Let L be a leaf of $\mathcal{F}'|_{E(K')_i}$. Let $(2E(K')_i, 2\mathcal{F}'|_{E(K')_i})$ be a double of $(E(K')_i, \mathcal{F}'|_{E(K')_i})$ along T'_i , i.e., $2E(K')_i = E(K')_i^+ \cup E(K')_i^-$, where $E(K')_i^\pm$ is a copy of $E(K')_i$ and $2E(K')_i$ is obtained from $E(K')_i^+$ and $E(K')_i^-$ by identifying their boundaries by the natural homeomorphism. Then $2\mathcal{F}'|_{E(K')_i}$ is the foliation on $2E(K')_i$ which is the image of the foliations on $E(K')_i^+$ and $E(K')_i^-$ each of which corresponds to $\mathcal{F}'|_{E(K')_i}$.

By Lemma 4.2.5, we see that $2\mathcal{F}'|_{E(K')_i}$ has no toral leaves. Then, by using [7, Theorem 2.2], we see that there exists a closed transverse curve τ in $(2E(K')_i, 2\mathcal{F}|_{E(K')_i})$ which meets $2L$ in one point, where $2L$ denotes the leaf of $2\mathcal{F}'|_{E(K')_i}$ which is the double of L. Without loss of generality, we may assume that the intersection of $2L$ and τ is contained in $\mathrm{int}E(K')_i^+$. Let φ be the natural involution on $2E(K')_i$, hence $\varphi(E(K')_i^{\pm}) = E(K')_i^{\mp}$. Then let $\tau' = (\tau \cap E(K')_i^+) \cup \varphi(\tau \cap E(K')_i^-)$. Then we deform τ' slightly in a small neighbourhood of $\partial E(K')_i^+(= \partial E(K')_i^- \subset 2E(K')_i)$ so that $\tau' \subset \mathrm{int}E(K')_i^+(\subset 2E(K')_i)$ and τ' is transverse to $2\mathcal{F}'|_{E(K')_i}$. Note that τ' meets $2L$ in one point which is contained in $E(K')_i^+$. This immediately implies that $\mathcal{F}'|_{E(K')_i}$ admits a closed transverse curve which meets L in one point. Hence $\mathcal{F}'|_{E(K')_i}$ is taut. $\square$

Lemma 4.2.7 *Suppose there is a compact leaf F_i of $\mathcal{F}'|_{E(K')_i}$. Then $F_i \cap \partial E(K')_i \neq \emptyset$ and each component of $\partial F_i(\subset T_i')$ is null-homologous in $E(K')_i$, i.e., corresponding to a longitude of K'.*

Proof. Since the union of compact leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$, it is clear that $F_i \cap \partial E(K')_i \neq \emptyset$. We note that each component of $F_i \cap \partial E(K')_i$ is an essential simple closed curve on $\partial E(K')_i$ (otherwise, $\mathcal{F}'|_{\partial E(K')_i}$ has a singularity). We also note that F_i is orientable and $\mathcal{F}$ is transversely oriented. This implies that for the homology boundary operator $\partial : H_2(E(K')_i, \partial E(K')_i) \longrightarrow H_1(\partial E(K')_i)$, $\partial[F_i] = r\alpha$, where α is an indivisible element of $H_1(\partial E(K')_i)$. By Lemma 4.2.6, there exists a closed transverse curve σ_i in $E(K')_i$ which meets F_i in one point. Thus the intersection number of σ_i and F_i is ± 1. This implies that $[F_i]$ is nonzero and is indivisible in $H_2(E(K')_i, \partial E(K')_i)$. Since $E(K')_i$ is an exterior of a knot, $H_2(E(K')_i, \partial E(K')_i) \cong \mathbb{Z}$ and any generator is the class represented by a Seifert surface for K'. Hence $[F_i]$ is a generator of $H_2(E(K')_i, \partial E(K')_i)$ and the homology class is represented by a Seifert surface for K'. Thus we see that $r = 1$ and α is the class represented by a longitude of K'. These imply that each component of ∂F_i is null-homologous in $E(K')_i$. $\square$

4.2.3 Putting $\mathbb{T}$ in a better position with assuming $q = 0$

Recall that K' is a non-cable knot. Let $\mathcal{F}$ be a codimension one, transversely oriented, taut C^0 foliation of finite depth on $M^{(n)}$ such that the union of the depth 0 leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$. Let $\mathbb{T}, M^{(n)\prime}, M_i, E(K')_i, Q_i, \mathcal{F}'$ are as in Section 4.2.2. That is, they satisfy the conditions of Lemmas 4.2.5$\sim$4.2.7. For each i $(i = 1, \dots, n)$, let $d_i(\mathbb{T}), L_i, L_i^*$ be as in Section 4.2.1. Recall that $Q_i \cong$ (disk with two holes) $\times\, S^1$. We say that a surface S in Q_i is *vertical* if S is ambient isotopic to a surface which is a union of S^1-fibers. Note that if S is vertical, then each component of S is either an annulus or a torus. We say that S is *horizontal* if S is ambient isotopic to a surface which is transverse to the S^1-fibers.

In the remainder of this subsection, we suppose that $q = 0$.

Lemma 4.2.8 *For each i $(1 \leq i \leq n)$, each component of $L_i^* \cap Q_i$ is a vertical annulus or a ∂-parallel annulus.*

Proof. Let $\widetilde{L}_i$ be a component of $L_i^* \cap Q_i$. Since L_i^* is compact (Lemma 4.2.4) and $\mathcal{F}$ is proper (Remark 2.1.3), $\widetilde{L}_i$ is compact (Claim 2 in Section 4.2.1). Since L_i^* is incompressible in M_i (Lemma 4.2.4) and each component of $L_i^* \cap T_i'$ is essential in L_i^* (otherwise T_i' is compressible), $\widetilde{L}_i$ is incompressible in Q_i. Hence by [8, VI.34], $\widetilde{L}_i$ is either vertical, horizontal or a ∂-parallel annulus. Hence for a proof of the lemma, it is enough to show that $\widetilde{L}_i$ is not horizontal. Suppose $\widetilde{L}_i$ is horizontal. Then it is clear that $\widetilde{L}_i \cap T_i' \neq \emptyset$. Since L_i^* is compact (Lemma 4.2.4) and $\mathcal{F}$ is proper (Remark 2.1.3), each component of $L_i^* \cap E(K')_i$ is compact. Hence by Lemma 4.2.7, each component of $\partial(L_i^* \cap E(K')_i)$ is a longitude of K'. Hence each component of $\widetilde{L}_i \cap T_i'$ is a longitude of K'. On the other hand, since $q = 0$, Proposition 4.1.1 together with Fact 4.1.1 in Section 4.1 implies that each longitude of K' is isotopic in $\partial E(K')$ to an S^1-fiber of Q_i. Hence each component of $\widetilde{L}_i \cap T_i'$ is isotopic in T_i' to an S^1-fiber of Q_i. However since $\widetilde{L}_i$ is transverse to the S^1-fibers of Q_i, $\widetilde{L}_i$ cannot have a boundary component which is isotopic in ∂Q_i to an S^1-fiber of Q_i, a contradiction. $\square$

Let $\widetilde{\mathbb{T}}$ be a union of tori which is ambient isotopic to $\mathbb{T}$ and transverse to $\mathcal{F}'$. Now, we define a complexity of $\widetilde{\mathbb{T}}$, denoted by $c(\widetilde{\mathbb{T}})$, as follows.

$$c(\widetilde{\mathbb{T}}) = \sum_{i=1}^{n} d_i(\widetilde{\mathbb{T}}).$$

In the remainder of this subsection, we discuss for $\mathbb{T}$ with $c(\mathbb{T})$ is maximal among all $\widetilde{\mathbb{T}}$ as above.

Lemma 4.2.9 *For each i $(1 < i < n)$, there exists a depth $d_i(\mathbb{T})$ leaf L such that there is a component of $L \cap Q_i$ which is a vertical annulus in Q_i and meets T_{i-1} or T_i.*

Proof. By Lemma 4.2.8, for each i, we have; for each depth $d_i(\mathbb{T})$ leaf L with $L \cap M_i \neq \emptyset$, each component of $L \cap Q_i$ is either a vertical annulus or a ∂-parallel annulus. Assume that for some m $(1 < m < n)$, we have; for each depth $d_m(\mathbb{T})$ leaf L with $L \cap M_m \neq \emptyset$, each component of $L \cap Q_m$ is a ∂-parallel annulus.

Claim 1 For each depth $d_m(\mathbb{T})$ leaf L with $L \cap M_m \neq \emptyset$, we have $L \cap T'_m = \emptyset$.

Proof of Claim 1. Assume that there exists a depth $d_m(\mathbb{T})$ leaf L with $L \cap M_m \neq \emptyset$ and $L \cap T'_m \neq \emptyset$. Since $\mathcal{F}$ is a finite depth foliation and the union of the compact leaves of $\mathcal{F}$ is $T_0 \cup T_n$, we see by Lemma 2.1.1 that L meets $\mathring{T}_0$ or $\mathring{T}_n$. This fact and the assumption that $L \cap T'_m \neq \emptyset$ imply that there is a component of $L \cap Q_m$, say A', such that $A' \cap T'_m \neq \emptyset$ and $A' \cap T_j \neq \emptyset$ $(j = m-1$ or $m)$. By the assumption, A' is a ∂-parallel annulus, but any ∂-parallel annulus cannot meet both T'_m and T_j, a contradiction. $\square$

By applying the arguments in the proof of Lemma 4.2.5 for the ∂-parallel annuli in Q_i, deform $\mathbb{T}$ by an ambient isotopy. Let $\mathbb{T}^\star$ be the new union of tori. Let $M^\star_i$ be the closure of the component of $M^{(n)\prime} \setminus \bigcup_{i=1}^{n-1} T^\star_i$ corresponding to M_i with $T^\star_i$ the component of $\mathbb{T}^\star$ corresponding to T_i. Then by the construction, it is easy to see the following.

(i) for any depth $d_m(\mathbb{T})$ leaf L with $L \cap M_m \neq \emptyset$, we have $L \cap M^\star_m = \emptyset$; and

(ii) $M^\star_m \subset M_m$.

By (ii), we have $d_m(\mathbb{T}^\star) \geq d_m(\mathbb{T})$. Moreover (i) implies $d_m(\mathbb{T}^\star) \neq d_m(\mathbb{T})$. Hence the following claim is established.

Claim 2 $d_m(\mathbb{T}^\star) > d_m(\mathbb{T})$.

Claim 3 If $j \neq m$, then $d_j(\mathbb{T}^\star) = d_j(\mathbb{T})$.

Proof of Claim 3. By the proof of Claim 1, we see that the components of $\mathbb{T}$, other than T_{m-1}, T_m are not changed by the deformation for obtaining $\mathbb{T}^\star$. Hence we can immediately see that for k $(1 \leq k \leq m-2$ or $m+2 \leq k \leq n)$,

$d_k(\mathbb{T}^\star) = d_k(\mathbb{T})$. Thus we will prove for $j = m-1,\ m+1$. Since the situation is symmetric, it is enough to prove for the case $j = m-1$. Suppose there does not exist a depth $d_m(\mathbb{T})$ leaf L such that $L \cap T_{m-1} \neq \emptyset$. Then $T^\star_{m-1} = T_{m-1}$ and we immediately have $d_{m-1}(\mathbb{T}^\star) = d_{m-1}(\mathbb{T})$. Suppose there exists a depth $d_m(\mathbb{T})$ leaf L with $L \cap T_{m-1} \neq \emptyset$. Note that the construction of $\mathbb{T}^\star$ implies $M^\star_{m-1} \supset M_{m-1}$ (cf. the above (ii)). Moreover by the deformation described in the proof of Lemma 4.2.5, we see that each component of $\overline{M^\star_{m-1} \setminus M_{m-1}}$ is a solid torus whose boundary is the union of an annulus in T_{m-1} and an annulus in $T^\star_{m-1}$. Since $M^\star_{m-1} \supset M_{m-1}$, $d_{m-1}(\mathbb{T}^\star) \leq d_{m-1}(\mathbb{T})$. Assume $d_{m-1}(\mathbb{T}^\star) < d_{m-1}(\mathbb{T})$. This implies that there exists a leaf L'' of $\mathcal{F}'$ such that $\mathrm{depth}(L'') < d_{m-1}(\mathbb{T})$ and L'' intersects one of the solid tori. Note that $L'' \cap M_{m-1} \neq \emptyset$, which implies $d_{m-1}(\mathbb{T}) \leq \mathrm{depth}(L'')$, a contradiction. Hence we have $d_{m-1}(\mathbb{T}^\star) = d_{m-1}(\mathbb{T})$. □

The above Claims 2 and 3 imply $c(\mathbb{T}^\star) > c(\mathbb{T})$, this contradicts the assumption that $c(\mathbb{T})$ is maximal. □

Let $\mathscr{L}_i$ be the union of the leaves L_i of $\mathcal{F}'$ such that $L_i \cap M_i \neq \emptyset$ and $\mathrm{depth}(L_i) = d_i(\mathbb{T})$.

Lemma 4.2.10 *By deforming $\mathbb{T}$ by an ambient isotopy, if necessary, in addition to the above conditions (i.e., $\mathbb{T}$ is transverse to $\mathcal{F}'$ and $c(\mathbb{T})$ is maximal), we may suppose that $\mathscr{L}_1 \subset M_1$ or $\mathscr{L}_n \subset M_n$.*

Proof. Let L be a depth 0 leaf. Then L meets either $\mathring{T}_0$ or $\mathring{T}_n$. Suppose $L \cap \mathring{T}_0 \neq \emptyset$. Then $d_1(\mathbb{T}) = 0$, and $\mathscr{L}_1$ is a union of depth 0 leaves. Assume that L meets both M_1 and M_2. By Lemma 4.2.8, a component of $\mathscr{L}_1 \cap Q_2$ is either vertical or a ∂-parallel annulus. Assume there is a component, say A_2, of $\mathscr{L}_1 \cap Q_2$ which is vertical in Q_2. Since each component of $\mathscr{L}_1$ intersects M_1, by retaking A_2, if necessary, we may suppose that A_2 intersects T_1. On the other hand, since $L \cap \mathring{T}_0 \neq \emptyset$ and $L \cap M_2 \neq \emptyset$, it is easy to see that there is a component of $L \cap Q_1$, say A_1, which is vertical in Q_1 and intersects T_1. Since A_1 (A_2 resp.) is a subset of $\mathscr{L}_1$, each component of $A_1 \cap T_1$ is either a component of $A_2 \cap T_2$ or disjoint from $A_2 \cap T_2$. However, this contradicts Lemma 4.2.1. Hence any component of $\mathscr{L}_1 \cap Q_2$ is ∂-parallel in Q_2. By the arguments of the proof of Lemma 4.2.5, we may suppose via an ambient isotopy that $\mathscr{L}_1 \subset M_1$. By applying the arguments for the case $L \cap \mathring{T}_n \neq \emptyset$ and $L \cap M_{n-1} \neq \emptyset$, we may suppose that $\mathscr{L}_1 \subset M_1$ or $\mathscr{L}_n \subset M_n$. □

4.2.4 Behavior of d_i's

In this subsection, we suppose $K, \mathcal{F}, \mathbb{T}, M^{(n)'}, \mathcal{F}'$ are as in Section 4.2.3, particularly as in Lemma 4.2.10. In what follows, to simplify the notation, we use d_i for $d_i(\mathbb{T})$. Let k be an integer ($1 \leq k \leq n$) such that $d_k = \max\{d_1, \ldots, d_n\}$. The purpose of this subsection is to prove the following (note that $c(\mathbb{T})$ is maximal).

Proposition 4.2.1 *Suppose $k = \min\{\ell | d_\ell = \max\{d_1, \ldots, d_n\}\}$. Then, we have the following:*

1. *if $2 \leq \ell \leq k$ ($k+1 \leq \ell \leq n-1$ resp.), then $d_{\ell-1} < d_\ell$ ($d_{\ell+1} < d_\ell$ resp.), hence $d_1 < \cdots < d_k \geq d_{k+1} > \cdots > d_n$; and*
2. *suppose $d_k = d_{k+1} = d$, then there exists a depth $d+1$ leaf of $\mathcal{F}'$.*

For the proof of Proposition 4.2.1, we first prove the following lemmas (Lemmas 4.2.11∼4.2.14).

Lemma 4.2.11 *For any m ($1 \leq m \leq k$) (j ($k \leq j \leq n$) resp.), we have the following.*

> ($*$) *For any m' ($1 \leq m' \leq m$) (j' ($j \leq j' \leq n$) resp.), $d_{m'} \leq d_m$ ($d_{j'} \leq d_j$ resp.).*

(Hence we have either $d_1 \leq \cdots \leq d_k \geq \cdots \geq d_n$, $d_1 \leq \cdots \leq d_n = d_k$ or $d_k = d_1 \geq \cdots \geq d_n$.)

Proof. Let L_k be a leaf of $\mathcal{F}'$ such that $\mathrm{depth}(L_k) = d_k$ and $L_k \cap M_k \neq \emptyset$. Since $\mathcal{F}$ is a finite depth foliation and the union of the compact leaves of $\mathcal{F}$ is $T_0 \cup T_n$, we see by Lemma 2.1.1 that L_k meets $\overset{\circ}{T}_0$ or $\overset{\circ}{T}_n$. Since the situation in symmetric, we may suppose without loss of generality that L_k meets $\overset{\circ}{T}_0$. We consider d_m for m less than k. If $k = 1$, then ($*$) is clear. Hence we may suppose $k > 1$. First, consider the case $m = k-1$. If $d_{k-1} = d_k$, then $d_{m'} \leq d_{k-1}$ for any $m' \leq k-2$ because d_k is maximal. Hence we suppose $d_{k-1} < d_k$. In this case, we can show that any depth d_{k-1} leaf does not meet T_{k-1}. In fact, if there is a depth d_{k-1} leaf L_{k-1} with $L_{k-1} \cap T_{k-1} \neq \emptyset$, then $L_{k-1} \cap M_k \neq \emptyset$. This implies that $d_k \leq \mathrm{depth}(L_{k-1}) = d_{k-1}$, a contradiction. Hence $L_{k-1} \cap T_{k-1} = \emptyset$. This implies that $L_{k-1} \cap \overset{\circ}{T}_0 \neq \emptyset$, and that for each m' ($1 \leq m' \leq k-2$), $L_{k-1} \cap M_{m'} \neq \emptyset$. This implies that $d_{m'} \leq d_{k-1}$, i.e., ($*$) holds for $m = k-1$.

Note that if $k = 2$, we have proved the lemma for $m \leq k$. Hence we suppose $k > 2$ in the remainder of the proof. Next consider the case $m = k - 2$. If $d_{k-2} = d_{k-1}$, then the above implies for any $m' \leq k - 3$, we have $d_{m'} \leq d_{k-2}$. If $d_{k-2} < d_{k-1}$, then we can apply the above arguments to show that $L_{k-2} \cap \overset{\circ}{T}_0 \neq \emptyset$, and that for each m' $(1 \leq m' \leq k - 3)$, $L_{k-2} \cap M'_{m'} \neq \emptyset$. Hence $d_{m'} \leq d_{k-2}$, i.e., $(*)$ holds for $m = k - 2$. Then we apply the above arguments repeatedly to have the conclusion of the lemma for $m \leq k$.

Next, we consider d_j for j greater than k. If there exists L'_k, a depth d_k leaf of $\mathcal{F}'$ with $L'_k \cap M_k \neq \emptyset$ and $L'_k \cap \overset{\circ}{T}_n \neq \emptyset$, we can apply the arguments for m $(1 \leq m \leq k)$ to show that $(*)$ holds.

Suppose there does not exist a leaf of $\mathcal{F}'$ satisfying the above conditions, i.e., for any depth d_k leaf L'_k of $\mathcal{F}'$ with $L'_k \cap M_k \neq \emptyset$, we have $L'_k \cap \overset{\circ}{T}_n = \emptyset$. First, consider the case $j = k+1$. If $d_{k+1} = d_k$, then $d_{j'} \leq d_{k+1}$ for any j' $(k+2 \leq j' \leq n)$ because d_k is maximal. Hence we suppose $d_{k+1} < d_k$. Then we can show that any depth d_{k+1} leaf intersecting M_{k+1} does not meet T_k by applying the above arguments. Thus $L_{k+1} \cap \overset{\circ}{T}_n \neq \emptyset$, and this implies that for any $j' \geq j$, $d_{j'} \leq d_j$. Then, we can apply the arguments for the case $m \leq k$ to show that $(*)$ holds for $j = k+1$. Then we apply the above arguments repeatedly to have the conclusion of the lemma for $j \geq k$. This completes the proof of the lemma. $\square$

Recall that $d_k = \max\{d_1, \ldots, d_n\}$. In what follows we further suppose $k = \min\{\ell \mid d_\ell = \max\{d_1, \ldots, d_n\}\}$.

Lemma 4.2.12 *Suppose $d_{k+1} = d_k$. Then by deforming $\mathbb{T}$ by an ambient isotopy, we may suppose that $\mathscr{L}_k \cap T_k = \emptyset$ and $\mathscr{L}_{k+1} \cap T_k = \emptyset$.*

Proof. Assume that $\mathscr{L}_k \cap T_k \neq \emptyset$ (hence $\mathscr{L}_{k+1} \cap T_k \neq \emptyset$). By Lemma 4.2.8, each component of $\mathscr{L}_k \cap Q_{k+1}$ is either a vertical or a ∂-parallel annulus.

If any component of $\mathscr{L}_k \cap Q_{k+1}$ intersecting T_k is a ∂-parallel annulus, by applying the arguments in the proof of Lemma 4.2.5, we may suppose via an ambient isotopy that $\mathscr{L}_k \cap T_k = \emptyset$. Let $\mathbb{T}^*$ be the new tori, T^*_i the component of $\mathbb{T}^*$ corresponding to T_i, and M^*_i the closure of the component of $M^{(n)} \setminus \cup T^*_i$ corresponding to M_i. Then we claim that there is a depth d_k leaf L_k of $\mathcal{F}$ such that $L_k \cap M^*_{k+1} \neq \emptyset$. In fact, if there does not exist a depth d_k leaf intersecting M^*_{k+1}, then we can show that $c(\mathbb{T}^*) > c(\mathbb{T})$ by applying the argument as in the proof of Lemma 4.2.9, a contradiction. Hence $d_{k+1}(\mathbb{T}^*) = d_{k+1}$, and $\mathscr{L}^*_{k+1} \cap T^*_k = \emptyset$, where $\mathscr{L}^*_{k+1}$ is the union of the leaves L' of $\mathcal{F}'$ such that $L' \cap M^*_{k+1} \neq \emptyset$ and $\mathrm{depth}(L') = d_{k+1}$.

Assume there is a component A_k of $\mathscr{L}_k \cap Q_{k+1}$ which is vertical in Q_{k+1} and $A_k \cap T_k \neq \emptyset$. By Lemma 4.2.8, each component of $\mathscr{L}_{k+1} \cap Q_k$ is either vertical or a ∂-parallel annulus. If any component of $\mathscr{L}_{k+1} \cap Q_k$ intersecting T_k is a ∂-parallel annulus, by applying the arguments as in the proof of Lemma 4.2.5, we may suppose via an ambient isotopy that $\mathscr{L}_{k+1} \cap T_k = \emptyset$, hence $\mathscr{L}_k \cap T_k = \emptyset$, giving the conclusion of the lemma. Assume there is a component A_{k+1} of $\mathscr{L}_{k+1} \cap Q_k$ which is vertical in Q_k and $A_{k+1} \cap T_k \neq \emptyset$. Since A_k (A_{k+1} resp.) is a subset of $\mathscr{L}_k$ ($\mathscr{L}_{k+1}$ resp.), each component of $A_k \cap T_k$ is either a component of $A_{k+1} \cap T_k$ or disjoint from $A_{k+1} \cap T_k$, this contradicts Lemma 4.2.1. This completes the proof of the lemma. $\square$

Lemma 4.2.13 *Suppose $d_{k+1} = d_k$. Then $d_1 = 0$ and $d_n = 0$. Moreover, by deforming $\mathbb{T}$ by an ambient isotopy, we may suppose $\mathscr{L}_1 \subset M_1$ and $\mathscr{L}_n \subset M_n$.*

Proof. Since $\mathcal{F}'$ is of finite depth (Lemma 4.2.3), there is a depth 0 leaf, say F_k, in $\overline{\mathscr{L}_k}$ and there is one, say F_{k+1}, in $\overline{\mathscr{L}_{k+1}}$ (see Lemma 2.1.1). By Lemma 4.2.12, we may suppose that $\mathscr{L}_k \cap T_k = \emptyset$ and $\mathscr{L}_{k+1} \cap T_k = \emptyset$. Since $\mathcal{F}'$ is transverse to $\mathbb{T}$, these imply $\overline{\mathscr{L}_k} \cap T_k = \emptyset$ and $\overline{\mathscr{L}_{k+1}} \cap T_k = \emptyset$. Thus $F_k \subset M_1 \cup M_2 \cup \cdots \cup M_k$ and $F_{k+1} \subset M_{k+1} \cup \cdots \cup M_n$. These show $F_k \cap M_1 \neq \emptyset$, and $F_{k+1} \cap M_n \neq \emptyset$, which imply $d_1 = 0$ and $d_n = 0$.

By the arguments in the proof of Lemma 4.2.10, $F_k \subset M_1 \cup M_2 \cup \cdots \cup M_k$ implies via an ambient isotopy that $\mathscr{L}_1 \subset M_1$, and $F_{k+1} \subset M_{k+1} \cup \cdots \cup M_n$ implies via an ambient isotopy that $\mathscr{L}_n \subset M_n$. $\square$

Lemma 4.2.14 *Suppose $n \geq 3$. If $d_{k+1} = d_k$, then $k+2 \leq n$ and $d_{k+2} < d_{k+1}$.*

Proof. By Lemma 4.2.13, for $i \neq 1, n$, we immediately have $d_i > 0$. Hence $d_{k+1}(= d_k) > 0$. Note that $d_n = 0$. Hence $k+1 \leq n-1$, i.e., $k+2 \leq n$. Assume $d_{k+2} = d_{k+1}$. By Lemma 4.2.9, there is a component A_{k+1} of $\mathscr{L}_{k+1} \cap Q_{k+1}$ which is vertical in Q_{k+1} and satisfies $A_{k+1} \cap T_k \neq \emptyset$ or $A_{k+1} \cap T_{k+1} \neq \emptyset$. We have the following cases.

Case 1 Any component of $\mathscr{L}_{k+1} \cap Q_{k+1}$ does not intersect T_k.

In this case, there exists a component A'_{k+1} of $\mathscr{L}_{k+1} \cap Q_{k+1}$ which is vertical and $A'_{k+1} \cap T_{k+1} \neq \emptyset$ (Lemma 4.2.9). If any component of $\mathscr{L}_{k+1} \cap Q_{k+2}$ intersecting T_{k+1} is ∂-parallel in Q_{k+2}, by the arguments in the proof of Lemma 4.2.5, we can deform T_{k+1} by an ambient isotopy so that (with abusing notations) $\mathscr{L}_{k+1} \cap$

$T_{k+1} = \emptyset$. However this implies $\mathscr{L}_{k+1} \subset M_{k+1}$, contradincting the fact that $\mathcal{F}$ is a finite depth foliation and the union of the depth 0 leaves of $\mathcal{F}$ is $T_0 \cup T_n$ (see Lemma 2.1.1). Hence there is a component of $\mathscr{L}_{k+1} \cap Q_{k+2}$ which is vertical in Q_{k+2} intersecting T_{k+1}. However this contradicts Lemma 4.2.1.

Case 2 There is a component of $\mathscr{L}_{k+1} \cap Q_{k+1}$ which intersects T_k.

Case 2 is devided into the following two subcases.

Case 2.1 Any component of $\mathscr{L}_{k+1} \cap Q_{k+1}$ intersecting T_k is ∂-parallel in Q_{k+1}.

In this case, by applying the arguments in the proof of Lemma 4.2.5, we can deform by an ambient isotopy so that (with abusing notations) $\mathscr{L}_{k+1} \cap T_k = \emptyset$. Let $\mathbb{T}^*$ be the new union of tori corresponding to $\mathbb{T}$, $M_j^*(Q_j^*$ resp.) the manifold corresponding to $M_j(Q_j$ resp.), d_j^* the value corresponding to d_j. Note that $M_k^* \subset M_k \cup M_{k+1}$, $M_{k+1}^* \subset M_{k+1}$, $M_j^* = M_j$ $(j \neq k, k+1)$.

Claim $c(\mathbb{T}^*) = c(\mathbb{T})$.

Proof of Claim. The maximality of $c(\mathbb{T})$ implies that $c(\mathbb{T}^*) \leq c(\mathbb{T})$. On the other hand, $M_{k+1}^* \subset M_{k+1}$ implies $d_{k+1}^* \geq d_{k+1}$. Since $M_k^* \subset M_k \cup M_{k+1}$ and $d_k = d_{k+1}$, we see that $d_k^* \geq d_k$. Since $M_j^* = M_j$ for $j \neq k, k+1$, we see that for $j \neq k$, $d_j^* = d_j$. Thus we obtain $c(\mathbb{T}^*) \geq c(\mathbb{T})$, hence $c(\mathbb{T}^*) = c(\mathbb{T})$. □

Let $\mathscr{L}_i^*$ be the union of the leaves L^* of $\mathcal{F}'$ such that $L^* \cap M_i^* \neq \emptyset$, and $\mathrm{depth}(L^*) = d_i^*$. By Claim and Lemma 4.2.9, we see that there is a component A_{k+1}^* of $\mathscr{L}_{k+1}^* \cap Q_{k+1}^*$ which is vertical in Q_{k+1}^* and satisfies $A_{k+1}^* \cap T_k^* \neq \emptyset$ or $A_{k+1}^* \cap T_{k+1}^* \neq \emptyset$. Recall that $\mathscr{L}_{k+1}^* \cap T_k^* = \emptyset$. Hence $A_{k+1}^* \cap T_{k+1}^* \neq \emptyset$, we can apply the argument of Case 1 to have a contradiction.

Case 2.2 There is a component A_{k+1} of $\mathscr{L}_{k+1} \cap Q_{k+1}$ which meets T_k and vertical.

Since $d_{k+1} = d_k$, each component of $\mathscr{L}_{k+1} \cap Q_k$ is either vertical or a ∂-parallel annulus (see the proof of Lemma 4.2.8). If there exists a component of $\mathscr{L}_{k+1} \cap Q_k$ which is vertical in Q_k intersecting T_k, this contradicts Lemma 4.2.1. Suppose any component of $\mathscr{L}_{k+1} \cap Q_k$ intersecting T_k is ∂-parallel. Then we can apply the arguments of Case 2.1 to have a contradiction.

These contradictions completes the proof of the lemma. □

By Lemmas 4.2.11 and 4.2.14, we see that if $n \geq 3$, $d_1, d_2, \ldots, d_n$ has a unique maximum d_k, or two successive maxima d_k, d_{k+1}.

Proof of Proposiotion 4.2.1.

Proof of 1. We give a proof for the case $2 \leq \ell \leq k$. Assume there exists ℓ ($2 \leq \ell \leq k$) such that $d_{\ell-1} = d_\ell$. Note that by the definition of k, we have $d_{k-1} < d_k$. Hence $n \geq 3$ and by retaking ℓ, if necessary, we may suppose $d_{\ell-1} = d_\ell < d_{\ell+1}$. Note that $d_\ell < d_{\ell+1}$ implies $\mathscr{L}_\ell \cap T_\ell = \emptyset$. Since $d_{\ell-1} = d_\ell$, we see that each component of $\mathscr{L}_\ell \cap M_{\ell-1}$ is compact (see the proof of Lemma 4.2.4), hence each component of $\mathscr{L}_\ell \cap Q_{\ell-1}$ is either vertical or a ∂-parallel annulus. If any component of $\mathscr{L}_\ell \cap Q_{\ell-1}$ intersecting $T_{\ell-1}$ is ∂-parallel in $Q_{\ell-1}$, then we can show that $\mathbb{T}$ can be isotoped so that $\mathscr{L}_\ell \subset M_\ell$ as in the proof of Lemma 4.2.14, a contradiction. Hence there exists a component of $\mathscr{L}_\ell \cap Q_{\ell-1}$ which is vertical in $Q_{\ell-1}$ intersecting $T_{\ell-1}$. However as in the proof of Lemma 4.2.14, we can show that there is a component of $\mathscr{L}_\ell \cap Q_\ell$ which is vertical and intersects T_ℓ, contradicting Lemma 4.2.1. This completes the proof of the lemma for ℓ ($2 \leq \ell \leq k$). We can prove for ℓ ($k+1 \leq \ell \leq n-1$) by the same way as above.

Proof of 2. Let L be a leaf of $\mathcal{F}'$ intersecting T_k. Since $d_k = d_{k+1}$, we have $\mathscr{L}_k \cap T_k = \emptyset$, $\mathscr{L}_{k+1} \cap T_k = \emptyset$ (Lemma 4.2.12). This implies that $\mathrm{depth}(L) \neq d$. On the other hand, by the defenition of d_i, we have $\mathrm{depth}(L) \geq d_k = d$. These imply $\mathrm{depth}(L) > d$. $\square$

Corollary 4.2.1 *The depth of $\mathcal{F}'$ is greater than or equal to $[\frac{n}{2}]$.*

Proof. Suppose $n = 1$. Then clearly we have $\mathrm{depth}(\mathcal{F}') \geq [\frac{1}{2}] = 0$. Suppose $n = 2$. By Lemma 4.2.10, without loss of generality we may suppose that $d_1 = 0$. If $d_2 = 0$, by 2 of Proposition 4.2.1, there exists a depth 1 leaf of $\mathcal{F}'$. Thus $\mathrm{depth}(\mathcal{F}') \geq 1 = [\frac{2}{2}]$. If $d_2 > 0$, then it is clear that $\mathrm{depth}(\mathcal{F}') \geq d_2 \geq 1 = [\frac{2}{2}]$. Suppose $n \geq 3$. If n is odd, 1 of Proposition 4.2.1 implies that $d_k \geq \frac{n-1}{2}$, i.e., $\mathrm{depth}(\mathcal{F}') \geq \frac{n-1}{2} = [\frac{n}{2}]$. If n is even, 1 of Proposition 4.2.1 implies that $d_k \geq \frac{n}{2} - 1$. Note that $d_k = \frac{n}{2} - 1$ holds if and only if $d_k = d_{\frac{n}{2}} = d_{\frac{n}{2}+1} = \frac{n}{2} - 1$. Hence in this case, by 2 of Proposition 4.2.1, we have $\mathrm{depth}(\mathcal{F}') \geq \frac{n}{2} = [\frac{n}{2}]$. Hence, for any n ($n \geq 1$), we have $\mathrm{depth}(\mathcal{F}') \geq [\frac{n}{2}]$. $\square$

4.3 Proof of Theorem 1.0.1

Let K, $\Sigma^{(n)}(K, 0)$, α be as in Theorem **??**. Let $\check{\mathcal{F}}$ be a codimension one, transversely oriented, taut C^0 foliation on $\Sigma^{(n)}(K, 0)$ of finite depth which has exactly one depth 0 leaf $\hat{T}$ representing α such that $\mathrm{depth}(\check{\mathcal{F}}) = \mathrm{depth}^0_{1,\alpha}(\Sigma^{(n)}(K, 0))$.

Since $\hat{T}$ is the compact leaf of the taut foliation $\check{\mathcal{F}}$, $\hat{T}$ is taut, i.e., incompressible and norm minimizing. Note that $[\hat{T}] = \alpha = \pm[T_i]$ $(1 \le i \le n)$ and each T_i is a torus. Hence $\hat{T}$ is a torus or a 2-sphere. Assume that $\hat{T}$ is a 2-sphere. Theorem 3 of [15] implies that $\Sigma^{(n)}(K,0) \cong S^2 \times S^1$. By Theorem 7 of [9] , $\Sigma^{(n)}(K,0) \cong S^2 \times S^1$ implies that $S^3(K,0) \cong S^2 \times S^1$. However since K is a non-trivial knot, Property R [6] implies that $S^3(K,0) \not\cong S^2 \times S^1$. Hence $\check{T}$ is an incompressible torus.

Claim The compact leaf $\check{T}$ is isotopic to some T_i $(1 \le i \le n)$.

Proof. Let $\hat{\mathbb{T}} = (\bigcup_{i=1}^n T_i) \cup (\bigcup_{i=1}^n T'_i)$ $(\subset \Sigma^{(n)}(K,0))$. Since $\check{\mathcal{F}}$ is taut, Theorem 4 of [17] implies that by an ambient isotopy in $\Sigma^{(n)}(K,0)$, we can deform $\hat{\mathbb{T}}$ so that either $\hat{\mathbb{T}}$ is transverse to $\check{\mathcal{F}}$ or there exists $T^{(n)}$, a component of $\hat{\mathbb{T}}$ which coincides with $\check{T}$. Suppose $\hat{\mathbb{T}}$ is transverse to $\check{\mathcal{F}}$. By applying the arguments as in the proof of Lemma 4.2.5, we may suppose for any i $(1 \le i \le n)$, $\check{\mathcal{F}}\mid_{E(K')_i}$ has no annular leaves, particularly $\check{T} \cap (\cup T'_i) = \emptyset$. We suppose $\mid \check{T} \cap \hat{\mathbb{T}} \mid$ is minimal among all union of tori which is isotopic to $\hat{\mathbb{T}}$ and transverse to $\check{\mathcal{F}}$. If $\mid \check{T} \cap \hat{\mathbb{T}} \mid > 0$, i.e., there is a torus T_j with $\check{T} \cap T_j \neq \emptyset$, there exist A_j a component of $\check{T} \cap Q_j$, and A_{j+1} a component of $\check{T} \cap Q_{j+1}$ such that $A_j \cap A_{j+1} \neq \emptyset$. Since $\hat{\mathbb{T}}$ is transverse to $\check{\mathcal{F}}$, each component of $\hat{\mathbb{T}} \cap \check{T}$ is an essential simple closed curve in $\hat{\mathbb{T}}$. This shows that $\check{T} \cap Q_i$, if exists, is incompressible in Q_i. Since $\check{T} \cap T'_i = \emptyset$, $\check{T}$ is not horizontal in Q_i. Hence by [8, VI.34], each component of $\check{T} \cap Q_i$ is either vertical or a ∂-parallel annulus. Then we note that the arguments in the proof of Lemma 4.2.5 for removing ∂-parallel annuli work for ∂-parallel components of $\check{T} \cap Q_i$. This together with the minimality of $\mid \check{T} \cap \hat{\mathbb{T}} \mid$ shows that each component of $\check{T} \cap Q_i$ is vertical. Particularly A_j and A_{j+1} are vertical, contradicting Lemma 4.2.1. Hence $\hat{\mathbb{T}} \cap \check{T} = \emptyset$. This shows either $\check{T} \subset E(K')_i$ or $\check{T} \subset Q_i$. If $\check{T} \subset E(K')_i$, then $\check{T}$ is separating in $E(K')_i$ hence separating in $\Sigma^{(n)}(K,0)$. Thus $\check{T}$ can not be a leaf of the taut foliation $\check{\mathcal{F}}$, a contradiction. Hence $\check{T} \subset Q_i$. Since Q_i is homeomorphic to (disk with two holes) $\times S^1$, $\check{T}$ is isotopic to either T'_i, T_{i-1} or T_i. Since T'_i is separating, $\check{T}$ can not isotopic to T'_i, so $\check{T}$ is isotopic to T_{i-1} or T_i. □

We may suppose without loss of generality that $\check{T}$ is isotopic to T_n. Then we can obtain a foliation $\hat{\mathcal{F}'}$ on $M^{(n)}$ by cutting $\check{\mathcal{F}}$ along $\check{T}$. By Corollary 4.2.1 and Lemma 4.2.3, we have

$$\mathrm{depth}(\check{\mathcal{F}}) = \mathrm{depth}(\hat{\mathcal{F}'}) \ge 1 + \left[\frac{n}{2}\right] .$$

This completes the proof of Theorem 1.0.1.

4.4 Proof of Theorem 1.0.2

Let $\mathcal{F}$ be a codimension one, transversely oriented, taut C^0 foliation of finite depth, with C^∞ leaves on $\Sigma^{(n)}(K,0)$ with exactly one depth 0 leaf representing α, where α is corresponding to a generator of $H_1(S^3(K,0)) \cong \mathbb{Z}$. Let $\widetilde{\mathcal{F}}$ be the foliation obtained by modifying $\mathcal{F}$ as in Section 3, hence $\widetilde{\mathcal{F}}$ satisfies the conditions given soon after Remark 2.1.4 in Section 2.1.2. Then, by applying the argument as in the proof of Claim in Section 4.3, we may assume that the compact leaf of $\widetilde{\mathcal{F}}$ is isotopic to T_n. Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K,0)$ by cutting along T_n (in this subsection, we basicaly use the same three dimensional manifolds and surfaces as in Section 4.2 and adopt the same notations for denoting the manifolds, e.g., $M^{(n)}$, M_i, T_j, T_i', etc.) Then, $M^{(n)}$ clearly corresponds to $\hat{M}$ which appears in the paragraph preceding Lemma 2.1.6 in Section 2.1.2. Recall that $\partial M^{(n)} = T_0 \cup T_n$. Let $\hat{\mathcal{F}}$ be the foliation on $M^{(n)}$ induced from $\widetilde{\mathcal{F}}$. Then, for the above folations $\widetilde{\mathcal{F}}$ and $\hat{\mathcal{F}}$, we can consider the graph $G(\widetilde{\mathcal{F}})$ and $\hat{G}(\hat{\mathcal{F}})$ in Definitions 2.1.10 and 2.1.11. Let $g = \mathrm{gap}(\widetilde{\mathcal{F}})$. Since $M^{(n)}$ is not homeomorphic to (torus)$\times[0,1]$, we see that $\widetilde{\mathcal{F}}$ is not a foliation given by a surface bundle structure over S^1. Thus $G(\widetilde{\mathcal{F}})$ has an edge, hence $g \geq 1$. By the construction of $\widetilde{\mathcal{F}}$ described in Section 3, we see that $\widetilde{\mathcal{F}}$ contains exactly one depth 0 leaf. These imply that $\hat{G}(\hat{\mathcal{F}})$ contains exactly two vertices at depth 0.

Lemma 4.4.1 *$\hat{G}(\hat{\mathcal{F}})$ is connected.*

Proof. Assume that $\hat{G}(\hat{\mathcal{F}})$ is not connected. Since the union of the compact leaves of $\hat{\mathcal{F}}$ is $T_0 \cup T_n$, there are exactly two vertices at depth 0. By Remark 2.1.5, any component of $\hat{G}(\hat{\mathcal{F}})$ must have a depth 0 vertex. Hence $\hat{G}(\hat{\mathcal{F}})$ consists of two components, say G_1 and G_2. Let $\{u_j\}$ ($\{v_k\}$ resp.) be the vertices of G_1 (G_2 resp.). Let U_j (V_k resp.) be the union of the leaves representing u_j (v_k resp.). Then we show that $\bigcup_j U_j$ is closed. Let $\{x_i\}_{i=1,2,\dots}$ be a Cauchy sequence in $\hat{M}$ such that each x_i is contained in $\bigcup_j U_j$ and converges to x_∞. We show that $x_\infty \in \bigcup_j U_j$. Let L_∞ be the leaf of $\hat{\mathcal{F}}$ which contains x_∞. Let P be a plaque of $\hat{\mathcal{F}}^\perp$ through x_∞. We may suppose each x_i is contained in P. Let P_+, P_- be the components of $P \setminus x_\infty$. Then, by retaking x_i if necessary, we may suppose that each x_i is contained in P_+. If there is a leaf L of $\hat{\mathcal{F}}$ which contains infinitely many x_i, this implies that $\overline{L} \setminus L \supset L_\infty$. Thus there exists a path connecting u_j and the vertex representing L_∞. Hence we have $x_\infty \in \bigcup_j U_j$. Suppose there does not exist such L. Let L_i be the leaf of $\hat{\mathcal{F}}$ which contains x_i. Since L_i intersects P finitely many times, we may suppose that x_i is the nearest to x_∞ among all the points of $L_i \cap P$. By applying the arguments as in the proof of Claim given soon after the proof of Lemma 3.1.1, we can show that L_∞ is equivalent to L_i. This implies that $x_\infty \in \bigcup_j U_j$. Hence $\bigcup_j U_j$ is closed. By applying the arguments as above, we can show that $\bigcup_k V_k$ is also closed. Note that $M^{(n)} = (\bigcup_j U_j) \cup (\bigcup_k V_k)$ and that $(\bigcup_j U_j) \cap (\bigcup_k V_k) = \emptyset$, contradicting the fact that $M^{(n)}$ is connected. $\square$

We say that graph Γ is a *tree* if Γ is connected and Γ does not contain a cycle.

Lemma 4.4.2 *The following two conditions are equivalent to each other.*

1. *$\hat{G}(\hat{\mathcal{F}})$ is a tree.*
2. *The number of the cycles of $G(\widetilde{\mathcal{F}})$ is one.*

Proof. We first show that 1 implies 2. Suppose 1 holds. Recall that the number of the depth 0 vertices of $\hat{G}(\hat{\mathcal{F}})$ is two, and $G(\widetilde{\mathcal{F}})$ is obtained from $\hat{G}(\hat{\mathcal{F}})$ by identifying them. Since $\hat{G}(\hat{\mathcal{F}})$ is a tree, $\hat{G}(\hat{\mathcal{F}})$ does not have a cycle and there is a unique path in $\hat{G}(\hat{\mathcal{F}})$ connecting the depth 0 vertices, the path becomes a cycle in $G(\widetilde{\mathcal{F}})$ and this is the only cycle in $G(\widetilde{\mathcal{F}})$.

Suppose 2 holds. Since $\hat{G}(\hat{\mathcal{F}})$ is connected (Lemma 4.4.1), we only need to prove that $\hat{G}(\hat{\mathcal{F}})$ does not have a cycle. Assume that $\hat{G}(\hat{\mathcal{F}})$ has a cycle. By applying the argument as above, the path in $\hat{G}(\hat{\mathcal{F}})$ connecting the depth 0 vertices become a cycle in $G(\widetilde{\mathcal{F}})$ and since the operation obtaining $G(\widetilde{\mathcal{F}})$ does not remove a cycle, this implies that the number of the cycles of $G(\widetilde{\mathcal{F}})$ is two, a contradiction. $\square$

In the following, we suppose that $\hat{G}(\hat{\mathcal{F}})$ is a tree. Let Γ be the path connecting the depth 0 vertices of $\hat{G}(\hat{\mathcal{F}})$. Then, clearly $\mathrm{gap}(\hat{\mathcal{F}}) = g$. Suppose $g = 1$. Note that Theorem 1.0.1 implies that $\mathrm{depth}(\mathcal{F}) \geq 1 + \left[\frac{n}{2}\right] \geq \frac{1+n}{2}$. Thus Theorem 1.0.2 holds.

Hence in the remainder of this proof, we suppose $g > 1$.

Lemma 4.4.3 *There exists exactly one edge, say e, of $\hat{G}(\hat{\mathcal{F}})$ with length$(e) > 1$. (Hence we have length$(e) = g$.)*

Proof. Let e be an edge of $\hat{G}(\hat{\mathcal{F}})$ such that $\mathrm{length}(e) > 1$. Let v, v' be the endpoints of e such that $\mathrm{depth}(v) < \mathrm{depth}(v')$. Let Γ_1 (Γ_2 resp.) be a directed path from v (v' resp.) to a vertex at depth 0 such that each edge of Γ_1 (Γ_2 resp.) has length one (Remark 2.1.5). Here we regard the vertex v as Γ_1 if $\mathrm{depth}(v) = 0$. Then, since $\hat{G}(\hat{\mathcal{F}})$ is a tree and the number of the vertices at depth 0 is two, it is clear that $\Gamma = \Gamma_1 \cup e \cup \Gamma_2$. Take any edge e' of $\hat{G}(\hat{\mathcal{F}})$ with length$(e') > 1$. By applying the argument as above, we can show that there exist directed paths Γ'_1, Γ'_2 from the endpoints of e' to the depth 0 vertices, each edge of which has length one. Moreover we have $\Gamma = \Gamma'_1 \cup e' \cup \Gamma'_2$. Since each edge of Γ_1, Γ_2, Γ'_1, Γ'_2 has length one, this shows that $e' = e$. Hence e is the only edge of length greater than one, thus we have length$(e) = g$. $\square$

Let v, v', Γ_1, Γ_2 be as in the proof of Lemma 4.4.3. Since the situation is symmetric, we may suppose Γ_1 (Γ_2 resp.) contains the vertex representing T_0 (T_n resp.). Let m be the number of edges of Γ_1. Since gap$(\hat{\mathcal{F}}) = g$, the number of edges of Γ_2 is $m+g$. Rename the vertices in $\Gamma_1 \cup \Gamma_2$ by $v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m+g+1}$ so that v_i ($0 \le i \le 2m+g+1$) are on Γ in this order, and that $v_0 = [T_0]$, $v_{2m+g+1} = [T_n]$.

Let L_k be a leaf representing v_k. Let $\mathbb{T} = (\bigcup_{j=1}^{n-1} T_j) \cup (\bigcup_{i=1}^{n} T'_i)$ be as in Section 4.2.1. Let M_i be as in Section 4.2. Let $M^{(n)\prime}$ be as in Section 4.2.1. Let M'_i be the closure of the component of $M^{(n)\prime} \setminus \mathbb{T}$ corresponding to M_i. Note that for $i \neq 1, n$, we have $M'_i = M_i$.

Claim depth$(\mathcal{F}) \ge m + g$.

Proof of Claim. It is clear that depth$(\hat{\mathcal{F}}) \ge \max\{\text{depth}(v_i)\}$. Note that v_{m+1} corresponds to v' in the proof of Lemma 4.4.3. Hence $\max\{\text{depth}(v_i)\} =$ depth(v_{m+1}). Note that depth$(v_{m+1}) = \Sigma_{\epsilon:\text{edges of }\Gamma_1}\text{length}(\epsilon) + \text{length}(e)$ (see Figure 4.15). Since the length of each edge of Γ_1 is one, this implies that depth$(v_{m+1}) =$ (number of edges of Γ_1)$+g = m+g$. Hence we have depth$(\hat{\mathcal{F}}) \ge m + g$. By Lemma 2.1.6 and Fact 3.3.1, we see that depth$(\mathcal{F}) \ge$ depth$(\hat{\mathcal{F}}) \ge m + g$. $\square$

Now we estimate the value $m+g$. If $m \ge n$, we have $m+g \ge n+g > \frac{n+g}{2}$. By the above Claim, this shows that Theorem 1.0.2 holds. Hence in the remainder of this subsection, we suppose $m < n$.

Lemma 4.4.4 *There is an ambient isotopy f_t ($0 \le t \le 1$) of $M^{(n)}$ whose support is contained in $\bigcup_{i=1}^{m+1} M_i$ satisfying the following two conditions:*

1. *$f_1(\mathbb{T})$ is transverse to $\hat{\mathcal{F}}$;*
2. *for k ($1 \le k \le m$), $L_k \subset \bigcup_{i=1}^{k} \widetilde{M_i}$, where $\widetilde{M_i}$ is the closure of the component of $M^{(n)} \setminus f_1(\cup_{j=1}^{n-1} T_j)$ corresponding to M_i.*

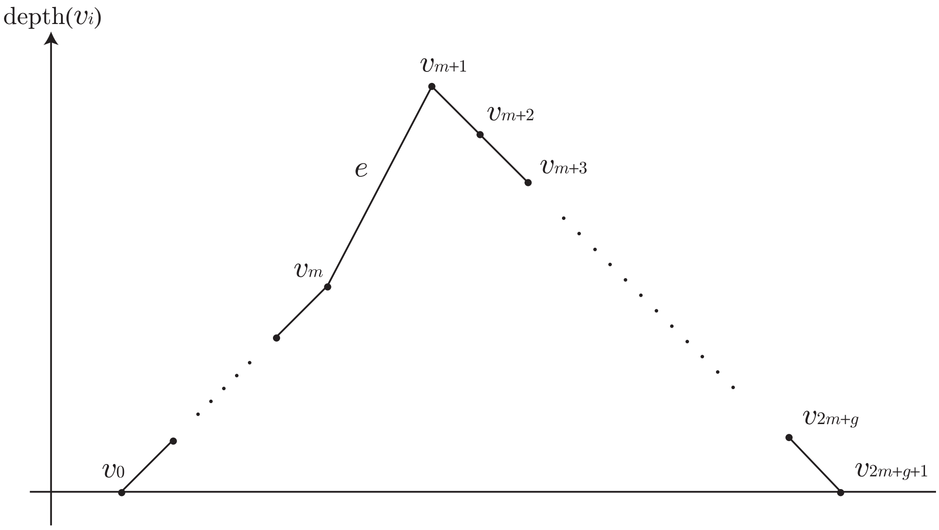


Figure 4.15

Proof. We consider for $k = 1$. By applying the argument as in the proof of Assertion(i) in the proof of Lemma 4.2.3, we see that $L_1 \cap M^{(n)\prime} = \emptyset$ or $L_1 \cap M^{(n)\prime}$ is compact. Suppose $L_1 \cap M_2 \neq \emptyset$. Then, by applying the argument as in the proof of Lemma 4.2.10, we can show that there is an ambient isotopy f_t^1 $(0 \leq t \leq 1)$ whose support is contained in $M_1 \cup M_2$ such that $L_1 \subset \check{M}_1$, where $\check{M}_i$ is the closure of the component of $M^{(n)} \setminus f_1^1(\cup_{j=1}^{n-1} T_j)$ corresponding to M_i, and that $f_1^1(\mathbb{T})$ is transverse to $\hat{\mathcal{F}}$. Suppose $L_1 \cap M_2 = \emptyset$. Then we let $f_t^1 = \mathrm{id}_{M^{(n)}}$ $(0 \leq t \leq 1)$. Then, we consider for $k = 2$. Suppose $L_2 \cap \overline{M^{(n)} \setminus \check{M}_1} \neq \emptyset$. If $L_2 \cap \overline{M^{(n)} \setminus \check{M}_1}$ is noncompact, then there exists a depth 1 leaf L_1' such that $L_1' \subset \overline{L_2}$ and $L_1' \cap \overline{M^{(n)} \setminus \check{M}_1} \neq \emptyset$. Now, since $L_1 \subset \check{M}_1$, $L_1' \neq L_1$. Let v_1' be the vertex representing L_1'. We claim that $v_1' \neq v_1$. In fact, if $v_1' = v_1$, then there is an embedding $\phi : L_1 \times [0,1] \to M^{(n)}$ giving equivalence relation between L and L'. Note that $\hat{\mathcal{F}}|_{\phi(L \times [0,1])}$ is a product foliation, and $L_1 \cap T_1 = \emptyset$. These imply that there is a point x in $T_1 \cap \phi(L_1 \times [0,1])$ such that $\hat{\mathcal{F}}$ and T_1 are not transverse at x, a contradiction. By Remark 2.1.5, there exists a directed path Γ_1' from v_1' to v_0 or v_{2m+g+1}. This contradicts the assumption that $\hat{G}(\hat{\mathcal{F}})$ is a tree. Hence $L_2 \cap \overline{M^{(n)} \setminus \check{M}_1}$ is compact. Suppose $L_2 \cap \check{M}_3 \neq \emptyset$. Then by applying the argument as in the proof of Lemma 4.2.10, we can show that there is an ambient isotopy f_t^2 whose support is contained in $\check{M}_2 \cup \check{M}_3$ such that $L_2 \subset \check{\check{M}}_2$, where $\check{\check{M}}_i$ is the closure of the component of $M^{(n)} \setminus f_1^2(f_1^1(\cup_{j=1}^{n-1} T_j))$ corresponding to M_i, and $f_1^2(f_1^1(\mathbb{T}))$ is transverse to $\hat{\mathcal{F}}$. Suppose $L_2 \cap \check{M}_3 = \emptyset$. Then we let

$f_t^2 = \mathrm{id}_{M^{(n)}}$ $(0 \le t \le 1)$. By applying the argument as above, we can obtain a sequence of ambient isotopies $f_t^1, f_t^2, \ldots, f_t^{m-1}, f_t^m$. Then, the desired ambient isotopy f_t is obtained by applying $f_t^1, f_t^2, \ldots, f_t^m$ successively in this order (with reparametrizing the parameter t). $\square$

In the following, we abuse notation $\mathbb{T}$ for denoting $f_1(\mathbb{T})$ for simplicity, hence for k $(1 \le k \le m)$, $L_k \subset \bigcup_{i=1}^k M_i$ holds. For k $(1 \le k \le m)$, let j_k be the integer which satisfies $L_k \cap M_{j_k} \neq \emptyset$ and $L_k \cap M_{j_k+1} = \emptyset$. We extend the definition of j_k by putting $j_0 = 0$. Since $L_k \subset \bigcup_{i=1}^k M_i$, we immediately have the following.

Lemma 4.4.5 *For k $(1 \le k \le m)$, we have $j_k \le k$.*

Suppose $j_m \ge n - m - g + 1$. By applying Lemma 4.4.5 for the case $k = m$, we have $j_m \le m$. These inequalities imply $m + g \ge \frac{n+g+1}{2} > \frac{n+g}{2}$. This together with the claim in this subsection shows that Theorem 1.0.2 holds. Hence in the remainder of this subsection, we may suppose $j_m < n - m - g + 1$. Note that Γ_2 contains $m + g + 1$ vertices, $v_{m+1}, v_{m+2}, \ldots, v_{2m+g+1}$. By applying the argument as in the proof of Lemma 4.4.4 to leaves corresponding to the $m+g-1$ vertices $v_{2m+g}, v_{2m+g-1}, \ldots, v_{m+2}$, we can obtain the following lemma. (Note that $L_{2m+g+1-k'}(M_{n+1-k'}$ resp.) in Lemma 4.4.6 corresponds to $L_{k'}$ ($M_{k'}$ resp.) in Lemma 4.4.4.)

Lemma 4.4.6 *There is an ambient isotopy f'_t $(0 \le t \le 1)$ of $M^{(n)}$ whose support is contained in $\bigcup_{i=1}^{m+g} M_{n+1-i}$ satisfying the following two conditions:*

1. *$f'_1(\mathbb{T})$is transverse to $\hat{\mathcal{F}}$;*
2. *for k' $(1 \le k' \le m + g - 1)$, $L_{2m+g+1-k'} \subset \bigcup_{i=1}^{k'} \widetilde{\widetilde{M}}_{n+1-i}$, where $\widetilde{\widetilde{M}}_{n+1-i}$ is the closure of the component of $M^{(n)} \setminus f'_1(\cup_{j=1}^{n-1} T_j)$ corresponding to M_{n+1-i}.*

Note that since $j_m < n - m - g + 1 = n + 1 - (m + g)$, f'_t does not change $\bigcup_{i=0}^m L_i$. In the following, we abuse notation $\mathbb{T}$ for denoting $f'_1(\mathbb{T})$ for simplicity, i.e., for k' $(1 \le k' \le m + g - 1)$, $L_{2m+g+1-k'} \subset \bigcup_{i=1}^{k'} M_{n+1-i}$. For k' $(1 \le k' \le m + g - 1)$, let $j'_{k'}$ be the integer which satisfies $L_{2m+g+1-k'} \cap M_{n-j'_{k'}+1} \neq \emptyset$, and $L_{2m+g+1-k'} \cap M_{n-j'_{k'}} = \emptyset$. Since $L_{2m+g+1-k'} \subset \bigcup_{i=1}^{k'} M_{n+1-i}$, we immediately have the following.

Lemma 4.4.7 *For $1 \le k' \le m + g - 1$, we have $j'_{k'} \le k'$.*

Then, we have the following.

Lemma 4.4.8 $n - j'_{m+g-1} \leq j_m + 1$.

Proof. Assume that $n - j'_{m+g-1} \geq j_m + 2$. By the definition of j_k, we see that $(M_{j_m+1} \cup M_{j_m+2}) \cap L_m = \emptyset$. On the other hand, the above inequolity implies that $(M_{j_m+1} \cup M_{j_m+2}) \cap L_{m+2} = \emptyset$. Since L_{m+1} approaches both L_m and L_{m+2}, L_{m+1} intersects both M_{j_m+1} and M_{j_m+2}. By applying the argument as in the proof of Lemma 4.4.4, we can show that $L_{m+1} \cap M_{j_m+1}$ and $L_{m+1} \cap M_{j_m+2}$ are compact. By applying the argument as in the proof of Lemma 4.2.8, we can show that each component of $L_{m+1} \cap Q_{j_m+1}$ ($L_{m+1} \cap Q_{j_m+2}$ resp.) is a vertical or boundary parallel annulus in Q_{j_m+1} (Q_{j_m+2} resp.), where Q_i is as in Section 4.2. By applying the argument as in the proof of Lemma 4.2.9, there exists a vertical annulus $A \subset L_{m+1} \cap Q_{j_m+1}$ such that $A \cap T_{j_m} \neq \emptyset$ or $A \cap T_{j_m+1} \neq \emptyset$, and there exists a vertical annulus $A' \subset L_{m+1} \cap Q_{j_m+2}$ such that $A \cap T_{j_m+1} \neq \emptyset$ or $A \cap T_{j_m+2} \neq \emptyset$. Since $\overline{L_{m+1}} \supset T_n$, we may suppose that $A \cap T_{j_m+1} \neq \emptyset$, and since $\overline{L_{m+1}} \supset T_0$, we may suppose that $A' \cap T_{j_m+1} \neq \emptyset$. However these contradict Lemma 4.2.1. $\square$

By Lemma 4.4.5, we see that $j_m \leq m$ and by Lemma 4.4.7, we see that $j'_{m+g-1} \leq m + g - 1$. These together with Lemma 4.4.8 imply that

$$n - m - g + 1 \leq m + 1.$$

Thus we obtain

$$m + g \geq \frac{n+g}{2}.$$

This together with the claim of this subsection, we can show that Theorem 1.0.2 holds. $\square$

This completes the proof of Theorem 1.0.2.

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