Depths of
codimension one foliations

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1 Introduction

Theory of foliations is originated from studies of global behavior of solutions of differential equations. The development of the theory was provoked by the following question; “Does there exist a two dimensional foliation on the three sphere $S^3$?” which had been proposed by H.Hopf in a different form in about 1935. This question was answered affirmatively by G.Reeb, and this is regarded as the initiation of the research field “foliation”. In 1970’s two new concepts on foliations were introduced. In [16], D.Sullivan called a foliation on a Riemannian manifold geometrically taut if every leaf is a minimal surface with respect to the Riemannian metric, and showed that codimension one foliation is geometrically taut if and only if each compact leaf admits a closed transversal curve. This geometric property is simply called taut, and turned out to be very useful in three dimensional topology via the theory of Thurston norm defined by W.P.Thurston [17]. The other notion, depth of codimension one foliation was introduced by Nishimori in 1977 [12]. The original definition of the depth of $\mathcal{F}$, denoted by $d(\mathcal{F})$, is as follows.

For leaves $F_1, F_2$ of $\mathcal{F}$, we say $F_1 \leq F_2$ if and only if $F_1 \subset \overline{F_2}$. $F_1 < F_2$ if and only if $F_1 \leq F_2$ and $F_1 \neq F_2$. For a leaf $F$ of $\mathcal{F}$, $d(F)$ is the supremum of $k$ such that there are $k$ leaves $F_1, \ldots, F_k$ such that $F_1 < \cdots < F_k = F$. The depth of $\mathcal{F}$, $d(\mathcal{F})$ is the supremum of $d(F)$ for the leaves $F$ of $\mathcal{F}$.

(Note that the value $d(\mathcal{F})$ given by this definition differs by one from the value of the depth given in Definition 2.1.8 of this paper.) It can be regarded that depth is a quantity which describes how far from a fiber bundle structure the foliation is. Nishimori described some fundamental properties of the depth and several authors studied the invariant afterwards.

In 1980’s D.Gabai developed the theory of codimension one foliations on three manifolds. He gave a powerful method which is called sutured manifold theory, for constructing taut foliations on three dimensional manifolds [5]. Particularly in [6], he showed that for any knot $K$ in $S^3$, there exists a codimension one, transversely oriented, taut $C^0$ foliation $\mathcal{F}$ of finite depth on the knot exterior $E(K)$ such that $\mathcal{F}|_{\partial E(K)}$ is a foliation by circles. (As a consequence of this theorem, Property R Conjecture, which was one of the most important problem in knot theory, follows immediately.) Inspired by this result, Cantwell-Conlon [2] introduced an invariant for knots called depth of knots which is the minimal
depth of the foliations on the knot exterior and studied the invariant in a sequence of papers ([2, 3], etc.). (For example, in [2], they showed that for each \( n \geq 0 \), there exists a knot at depth \( n \).) The purpose of this paper is, as a sequel of these researches, to propose more delicate treatments of depth. In the first result, we pay our attention to the number of depth 0 leaves of the foliation under consideration. Here, we note that Cantwell-Conlon often assumed that each of the foliations under consideration has exactly one depth 0 leaf. The motivation of the first research is the following question.

**Question.** For the finite depth foliations on a given 3-manifold \( M \), is there a difference between the minimal value of the depths of the foliations on \( M \) each of which admits exactly one depth 0 leaf and the minimal value of the depths of the foliations on \( M \) without such an assumption?

In this paper, we discuss this question for \( \Sigma^{(n)}(K, 0) \), the \( n \)-fold cyclic covering space of \( S^3(K, 0) \), where \( S^3(K, 0) \) denotes the manifold obtained from \( S^3 \) by performing 0-surgery on a knot \( K \). The main result is as follows.

**Theorem 1.0.1** Let \( \Sigma^{(n)}(K, 0) \) be as above. Suppose that \( K \) is a 0-twisted double of a non-cable knot. Then for each \( n \), we have

\[
\text{depth}^{0}_{1, \alpha}(\Sigma^{(n)}(K, 0)) \geq 1 + \left\lceil \frac{n^2}{2} \right\rceil,
\]

where \( \text{depth}^{0}_{1, \alpha}(\Sigma^{(n)}(K, 0)) \) denotes the minimal depth of codimension one, transversely oriented, taut \( C^0 \) foliations on \( \Sigma^{(n)}(K, 0) \) each of which admits exactly one depth 0 leaf representing the homology class corresponding to a generator \( \alpha \) of \( H_1(S^3(K, 0)) \) and \( \left\lceil x \right\rceil \) denotes the greatest integer among the integers which are not greater than \( x \).

Let \( k \) be the minimal depth of the codimension one, transversely oriented, taut \( C^0 \) foliation on \( S^3(K, 0) \) (note that by Gabai [6], \( k \) is always finite). Suppose \( n > 1 \). Then, by lifting the depth \( k \) foliation on \( S^3(K, 0) \), we see that \( \Sigma^{(n)}(K, 0) \) admits a codimension one, transversely oriented, taut \( C^0 \) foliation of depth \( k \) with more than one (in fact, at least \( n \)) depth 0 leaves. This together with Theorem 1.0.1 gives an affirmative answer to Question.

In the second research, we introduce a quantity called “gap” of the foliation to deal with behaviors of depths of leaves of foliations of finite depth. We know by the definition of depth of leaves (see Section 2.1.2) that each depth \( k(\geq 1) \)
leaf of $\mathcal{F}$ is adjacent to a depth $k - 1$ leaf. Note that if $k$ does not represent the maximal depth in $\mathcal{F}$, it is not necessary the case that there exists a depth $k + 1$ leaf which is adjacent to the leaf. However even if there does not exist such a leaf, there could be a leaf with depth more than $k + 1$ which is adjacent to the depth $k$ leaf. We phrase this situation “There is a gap between the depths of the leaves.” More precisely, for a leaf $L$, we consider the minimum value of the differences between the depth of $L$ and the depths of leaves which are adjacent to $L$ and of depth greater than that of $L$. Then the gap of the foliation is the maximum of such values among the leaves of the foliation. For the formal definition of gap, see Section 2.1.2. By using this invariant, we give an estimation of depth of foliations on $\Sigma^{(n)}(K, 0)$ above.

**Theorem 1.0.2** Let $\mathcal{F}$ be a codimension one, transversely oriented, taut, $C^0$ foliation with $C^\infty$ leaves on $\Sigma^{(n)}(K, 0)$ with exactly one depth 0 leaf representing $[\alpha]$, where $\alpha$ is corresponding to a generator of $H_1(S^3(K, 0)) \cong \mathbb{Z}$. Suppose $\hat{G}(\hat{\mathcal{F}})$ is a tree. Then for each $n$, we have:

$$\text{depth}(\mathcal{F}) \geq \frac{n + \text{gap}(\hat{\mathcal{F}})}{2}.$$  

For the notations $\hat{G}(\hat{\mathcal{F}})$ and $\text{gap}(\hat{\mathcal{F}})$, see Section 2.1.2.

This paper is organized as follows. In Section 2.1, we give definitions concerning about foliations and describe some facts related to the concepts. We also introduce Semistability Theorem given by Dippolito [4, ] (Theorem 2.1.1). In Section 2.1.1, we give definition of depth of foliations and show some facts related to the concepts. In Section 2.1.2, we give the definition of gap of foliations. For the definition, we define equivalence relation on leaves and introduce graph of foliations. In Section 2.2, we give some definitions concerning topology of three dimensional manifolds (Thurston norm, knots and links, sutured manifolds, etc.). We also introduce a theorem given by Gabai [6, ] (Theorem 2.2.1) and give definition of depth of knots. Let $\mathcal{F}$ be a codimension one, transversely oriented $C^0$ foliation of finite depth with $C^\infty$ leaves on a compact, orientable manifold. In Section 3, we give a means of modifying $\mathcal{F}$ to obtain a “good” foliation to which we can define the gap in Section 2.1.2. In Section 4, we give the proofs of Theorems 1.0.1 and 1.0.2. In Section 4.1, for a $q$-twisted doubled knot $K_q$ with companion $K'_q$, and standard Seifert surface $S_q$ (for the definition of these terms, see Section 2.2), we show that the manifold $M_{S_q}$ obtained from the com-
premiminary sutured manifold of $S_q$ by attaching a 2-handle along its suture is homeomorphic to the exterior of a 2-component link obtained from $K'$ by taking parallel copies with linking number $q$ (Proposition 4.1.1). This implies that $M_{S_q}$ admits a decomposition $M_{S_q} = E(K') \cup Q$, where $E(K')$ is the exterior of $K'$, and $Q$ is the manifold homeomorphic to (disk with two holes) $\times S^1$. Note that $\Sigma^{(n)}(K, 0)$ is a union of $n$ copies of $M_{S_q}$, say $\Sigma^{(n)}(K, 0) = M_1 \cup \cdots \cup M_n$, where each $M_i$ is homeomorphic to $M_{S_q}$ (moreover, according to the above decomposition of $M_{S_q}$, each $M_i$ admits a decomposition $M_i = E(K'_i) \cup Q_i$). Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K, 0)$ by cutting along a torus obtained from a lift of $S_q$ by capping off the boundary by a meridian disk of the solid torus (for the 0-surgered manifold $S^3(K, 0)$). Let $\mathcal{F}$ be a codimension one, transversely oriented, taut $C^0$ foliation of finite depth on $M^{(n)}$ such that the union of the compact leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$. In Section 4.2, for the proof of Theorem 1.0.1, we study the depth of $\mathcal{F}$. In Section 4.2.1, for a doubled knot $K$, we show that there exists a submanifold $M^{(n)'}$ which is obtained from $M^{(n)}$ by removing a regular neighbourhood of $\partial M^{(n)}$ such that $\mathcal{F}$ is transverse to $\partial M^{(n)'}$, hence $\mathcal{F} |_{M^{(n)'}}$ is a foliation on $M^{(n)'}$ transverse to $\partial M^{(n)'}$. Then we show that $\text{depth}(\mathcal{F}') \leq \text{depth}(\mathcal{F}) - 1$. These imply that our research can be reduced to the study of the depth of $\mathcal{F}'$. Since $M^{(n)'}$ is homeomorphic to $M^{(n)}$, we abuse notation by denoting $M^{(n)'} = M_1 \cup \cdots \cup M_n = (E(K'_1) \cup Q_1) \cup \cdots \cup (E(K'_n) \cup Q_n)$. In Section 4.2.2, we mimic the arguments of Cantwell-Conlon [2, Section 2] to show that if $K'$ is a non-cable knot, then via an ambient isotopy, we can put $\mathcal{F}'$ in a position which is nice with respect to the submanifolds $E(K'_i)$. Let $d_i$ be the minimal value of the depths of the leaves of $\mathcal{F}'$ which meet $M(K_i)$. In Section 4.2.3, we show with adding condition $q = 0$ that via an ambient isotopy, we can put $\mathcal{F}'$ in a position which is nice with respect to $Q_i$. In Section 4.2.4, we analyze the behaviors of $d_i$’s by using arguments introduced in Section 4.2.3 (for example, we give the inequalities $d_1 < \cdots < d_k \leq d_{k+1} > \cdots > d_n$), which imply a lower bound of $\mathcal{F}'$ (Corollary 4.2.1). In Section 4.3, we give the proof of Theorem 1.0.1 by showing that the depth 0 leaf of each foliation treated in Theorem 1.0.1 is isotopic to a lift of a torus which is obtained from $S_q$ by capping off the boundary by a meridian disk. In Section 4.4, we give the proof of Theorem 1.0.2. For a given foliation with exactly one depth 0 leaf on $\Sigma^{(n)}(K, 0)$ above, we can apply the modification in Section 3 to obtain the foliation to which we can define the gap. By applying the arguments as in Section 4.3, we can obtain a foliation treated in Section 4.2. For such a foliation, we show some properties
about the graph $\hat{G}(\hat{F})$ (Lemmas 4.4.1〜4.4.3). Then by considering positions of leaves representing vertices on the unique path joining the depth 0 vertices of $\hat{G}(\hat{F})$ when it is a tree, we give estimations of depths of $F$. By using these arguments, we obtain Theorem 1.0.2.
2 Preliminaries

2.1 Codimension one foliations

Let $M$ be a Riemannian manifold of dimension $n$. In this subsection, we suppose that $M$ is compact and orientable.

**Definition 2.1.1** A codimension $q$ (or dimension $n-q$) $C^r \,(0 \leq r \leq \infty)$ foliation on $M$ is a $C^r$ atlas $F$ on $M$ with the following properties.

1. If $(U, \varphi) \in F$, then $\varphi(U) = U_1 \times U_2 \subset \mathbb{R}^{n-q} \times \mathbb{R}^q$ where $U_1$ (resp.) is an open disk in $\mathbb{R}^{n-q}$ ($\mathbb{R}^q$ resp.).

2. If $(U, \varphi)$ and $(V, \psi) \in F$ are such that $U \cap V \neq \emptyset$, then the change of coordinates map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is of the form $\psi \circ \varphi^{-1}(x, y) = (h_1(x, y), h_2(y))$.

The charts $(U, \varphi) \in F$ will be called *foliation charts*. We call the pair $(M, F)$ a *foliated manifold*.

For the basic terminologies concerning foliations (holonomy, etc.), see [1]. Let $F$ be a codimension $q C^r \,(0 \leq r \leq \infty)$ foliation on $M$. Let $(U, \varphi)$ be a foliation chart. The sets of the form $\varphi^{-1}(U_1 \times \{c\}), c \in U_2$ are called *plaques* of $U$, or else plaques of $F$.

**Definition 2.1.2** A path of plaques of $F$ is a sequence of plaques $P_1, \ldots, P_k$ of $F$ such that $P_j \cap P_{j+1} \neq \emptyset$ for all $j \in \{1, \ldots, k-1\}$. Since $M$ is covered by plaques of $F$, we can define on $M$ the following equivalence relation: $p_1 \equiv p_2$ if there exists a path of plaques $P_1, \ldots, P_k$ with $p_1 \in P_1, p_2 \in P_k$. The equivalence classes of the relation are called *leaves* of $F$.

In the remainder of this section, suppose $q = 1$, i.e., $F$ is a codimension one foliation.

**Definition 2.1.3** We say that a leaf of $F$ is *proper* if its topology as a manifold coincides with the topology induced from that of $M$. A foliation $F$ is called *proper* if every leaf of $F$ is proper.

**Definition 2.1.4** We say that $F$ is *taut* if for any leaf $L$ of $F$, there is a properly embedded (possibly, closed) transverse curve which meets $L$. 


**Definition 2.1.5** Let $\mathcal{F}$ be a $C^r$ foliation. We say that $\mathcal{F}$ is a $C^r$ foliation with $C^\infty$ leaves on $M$ if each leaf is a $C^\infty$ immersed manifold.

By using a partition of unity argument, we can show that any codimension one, transversely oriented foliation with $C^\infty$ leaves has a one dimensional $C^\infty$ foliation which is transverse to $\mathcal{F}$. Let $\mathcal{F}^\perp$ be a one dimensional $C^\infty$ foliation which is transverse to $\mathcal{F}$.

Let $(M, \mathcal{F})$ be a foliated manifold. A subset $U$ of $M$ is called saturated if $U$ is a union of leaves of $\mathcal{F}$. It is clear that closed leaves are always proper, and it is easy to see that each proper leaf $L$ has an open saturated neighbourhood $U$ in which it is relatively closed ($\overline{L} \cap U = L$).

**Notation 2.1.1** Let $U$ be an open saturated set, and $i : U \to M$ be the inclusion. There is an induced Riemannian metric on $U \subset M$. Then $\hat{U}$ denotes the metric completion of $U$, and $\hat{i} : \hat{U} \to M$ denotes the extended isometric immersion. Let $\hat{\mathcal{F}} = \hat{i}^{-1}(\mathcal{F})$, and $\hat{\mathcal{F}}^\perp = \hat{i}^{-1}(\mathcal{F}^\perp)$ be the induced foliations on $\hat{U}$. The set $\delta U = \hat{i}(\partial \hat{U})$ is called the border of $U$.

In the remainder of this section, we suppose $\mathcal{F}$ is transversely oriented.

**Definition 2.1.6** An $(\mathcal{F}, \mathcal{F}^\perp)$ coordinate atlas is a locally finite collection of $C^r$ embeddings $\varphi_i : D^{n-1} \times [0, 1] \to M$ such that the interior of the images cover $M$, and the restriction of $\varphi_i$ to each $D^{n-1} \times \{t\}$ (to each $\{x\} \times [0, 1]$ resp.) is a $C^\infty$($C^r$ resp.) embedding into a leaf of $\mathcal{F}$($\mathcal{F}^\perp$ resp.).

The unit tangent bundle $q : \tilde{M} \to M$ of $\mathcal{F}^\perp$ is a $C^\infty$ double covering of $M$. Since $\mathcal{F}$ is transversely oriented, for each leaf $L$ of $\mathcal{F}$, $q^{-1}(L)$ consists of two components. Each component of $q^{-1}(L)$ is called a side of $L$.

**Definition 2.1.7** A side $\tilde{\mathcal{L}}$ of $q(\tilde{\mathcal{L}}) = L$ is proper if there are a transverse curve $\tau : [0, 1] \to M$ starting from $L$ in the direction of $\tilde{\mathcal{L}}$ and $\varepsilon(>0)$ such that $\tau(t) \notin L$ for $0 < t < \varepsilon$. Let $\tilde{L}$ be a proper side of $L$. The leaf $L$ has unbounded holonomy on the side $\tilde{L}$ if there are a transverse curve $\gamma : [0, 1] \to M$ starting from $L$ in the direction of $\tilde{L}$ and a sequence $h_1, h_2, \ldots$ of holonomy pseudogroup elements with domain containing $\text{im}(\gamma)$ such that $h_i(\text{im}(\gamma)) = \gamma([0, \varepsilon_i]), \varepsilon_i \searrow 0$.

The leaf $L$ is semistable on the side $\tilde{L}$ if there is a sequence $e_1, e_2, \ldots$ of $C^\infty$ immersions of $\tilde{L} \times [0, 1]$ (with its manifold structure) into $M$ such that $e_i(x, 0) =$
$q(x)$ for all $x$ and $i$, $e_i(\frac{\partial}{\partial t})|_{t=0}$ points in the direction $\tilde{L}$, $e_i(\frac{\partial}{\partial t})$ is always tangent to $\mathcal{F}^1$, each $e_i(\tilde{L} \times \{1\})$ is a leaf of $\mathcal{F}$, and $\bigcap_i e_i(\tilde{L} \times [0,1]) = L$.

In [4], Dippolito showed the following.

**Theorem 2.1.1 (Semistability Theorem [4, □])** Let $\mathcal{F}$ be a codimension one foliation with $C^\infty$ leaves on a closed manifold. If $\tilde{L}$ is a proper side of a leaf $L$ of $\mathcal{F}$, then $L$ either is semistable or has unbounded holonomy on the side $\tilde{L}$.

**Remark 2.1.1** Any side of a proper leaf is proper.

### 2.1.1 Depth of foliations

**Definition 2.1.8** A leaf $L$ of $\mathcal{F}$ is at depth 0 if it is compact. Inductively, when leaves of at depth less than $k$ are defined, $L$ is at depth $k \geq 1$ if $\mathcal{L} \setminus L$ consists of leaves at strictly less than $k$, and at least one of which is at depth $k - 1$. If $L$ is at depth $k$, we use the notation depth($L$) = $k$, and call $L$ a depth $k$ leaf. The foliation $\mathcal{F}$ is of depth $k < \infty$ if every leaf of $\mathcal{F}$ is at depth at most $k$ and $k$ is the least integer for which this is true. If $\mathcal{F}$ is of depth $k$, we use the notation depth($\mathcal{F}$) = $k$. If there is no integer $k < \infty$ which satisfies the above condition, the foliation $\mathcal{F}$ is of infinite depth.

In the remainder of this section, we suppose $\mathcal{F}$ is of finite depth.

**Remark 2.1.2** Let $L$, $L'$ be leaves of $\mathcal{F}$. By Definition 2.1.8, we see that if $\mathcal{L} \setminus L \subset \mathcal{L} \setminus L'$, then depth($L$) $\leq$ depth($L'$).

**Remark 2.1.3** It is known that any leaf of $\mathcal{F}$ is proper.

**Lemma 2.1.1** For any leaf $L$ of $\mathcal{F}$, there exists a depth 0 leaf of $\mathcal{F}$ in $\mathcal{L}$.

**Proof.** By the definition of depth of leaves (Definition 2.1.8), there exists a leaf $L_1$ of $\mathcal{F}$ such that $L_1 \subset \mathcal{L} \setminus L$, and depth($L_1$) $< \text{depth}(L)$. If depth($L_1$) $= 0$, then the lemma holds. Suppose depth($L_1$) $> 0$. We claim that $\mathcal{L} \supset \mathcal{L}_{1}$. Let $x$ be a point in $\mathcal{L}_{1}$. Since $M$ is a complete metric space, there is a Cauchy sequence \( \{x_i\}_{i=1,2,...} (x_i \in L_1) \) such that $x_i$ converges to $x$. Since $x_i \in \mathcal{L}$, there exists $y_i$ in $L$ such that $d(x_i, y_i) \leq \frac{1}{i}$. Clearly, $y_i$ converges to $x$, i.e., $x \in \mathcal{L}$. By the definition of depth of leaves (Definition 2.1.8), there exists a leaf $L_2$ in $\mathcal{L}_{1}$ such that depth($L_2$) $< \text{depth}(L_1)$. If depth($L_2$) $= 0$, then the lemma holds. Suppose that depth($L_2$) $> 0$. Since $\mathcal{L} \supset \mathcal{L}_{1} \supset L_2$, we apply the above argument to $L_2$ to
show that $\overline{L} \supset \overline{L}_2$. Then by applying the above arguments repeatedly, we see that the lemma holds. \qed

**Lemma 2.1.2** Let $L$ be a leaf of $\mathcal{F}$. Suppose $L$ has unbounded holonomy on the side $\overline{L}$ and let $\gamma$ be as in Definition 2.1.7. Then for any leaf $L'$ of $\mathcal{F}$ such that $L' \cap \gamma \neq \emptyset$, we have $\text{depth}(L) < \text{depth}(L')$.

**Proof.** Let $h_1, h_2, \ldots$ be as in Definition 2.1.7. Fix a point $x_0 \in L' \cap \gamma$. Let $x_i = h_i(x_0)$ ($i = 1, 2, \ldots$). Then $x_i \in L'$, and $x_i$ converges to the point $\gamma(0) \in L$. This shows that $L \subset \overline{L}$. Since $L \neq L'$, this implies $\text{depth}(L) < \text{depth}(L')$. \qed

**Lemma 2.1.3** Let $L$, $L'$ be leaves of $\mathcal{F}$. Suppose $L$ is semistable on the side $\overline{L}$ and let $e_1, e_2, \ldots$ be as in Definition 2.1.7. Suppose there exists $i$ such that $e_i(\overline{L} \times [0, 1]) \cap L'$. Then we have $\text{depth}(L) \leq \text{depth}(L')$.

**Proof.** If $L$ is compact, then obviously the lemma holds. Suppose $L$ is non-compact. Then $\overline{L} \setminus L \neq \emptyset$. Let $L^*$ be a leaf contained in $\overline{L} \setminus L$. Fix a point $x^*$ in $L^*$. Let $P$ be a plaque of $\mathcal{F}^\perp$ through $x^*$. Let $P'$ be the closure of a component of $P \setminus x^*$ such that $x^* \in \overline{P'} \cap \overline{L}$. Then we can take points $x_1, x_2, \ldots$ in $P' \cap L$ such that $x_i$ monotonously converges to $x^*$. Let $\overline{x}_j$ be the points in $\overline{L}$ such that $e_i(\overline{x}_j \times \{0\}) = x_j$. Let $P_j = e_i(\overline{x}_j \times [0, 1])$. Then $P_2, P_3, \ldots$ are mutually disjoint arcs embedded in $P'$. Since $L' \subset e_i(\overline{L} \times [0, 1])$, $L' \cap P_j \neq \emptyset$ ($j = 2, 3, \ldots$) Fix a point $x'_j \in L' \cap P_j$. Then $\{x'_j\}_{j=2,3,\ldots}$ converges to $x^*$. Hence $L^* \subset \overline{T}$. Since $L^* \neq L'$, this implies that $\overline{L} \setminus L \subset \overline{T} \setminus L'$. By Remark 2.1.2, we have $\text{depth}(L) \leq \text{depth}(L')$. \qed

**Lemma 2.1.4** Let $\{L_i^{(d)}\}$ be a set of depth $d$ leaves of $\mathcal{F}$, $U$ a component of $M \setminus \bigcup L_i^{(d)}$, and $F$ a component of $\partial U$. Then $\text{depth}(L) \leq d$, where $L$ denotes the leaf $i(F)$ of $\mathcal{F}$.

**Proof.** Fix a point $x$ in $L$. Let $P$ be a plaque of $\mathcal{F}^\perp$ through $x$. Let $P_1, P_2$ be the closures of the components of $P \setminus x$. We may suppose $P_2$ is contained in $i(U)$. If there exists a subarc $P'_i$ of $P_1$ with $x \in \partial P'_i$ such that $P'_i \subset i(U)$, then obviously $L \in \{L_i^{(d)}\}$. Hence we may suppose for any subarc $P'_i$ of $P_1$ with $x \in \partial P'_i$, we have $P'_i \not\subset i(U)$. If there exists a subarc $P''$ of $P_1$ with $x \in \partial P''$ such that $P''$ does not intersect $\bigcup L_i^{(d)}$, then $L \in \{L_i^{(d)}\}$, hence $\text{depth}(L) = d$.

Suppose for any subarc $P''$ of $P_1$ with $x \in \partial P''$, there exists $L' \in \{L_i^{(d)}\}$ such that $L' \cap P_1' \neq \emptyset$. Then the situation is divided into the following two cases.

Case 1 $L$ is semistable on the side $\overline{L}$ which contains $P_1$.  

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Let $e_1, e_2, \ldots$ be as in Definition 2.1.7. Take a subarc $P''$ of $P_1$ with $x \in \partial P''$ and $P'' \subset \text{im}(e_1)$. Let $L''$ be an element of $\{L_i^{(d)}\}$ such that $L'' \cap P'' \neq \emptyset$. By Lemma 2.1.3, we have $\text{depth}(L) \leq \text{depth}(L'') = d$.

Case 2 $L$ has unbounded holonomy on the side $\tilde{L}$ which contains $P_1$.

Take a subarc $\gamma''$ of $P_1$ with $x \in \partial \gamma''$ and satisfies the condition of $\gamma$ in Definition 2.1.7. By Lemma 2.1.2, we have $\text{depth}(L) < \text{depth}(L') = d$. □

**Lemma 2.1.5** Let $\{L_i^{(d)}\}, U$ be as in Lemma 2.1.4. Suppose there exists a pair of components of $\partial \hat{U}$ representing the same leaf $L$ of $F$. Then $L$ is an element of $\{L_i^{(d)}\}$.

**Proof.** Let $x, P_1$ be as in the proof of Lemma 2.1.4. It is clear that there exists a subarc $P'_1$ of $P_1$ with $x \in \partial P'_1$ such that $P'_1 \subset \hat{\iota}(\hat{U})$. This implies $L \in \{L_i^{(d)}\}$. □

### 2.1.2 Gap of foliations

**Definition 2.1.9** For leaves $L_1$ and $L_2$ of $F$, we say that $L_1$ is equivalent to $L_2$ if $L_1 = L_2$ or there exists an embedding $\phi : L_1 \times [0, 1] \rightarrow M$ such that the image of $L_1 \times \{0\}$ ($L_1 \times \{1\}$ resp.) coincides with $L_1$ ($L_2$ resp.), and the image of $\{x\} \times [0, 1]$ is contained in a leaf of $F$ for each $x \in L_1$. Moreover, if $\tilde{L}$ is the side of $L$ such that $\phi_*(\frac{\partial}{\partial t})|_{t=0} / ||\phi_*(\frac{\partial}{\partial t})|_{t=0}||$ is contained in $\tilde{L}$, then we say that $L$ is equivalent to $L'$ through the side $\tilde{L}$.

**Remark 2.1.4** Let $L$ be a leaf of $F$. Suppose that $L$ is semistable on the proper side $\tilde{L}$ and let $e_i : \tilde{L} \times [0, 1] \rightarrow M$ be as in Definition 2.1.7. It is clear that for each $i$, $L$ is equivalent to $e_i(\tilde{L} \times \{1\})$ through the side $\tilde{L}$.

Suppose further $M$ is closed. Let $\tilde{F}$ be a codimension one, transversely oriented $C^r$ foliation of finite depth with $C^\infty$ leaves on $M$ which satisfies the following conditions:

1. the number of equivalence classes of the leaves of $\tilde{F}$ is finite;

2. let $L_1, L_2$ be leaves of $\tilde{F}$. If $L_1$ is equivalent to $L_2$ through the side $\tilde{L}_1$, then the restriction of $\tilde{F}$ to the region between $L_1$ and $L_2$ containing the side $\tilde{L}_1$ is a product foliation with each leaf is homeomorphic to $L_1$.

Let $[L_0]$ be the equivalence classes of the depth 0 leaves of $\tilde{F}$. Let $\hat{M}$ be the union of the metric completions of the components of $M \setminus (\cup_i L_i^0)$. Let $\hat{F}$ be the
foliation on $\hat{M}$ induced from $\tilde{F}$. By the definition of depth, we immediately have the following.

**Lemma 2.1.6** Under the above notations, we have $\text{depth}(\hat{F}) = \text{depth}(\tilde{F})$.

Let $[L_j]$ be the equivalence classes of the leaves of $\hat{F}$.

**Definition 2.1.10** The graph of $\hat{F}$ denoted by $\hat{G}(\hat{F}) = \{V, E\}$ is the directed graph with the vertex set $V = \{v_j\}$ and the edge set $E = \{e_{k\ell}\}$ such that each $v_j$ corresponds to the equivalence class $[L_j]$ of the leaves of $\hat{F}$ and there is an edge $e_{k\ell}$ from $v_k$ to $v_\ell$ if $L_\ell \subset \overline{L_k \setminus L_k}$ and there does not exist a leaf $L$ such that $L \subset L_k \setminus L_k$ and $L_\ell \subset L \setminus L$.

By the construction, the foliated manifold $(M, \tilde{F})$ is recovered from $(\hat{M}, \hat{F})$ by identifying pairs of depth 0 leaves $L_0^+$ and $L_0^-$, each corresponding to $L_0^i$. Then we define the graph of $\tilde{F}$ as follows.

**Definition 2.1.11** The graph of $\tilde{F}$ denoted by $G(\tilde{F})$ is the graph obtained from $\hat{G}(\hat{F})$ by identifying pairs of vertices corresponding to $L_0^+$ and $L_0^-$ for each depth 0 leaf $L_0^i$.

By the definition, we immediately have the following.

**Lemma 2.1.7** The following three conditions are equivalent to each other.

1. $\hat{G}(\hat{F})$ is the graph consisting of exactly one vertex.
2. $\tilde{F}$ is a foliation given by a fiber bundle structure over $S^1$.
3. There exists a leaf $L_0^0$ of $\tilde{F}$ such that $L_0^0$ and $L_0^{-}$ corresponding to the same vertex of $\hat{G}(\hat{F})$.

**Definition 2.1.12** Let $v$ be a vertex of $\hat{G}(\hat{F})$ or $G(\tilde{F})$. We say that $v$ is at depth $k$ if $v$ represents a leaf at depth $k$. If $v$ is at depth $k$, we use the notation $\text{depth}(v) = k$, and call $v$ a depth $k$ vertex.

**Definition 2.1.13** Let $e$ be an edge of $\hat{G}(\hat{F})$ or $G(\tilde{F})$. Let $v$ be the initial point and $v'$ the terminal point of $e$. Then, we define the length of $e$ as follows:

$$\text{length}(e) = \text{depth}(v) - \text{depth}(v').$$
Remark 2.1.5 Let \( v \) be a vertex of \( \hat{G}(\hat{F}) \) or \( G(\tilde{F}) \). If \( \text{depth}(v) \neq 0 \), then by the definition of the depth, we see that there exists a directed path \( \Gamma = e_1 \cup \cdots \cup e_n \) from \( v \) to a depth 0 vertex such that \( \text{length}(e_i) = 1 \) (\( i = 1, \ldots, n \)).

Definition 2.1.14 We define the gap of the foliation \( \tilde{F} \) as follows:

\[
gap(\tilde{F}) = \begin{cases} 0 & \text{if } G(\tilde{F}) \text{ has no edges,} \\ \max_{\text{edges of } G(\tilde{F})} \{\text{length}(e)\} & \text{if } G(\tilde{F}) \text{ has an edge.} \end{cases}
\]

2.2 Topology of three dimensional manifolds

In this subsection, we introduce some basic terminologies concerning three dimensional manifolds. Throughout this subsection, we suppose submanifolds are differentiable, hence each submanifolds admits a regular neighbourhood. For a submanifold \( G \) of a manifold \( M \), \( N(G, M) \) denotes a regular neighbourhood of \( G \) in \( M \). When \( M \) is clear from the context, we often abbreviate \( N(G, M) \) by \( N(G) \). Let \( \beta \) be a simple closed curve embedded in a surface. We say that \( \beta \) is inessential if there exists a disk \( D \) in the surface such that \( \partial D = \beta \). The simple closed curve \( \beta \) is essential if it is not inessential.

In the remainder of this subsection, \( M \) denotes a three dimensional manifold. We say that \( M \) is irreducible if for any two-sphere \( S^2 \) embedded in \( M \), there exists a three-ball \( B^3 \) in \( M \) such that \( \partial B^3 = S^2 \). Let \( F \) be a surface properly embedded in \( M \). We say that \( F \) is compressible if there is a disk \( D \) in \( M \) such that \( D \cap F = \partial D \) and \( \partial D \) is essential in \( F \). The disk \( D \) is called a compression disk. The surface \( F \) is incompressible if it is not compressible. Let \( F_1, F_2 \) be surfaces embedded in \( M \) such that \( \partial F_1 = \partial F_2 \) or \( \partial F_1 \cap \partial F_2 = \emptyset \). We say that \( F_1 \) and \( F_2 \) are parallel (or \( F_2 \) is parallel to \( F_1 \)) if there is a submanifold \( N \) in \( M \) such that \( N \) is homeomorphic to \( F_1 \times [0, 1] \) where we have the following.

1. If \( \partial F_1 = \partial F_2 \), then \( F_1(F_2 \text{ resp.}) \) corresponds to the closure of the component of \( \partial(F_1 \times [0, 1]) \setminus (\partial F_1 \times \{1/2\}) \) that contains \( F_1 \times \{0\}(F_1 \times \{1\} \text{ resp.}). \)

2. If \( \partial F_1 \cap \partial F_2 = \emptyset \), then \( F_1(F_2 \text{ resp.}) \) corresponds to \( F_1 \times \{0\}(F_1 \times \{1\} \text{ resp.}) \) and \( N \cap \partial M \) corresponds to \( \partial F_1 \times [0, 1] \).

Suppose \( F \) is connected. We say that \( F \) is \( \partial \)-parallel if \( F \) is parallel to a subsurface of \( \partial M \). We say that \( F \) is essential if \( F \) is incompressible and not \( \partial \)-parallel.
2.2.1 Thurston norm

Let $F$ be a surface in $\partial M$. Then, for a connected surface $S$ properly embedded in $(M, F)$, let $\chi_-(S) = \max\{0, -\chi(S)\}$. In general, for a surface $\mathcal{S}$, let $\chi_-(\mathcal{S}) = \sum_{i=1}^{n} \chi_-(S_i)$ ($S_1, \ldots, S_n$ are the components of $\mathcal{S}$). For a nontrivial homology class $a \in H_2(M, F; \mathbb{Q})$, we define $x(a) = \min\{\chi_-(\mathcal{S}) \mid \mathcal{S}$ is a surface properly embedded in $(M, F)$ which represents $a \in H_2(M, F; \mathbb{Q})\}$. Let $G$ be a surface properly embedded in $(M, F)$. We say that $G$ is norm minimizing if $\chi_-(G) = x([G])$, where $[G]$ is the element of $H_2(M, F; \mathbb{Q})$ represented by $G$. Let $S$ be a surface properly embedded in $M$. We say that $S$ is taut if $S$ is incompressible and norm minimizing in $H_2(M, N(\partial S, \partial M))$.

Let $V$ be a solid torus in $M$. A simple closed curve $m$ in $\partial V$ is called a meridian of $V$ if $m$ is essential in $\partial V$ and there exists a disk $D$ properly embedded in $V$ such that $\partial D = m$. A simple closed curve $\ell$ is called a longitude of $V$ if it is null-homologous in $M \setminus V$.

Remark 2.2.1 For a taut foliation, it is known that any compact leaf is norm minimizing [17, Corollary 2]. We can show that any leaf of a taut foliation is incompressible by the arguments in the proof of Lemma 7 of [15], hence any compact leaf of a taut foliation is taut.

2.2.2 Knots and links

The union of finite number of mutually disjoint oriented simple closed curves in a 3-manifold $M$ is called a link. For a link $L$ in $M$, $E(L)$ denotes $M \setminus N(L, M)$. We call $E(L)$ an exterior of $L$. A link which consists of one component is called a knot. Let $K_1, K_2$ be knots in $M$. We say that $K_1$ and $K_2$ are equivalent (or $K_2$ is equivalent to $K_1$) if there exists a homeomorphism $h : M \to M$ such that $h(K_2) = K_1$. A Seifert surface for $K$ is an oriented connected surface $S$ embedded in $M$ such that $\partial S = K$. Note that $N(K)$ is a solid torus. Then a meridian (longitude resp.) of $N(K)$ is called a meridian (longitude resp.) of $K$. It is known that for any knot $K$ in the 3-sphere $S^3$, there exists a Seifert surface for $K$. This implies that there exists a longitude of $K$ intersecting a meridian of $K$ transversely in one point. If a Seifert surface for $K$ has minimal genus among all Seifert surfaces for $K$, it is called a minimal genus Seifert surface for $K$ and the genus is called the genus of $K$. A knot $K$ in $S^3$ is called a trivial knot if there exists a disk $D^2$ embedded in $S^3$ such that $\partial D^2 = K$, otherwise $K$ is called a non-trivial knot.
Definition 2.2.1 Let $R, R_1, R_2$ be oriented surfaces in $S^3$. We say that $R$ is obtained by plumbing $R_1$ and $R_2$ if they satisfy the following.

1. $R = R_1 \cup R_2$, where $R_1 \cap R_2$ is a rectangle $D$ with edges $a_1, b_1, a_2, b_2$ as in Figure 2.1 such that $a_1, a_2 \subset \partial R_1$ and $b_1, b_2 \subset \partial R_2$, $a_1$ and $a_2$ ($b_1$ and $b_2$ resp.) are arcs properly embedded in $R_2$ ($R_1$ resp.).

2. There exist 3-balls $B_1, B_2$ in $S^3$ which satisfy the following.
   (a) $B_1 \cup B_2 = S^3$.
   (b) $B_1 \cap B_2 = \partial B_1 = \partial B_2$.
   (c) $B_i \supset R_i$ ($i = 1, 2$).
   (d) $\partial B_1 \cap R_1 = \partial B_2 \cap R_2 = D$.

Let $K'$ be a knot in $S^3$, and $V = D^2 \times S^1$ an unknotted solid torus in $S^3$. Let $L$ be a link in $V$ such that $L$ is not contained in any 3-ball in $V$, and $h : V \rightarrow N(K')$ a homeomorphism. Then the link $h(L)$ is called a satellite for $K'$, and $K'$ is called a companion for $h(L)$. Let $\ell$ ($m$ resp.) be a longitude (a meridian resp.) of $V$. If $L$ is a knot in $\partial V$ representing a homology class $p[m] + q[\ell] (\in H_1(\partial V, \mathbb{Z}))$ (with $|q| \geq 2$) then $h(L)$ is called a cable of $K'$. We say that a knot $K$ is a cable knot if there exists a knot $K''$ such that $K$ is a cable of $K''$. Let $C$ be the knot in $V$ as in Figure 2.2 and $\ell'$ ($m'$ resp.) ($\subset \partial N(K')$) a longitude (a meridian resp.) of $N(K')$. For an integer $q$, let $h_q : V \rightarrow N(K')$ be a homeomorphism with $(h_q)_*([m]) = [m'], (h_q)_*([\ell]) = [\ell'] + q[m']$. Then, we
call the satellite $h_q(C)$ a $q$-twisted double of $K'$ (or simply, we say that $h_q(C)$ is a $q$-twisted doubled knot). Let $S$ be the genus one surface in $V$, as in Figure 2.3. Clearly $h_q(S)$ is a Seifert surface for $h_q(C)$. We often use the notation $S_q$ for denoting this Seifert surface. Note that if $K'$ is a non-trivial knot, then for any $q \in \mathbb{Z}$, $h_q(C)$ is a non-trivial knot [13, IV.10]. Since $S_q$ is of genus one, this implies that if $K'$ is a non-trivial knot, then $S_q$ is a minimal genus Seifert surface for $h_q(C)$. We call $S_q$ a standard Seifert surface for $h_q(C)$. An unknotted annulus in $S^3$ with positive or negative one full twist is called a Hopf annulus (see Figure 2.4). Let $A$ be the annulus embedded in $V$ as in Figure 2.3, and let $A_q = h_q(A)$. Then by Figure 2.3, we see that $S_q$ is obtained by plumbing $A_q$ and a Hopf annulus.
2.2.3 Depth of knots

Recall that $M$ denotes a 3-manifold. In this subsection, we suppose that $M$ is compact and oriented.

D. Gabai showed the following in [6].

**Theorem 2.2.1** ([6, Theorem 1]) Let $K$ be a knot in $S^3$, and $S$ a minimal genus Seifert surface for $K$. Then there exists a codimension one, transversely oriented, taut $C^0$ foliation $\mathcal{F}$ of finite depth on $E(K)$ such that $S \cap E(K)$ is a compact leaf of $\mathcal{F}$, and that $\mathcal{F} |_{\partial E(K)}$ is a foliation by circles.

Cantwell-Conlon [3] proposed the following.

**Definition 2.2.2** Let $K$ be a knot in $S^3$. For an integer $r$ ($0 \leq r \leq \infty$), $K$ is of $C^r$-depth at most $k$ if the knot exterior $E(K)$ admits a codimension one, transversely oriented, taut $C^r$ foliation which is transverse to $\partial E(K)$ and of depth at most $k$. If $k$ is the least integer for which this is true, then $K$ is of $C^r$-depth $k$. If such an integer does not exist, the knot $K$ is of infinite $C^r$-depth.

The following definition which gives a variation of depth is required for the statement of Theorem 1.0.1.

**Definition 2.2.3** Suppose a nontrivial element $\alpha$ of $H_2(M, \mathbb{Q})$ is represented by a connected, taut surface $F$. Then, we define $C^r$ 1-depth of $M$ associated to $\alpha$ which is denoted by depth$_{1,\alpha}^r(M)$ as follows.

$$\text{depth}_{1,\alpha}^r(M) = \min \{ \text{depth}(\mathcal{F}) \mid \mathcal{F} \text{ is a codimension one, transversely oriented, taut } C^r \text{ foliation with exactly one depth 0 leaf, which represents } \alpha \}$$

2.2.4 Sutured manifolds

In this subsection, we quickly recall the definition of sutured manifold introduced by Gabai [6] and some related concepts.

**Definition 2.2.4** Let $M$ be a compact, oriented 3-manifold. The pair $(M, \gamma)$ is a sutured manifold if $\gamma$ is a union of mutually disjoint annuli in $\partial M$ satisfying the following conditions.

1. Each component of $\gamma$ contains an oriented simple closed curve which is homologically non-trivial in $\gamma$. The simple closed curve is called a suture, and $s(\gamma)$ denotes the union of the sutures of $\gamma$.  

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2. Let \( R(\gamma) = \partial M \setminus \text{int}\gamma \). Then \( R(\gamma) \) is oriented as follows. For each component \( \delta \) of \( \partial R(\gamma) \) and each component \( A_1 \) of \( \gamma \) such that \( A_1 \cap \delta \neq \emptyset \), we have the following. If \( s_1 \) is the suture in \( A_1 \), then \( \delta \) and \( s_1 \) are homologous in \( A_1 \).

Then \( R_+(\gamma) \) (\( R_-(\gamma) \) resp.) denotes the union of the components of \( R(\gamma) \) with its normal vectors point outward (inward resp.).

Note that in the original definition of sutured manifold, \( \gamma \) may have torus components. However in our setting this situation does not occur. Hence we gave a definition for the restricted case.

Let \( S \) be a compact surface with \( \partial S \neq \emptyset \), \( M = S \times [0, 1] \), and \( \gamma = \partial S \times [0, 1] \). It is easy to see that \((M, \gamma)\) admits a sutured manifold structure with \( R_+(\gamma) = S \times \{0\} \) (see Figure 2.5). We call the sutured manifold a \textit{product sutured manifold}.

Let \( K \) be a knot in an oriented 3-manifold and \( S \) a Seifert surface for \( K \). We may suppose that \( S \cap E(K) = S \setminus N(\partial S, S) \) and \( S \cap E(K) \) is properly embedded in \( E(K) \). Then \( S_E \) denotes the surface \( S \cap E(K) \). Let \( M_1 \) be a regular neighbourhood of \( S_E \) in \( E(K) \) and \( \gamma_1 = M_1 \cap \partial E(K) \). Clearly, \((M_1, \gamma_1)\)
is a product sutured manifold. Let $X_S = \overline{E(K)} \setminus M_1$, $\gamma_S = \overline{\partial E(K)} \setminus \gamma_1$ ($= X_S \cap \partial E(K)$). It is easy to see that $(X_S, \gamma_S)$ admits a sutured manifold structure with $R_{\pm}(\gamma_S) = R_{\mp}(\gamma_1)$. We call the sutured manifold $(X_S, \gamma_S)$ the complementary sutured manifold of $S$. 
3 Modifying foliations

Let $M$ be a compact, oriented $n$ dimensional manifold and $\mathcal{F}$ a codimension one, transversely oriented $C^\infty$ foliation of finite depth with $C^\infty$ leaves on $M$. We further suppose depth$(\mathcal{F}) \neq 0$. In this section, we show that for any foliation as above, we can modify $\mathcal{F}$ to obtain a foliation with the condition soon after Remark 2.1.4 in Section 2.1.2. Let $\mathcal{F}^\perp$ be a one dimensional $C^\infty$ foliation on $M$ which is transverse to $\mathcal{F}$. Since $M$ is compact, we can take an $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas $\{\varphi_i\}$ of $m(< \infty)$ components.

3.1 First step of Modification

In this subsection, we describe a procedure for modifying $\mathcal{F}$ by using depth 0 leaves of $\mathcal{F}$.

Lemma 3.1.1 Under the equivalence relation of Definition 2.1.9, the number of the equivalence classes represented by the depth 0 leaves is at most $2m + \text{rank } H_1(M, \mathbb{R}) - 1$.

Proof. Let $\{L_j^{(0)}\}$ be representatives of the equivalence classes of the depth 0 leaves of $\mathcal{F}$. We assume that $\{L_j^{(0)}\}$ has $2m + \text{rank } H_1(M, \mathbb{R})$ elements. By slightly modifying the $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas $\{\varphi_i\}$ if necessary, we may suppose that $(\cup L_j^{(0)}) \cap (\cup \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Note that if we take any subset of $\{L_j^{(0)}\}$ consisting of at least $\text{rank } H_1(M, \mathbb{R}) + 1$ elements, then the union of them separates $M$. Hence the number of the components of $M \setminus \cup L_j^{(0)}$ is at least $2m+1$. Hence we can find $U$, a component of $M \setminus \cup L_j^{(0)}$ such that $U \cap (\cup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Note that $U$ is a saturated set. Hence we use notations in Notation 2.1.1 in Section 2.1. For any point $x$ in $\partial \hat{U}$, let $\hat{\tau}_x$ be the leaf of $\hat{\mathcal{F}}^\perp$ which meets $x$. Since $U \cap (\cup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset, \hat{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some $i$ and $c \in D^{n-1}$. Hence $\hat{\tau}_x$ is an arc properly embedded in $\hat{U}$ with endpoints $x$ and $y$, say. Since $\mathcal{F}$ is transversely oriented, $x$ and $y$ are contained in different components of $\partial \hat{U}$. If $\hat{i}(x)$ and $\hat{i}(y)$ are contained in the same leaf of $\hat{\mathcal{F}}$, this implies that $\{L_j^{(0)}\}$ consists of one element, contradicting the assumption that $\{L_j^{(0)}\}$ has $2m + \text{rank } H_1(M, \mathbb{R})$ elements. Thus $\hat{i}(x)$ and $\hat{i}(y)$ are contained in different leaves, say $F_x$ and $F_y$ of $\mathcal{F}$. Obviously, we can take an embedding $\phi : F_x \times [0,1] \to M$ which gives equivalence relation between $F_x$ and $F_y$, this contradicts the assumption that each pair of elements of $\{L_j^{(0)}\}$ is not mutually equivalent. □

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For an equivalence class \([L]\) represented by a depth 0 leaf \(L\), \(\cup L_\alpha\) denotes the union of the leaves of \(\mathcal{F}\) representing \([L]\).

**Claim** Under the above notations, \(\cup L_\alpha\) is closed.

**Proof of Claim.** Let \(\{x_i\}_{i=1,2,\ldots}\) be a Cauchy sequence in \(M\) such that each \(x_i\) is contained in \(\cup L_\alpha\) and converges to \(x_\infty\). We show that \(x_\infty \in \cup L_\alpha\). Let \(L_\infty\) be the leaf of \(\mathcal{F}\) which contains \(x_\infty\). Let \(P\) be a plaque of \(\mathcal{F}^\perp\) through \(x_\infty\). We may suppose each \(x_i\) is contained in \(P\). Let \(P_+, P_-\) be the components of \(P \setminus x_\infty\). Then, by retaking \(x_i\) if necessary, we may suppose that each \(x_i\) is contained in \(P_+\). Let \(L_i\) be the leaf of \(\mathcal{F}\) which contains \(x_i\). Since \(L_i\) is compact, \(L_i\) intersects \(P\) finitely many times. Thus we may suppose that \(x_i\) is the nearest to \(x_\infty\) among all the points of \(L_i \cap P\). Suppose \(L_\infty\) has unbounded holonomy. Let \(\gamma, h_1, h_2, \ldots\) be as in Definition 2.1.7. Since \(x_i\) converges to \(x_\infty\), we may suppose \(x_n \in \text{im}(\gamma)\), for \(n \gg 0\). Fix such \(n\). Take \(\delta_n\) such that \(\gamma([0, \delta_n])\) is the subarc of \(\text{im}(\gamma)\) with endpoints \(x_\infty, x_n\). Since \(x_n\) is the nearest to \(x_\infty\), \(h_i(\text{im}(\gamma)) \not\subset \gamma([0, \delta_n])\) for any \(n\), a contradiction. Hence \(L_\infty\) does not have unbounded holonomy on the side which contains \(x_i\). By Theorem 2.1.1, \(L_\infty\) is semistable on the side which contains \(x_i\). Hence \(L_\infty\) is equivalent to \(L_i\) (Remark 2.1.4). Thus \(x_\infty\) is contained in \(\cup L_\alpha\). \(\square\)

Now, we describe how to modify \(\mathcal{F}\) near \(L\) to obtain a new finite depth foliation \(\mathcal{F}^1\). The situation is divided into the following two cases.

**Case 1** There exists more than one leaves of \(\mathcal{F}\) representing \([L]\).

**Case 2** There exists exactly one leaf of \(\mathcal{F}\) representing \([L]\).

In Case 1, let \(\{\phi_\beta\}\) be the set of all the embeddings which give equivalence relations between \(L\) and the leaves representing \([L]\). Let \(\mathcal{U} = \cup \phi_\beta(L \times [0, 1])\). The situation is divided into the following two subcases.

**Case 1.1** \(\mathcal{U} = M\).

**Claim 1** For each side of \(L\), there is a leaf \(L'\) to which \(L\) is equivalent through the side.

**Proof of Claim 1.** Let \(P\) be a plaque of \(\mathcal{F}^\perp\) through \(x \in L\). For a side \(\tilde{L}\) of \(L\), let \(P_+\) be the component of \(P \setminus x\) which is contained in the side \(\tilde{L}\). Let \(x_i\) be a sequence of points in \(P_+\) which converges to \(x\). Since \(\mathcal{U} = M\), for each \(i\), there exists a leaf \(L_i\) to which \(L\) is equivalent via embedding \(\phi_i : L \times [0, 1] \to M\) such that \(\phi_i(L \times [0, 1]) \ni x_i\). If there exists \(i\) such that \(\phi_i(L \times [0, 1])\) contains the side \(\tilde{L}\), then the leaf \(\phi_i(L \times \{1\})\) is equivalent to \(L\) through the side \(\tilde{L}\). Suppose for each \(i\), \(\phi_i(L \times [0, 1])\) does not contain the side \(\tilde{L}\). Let \(L_i = \phi_i(L \times \{1\})\). Then
Proof of Claim 2. Since the situation is symmetric, we give the proof for both Claim 2 and \( \varphi \). Let \( L \) be a leaf of \( \mathcal{F} \), \( U \), \( P \), \( L_n \) and \( \frac{1}{2} L_n \) be defined as above. Let \( \phi \) be an embedding which gives equivalence relations between \( L \) and \( \mathcal{F} \). Let \( U^\pm = \cup \phi^\pm(L \times [0, 1]). \)

Claim 2 Both \( U^+ \) and \( U^- \) are closed.

Proof of Claim 2. Since the situation is symmetric, we give the proof for \( U^+ \). Let \( \{ x_i \}_{i=1,2,...} \) be a Cauchy sequence in \( M \) such that each \( x_i \) is contained in \( U^+ \) and converges to \( x_\infty \). We show that \( x_\infty \in U^+ \). Assume \( x_\infty \notin U^+ \). This implies that there does not exist \( j \) such that \( x_\infty \in \phi^+_j(L \times [0, 1]) \). For each \( i \), we fix an embedding \( \phi^+_i \) giving an equivalence relation through the side \( \bar{L}_+ \), such that \( x_i \in \phi^+_i(L \times [0, 1]) \). Let \( L_\infty \) be the leaf of \( \mathcal{F} \) which contains \( x_\infty \). Let \( L_i = \phi^+_i(L \times \{1\}) \). Let \( P \) be a plaque of \( \mathcal{F}^\bot \) through \( x_\infty \). We may suppose each \( x_i \) is contained in \( P \). Let \( P_+ \), \( P_- \) be the components of \( P \setminus x_\infty \). Since \( x_\infty \notin \phi^+_i(L \times [0, 1]) \), all of the \( x_i \)'s are contained in \( P_+ \) or \( P_- \), say \( P_+ \). Let \( y_i \) be the point of \( L_i \cap P_+ \) which is the nearest to \( x_\infty \). By applying the argument in the proof of Claim given soon after the proof of Lemma 3.1.1, we can show that \( L_\infty \) does not have unbounded holonomy on the side which contains \( x_i \). By Theorem 2.1.1, \( L_\infty \) is semistable on the side which contains \( x_i \). Hence for large \( n \), \( L_\infty \) is equivalent to \( L_n \) through the side which contains \( x_i \). Let \( \phi' : L_n \times [0, 1] \rightarrow M \) be an embedding which gives the equivalence relation. Then, by composing \( \phi^+_n \) and \( \phi' \), we can obtain an embedding which gives equivalence relation between \( L \) and \( L_\infty \) through the side \( \bar{L}_+ \). Hence \( x_\infty \notin U^+ \), a contradiction.

Claim 3 There exists a leaf \( L_* \) to which \( L \) is equivalent through both sides of \( L \).

Proof of Claim 3. Since \( U = \mathcal{F} \) and \( M \) is connected, the above Claim 2 implies that \( U^+ \cap U^- \neq \emptyset \). For a point \( y \) in \( U^+ \cap U^- \), let \( \phi^\pm \) be an embedding from \( L \times [0, 1] \) to \( M \) which gives equivalence relation through \( \bar{L}_\pm \) such that \( \phi^\pm(L \times [0, 1]) \) contains \( y \). Let \( L^\pm = \phi^\pm(L \times \{1\}) \). If \( L^+ = L^- \), then \( L \) is equivalent to \( L^+ = L^- \) through both sides of \( L \). Suppose \( L^+ \subset \text{int} \phi^-(L \times [0, 1]) \). Since \( L^+ \) is transverse to \( \mathcal{F}^\bot \), \( L \) is equivalent to \( L^+ \) through the side \( \bar{L}^- \). Thus
$L^+$ satisfies the condition of $L_*$. The case of $L^- \subset \text{int}\phi^+(L \times [0,1])$ is treated in the same manner, these complete the proof of the claim. □

By joining the embeddings giving equivalence relation between $L$ and $L_*$ in the above Claim 3, we see that there is an immersion $\phi' : L \times [0,1] \to M$ such that the image of $\{x\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$, $\phi'(L \times \{0\}) = \phi'(L \times \{1\}) = L$. Hence $M$ admits a fiber bundle structure over $S^1$ with each fiber homeomorphic to $L$ and transverse to $\mathcal{F}^\perp$, and $L$ is a fiber. In this case, we let $\mathcal{F}^1$ be the foliation given by this bundle structure, i.e., each leaf of $\mathcal{F}^1$ is a fiber of the fibration.

**Case 1.2** $U \neq M$.

In this case, we first show the following claims (Claims 1–3).

By applying the argument as in the proof of Claim 2 of Case 1.1, we can show the following.

**Claim 1** $U$ is closed.

Let $\tau$ be a leaf of $\mathcal{F}^\perp|_U$ which meets a component of $\partial U$, say $L_0$.

**Claim 2** The leaf $\tau$ is an arc properly embedded in $U$.

**Proof of Claim 2.** Assume not, i.e., $\tau$ meets $\partial U$ in one point. Let $\{x_i\}_{i=1,2,...}$ be a sequence of points on $\tau$ such that $d_\tau(x_0, x_i) > i$, where $d_\tau$ is the path metric on $\tau$ induced from $M$. By the above Claim 1, $U$ is compact. Hence there exists an accumulating point of $\bigcup x_i$.

If necessary, we may suppose that $x_i$ converges to $x_\infty$. Since $U$ is closed, $x_\infty \in U$. Then $x_\infty \in L$ or there exists $\phi_\infty \in \{\phi_\beta\}$ such that $x_\infty \in \phi_\infty(L \times [0,1])$. Since $\tau$ meets $\partial U$ in one point, $x_\infty \notin \partial U$.

**Case 1.2.1** $x_\infty \in L$.

In this case, $L_0 \neq L$. Let $\phi_0$ be an embedding which gives equivalence relation between $L$ and $L_0$. Since $x_i$ converges to $x_\infty$, and $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_0(L \times [0,1])$ is a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_0(L \times [0,1])}$ contained in $\tau$, contradicting that $\tau$ meets $\partial U$ in one point.

**Case 1.2.2** $x_\infty \notin L$.

In this case, there exists $\phi_\infty \in \{\phi_\beta\}$ such that $x_\infty \in \phi_\infty(L \times [0,1])$. Let $L_\infty = \phi_\infty(L \times \{1\})$. On the other hand, since $U$ is closed, $L_0 \subset U$. This implies that $L_0$ is equivalent to $L$. Suppose $L_0 = L$. Since $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_\infty(L \times [0,1])$ is a union of infinitely many fibers of $\mathcal{F}^\perp|_{\phi_0(L \times [0,1])}$ those are contained in $\tau$, contradicting that $\tau$ meets $\partial U$ in one point.
Suppose $L_0 \neq L$. Let $\phi_0$ be an embedding which gives equivalence relation between $L$ and $L_0$. The situation is divided into the following two cases.

Case 1.2.2.1 $\phi_0(L \times [0, 1]) \cap \phi_\infty(L \times [0, 1]) = \phi_\infty(L \times [0, 1])$.

In this case, since the length of each fiber of $\mathcal{F}^{\perp}|_{\phi_\infty(L \times [0, 1])}$ is finite, and $d_\tau(x_0, x_i) > i$, we see that $\tau \cap \phi_\infty(L \times [0, 1])$ is a union of infinitely many fibers of $\mathcal{F}^{\perp}|_{\phi_\infty(L \times [0, 1])}$. Hence $\tau \cap \phi_0(L \times [0, 1])$ is also a union of infinitely many subarcs of $\tau$ which are properly embedded in $\phi_0(L \times [0, 1])$, contradicting that $\tau$ meets $\partial \mathcal{U}$ in one point.

Case 1.2.2.2 $\phi_0(L \times [0, 1]) \cap \phi_\infty(L \times [0, 1]) = L$.

In this case, by applying the argument as in Case 1.2.2.1, we can show that $\tau \cap \phi_\infty(L \times [0, 1])$ is a union of infinitely many fibers of $\mathcal{F}^{\perp}|_{\phi_\infty(L \times [0, 1])}$. Since each fiber of $\mathcal{F}^{\perp}|_{\phi_0(L \times [0, 1])}$ is adjacent to a fiber of $\mathcal{F}^{\perp}|_{\phi_\infty(L \times [0, 1])}$, $\tau \cap \phi_0(L \times [0, 1])$ is also a union of infinitely many fibers of $\mathcal{F}^{\perp}|_{\phi_0(L \times [0, 1])}$. This contradicts the assumption that $\tau$ meets $\partial \mathcal{U}$ in one point. \hfill \Box

Claim 3 The boundary of $\mathcal{U}$ consists of two components.

Proof of Claim 3. Since $\mathcal{F}$ is transversely oriented, we see by Claim 2 of Case 1.2 that $\partial \mathcal{U}$ consists of at least two components. Suppose $L \subset \partial \mathcal{U}$. Let $L''$ be another component of $\partial \mathcal{U}$. Since $\mathcal{U}$ is closed, $L'' \subset \mathcal{U}$. Hence $L''$ is equivalent to $L$. Let $\phi''$ be an embedding which gives equivalence relation between $L$ and $L''$. Then, it is obvious that $\mathcal{U} = \phi''(L \times [0, 1])$, hence $\partial \mathcal{U} = L \cup L''$. Suppose $L \subset \text{int} \mathcal{U}$. Let $L_1, L_2$ be different components of $\partial \mathcal{U}$. Let $\phi_1$ ($\phi_2$ resp.) be an embedding which gives equivalence relation between $L$ and $L_1$ ($L_2$ resp.). Then it is obvious that $\mathcal{U} = \phi_1(L \times [0, 1]) \cup \phi_2(L \times [0, 1])$. Hence $\partial \mathcal{U} = L_1 \cup L_2$. This completes the proof of the claim. \hfill \Box

Let $\partial \mathcal{U} = L_\infty \cup L_{-\infty}$. Obviously, $L_\infty$ is equivalent to $L_{-\infty}$, i.e., there exists $\phi^*: L_\infty \times [0, 1] \to M$ such that $\phi^*(L_\infty \times [0, 1]) = \mathcal{U}$. Now, we modify $\mathcal{F}$ by replacing $\mathcal{F}|_\mathcal{U}$ with the image of the product foliation on $L_\infty \times [0, 1]$. The modification near the depth 0 leaf $L$ is completed.

In Case 2 (the case that there exists exactly one leaf of $\mathcal{F}$ representing $[L]$), let $U = M \setminus L$. Then $\partial \mathcal{U} = L_+ \cup L_-$, where $L_+$ ($L_-$ resp.) is homeomorphic to $L$. The situation is divided into the following two subcases.

Case 2.1 There exists a homeomorphism $h: L \times [0, 1] \to \hat{U}$ such that the image of each $x \times [0, 1]$ is a leaf of $\mathcal{F}^{\perp}$.

In this case $M$ admits a fiber bundle structure over $S^1$ with each fiber homeomorphic to $L$ and transverse to $\mathcal{F}^{\perp}$, and $L$ is a fiber. Then, $\mathcal{F}^{\perp}$ is the foliation
given by this bundle structure,

Case 2.2 There does not exist a homeomorphism from \( L \times [0, 1] \) to \( \hat{U} \) as in Case 2.1.

In this case \( F \) is unchanged by the modification.

In Cases 1.2 and 2.2, we further modify the foliation by using another equivalence class of the depth 0 leaves. By Lemma 3.1.1, this terminates in finitely many steps. Let \( F^1 \) be the foliation which is obtained by repeating the procedure for all equivalence classes of the depth 0 leaves. Note that this modification does not change the transverse foliation \( F^\perp \), i.e., \( F_1^\perp = F^\perp \).

### 3.2 Second step of Modification

In this subsection, we describe a procedure for modifying \( F^1 \) obtained in Section 3.1 by using depth 1 leaves of \( F^1 \).

**Lemma 3.2.1** For the modified foliation \( F^1 \), the number of the equivalence classes represented by the depth 1 leaves is finite.

**Proof.** Let \( \{L_k^{(1)}\} \) be a set of depth 1 leaves of \( F^1 \) such that each pair of elements is not mutually equivalent. We assume that \( \{L_k^{(1)}\} \) has infinitely many elements. By Lemma 2.1.2 and Theorem 2.1.1, we can show that each leaf of \( \{L_k^{(1)}\} \) is isolated in \( \bigcup L_k^{(1)} \). By slightly modifying the \((F, F^\perp)\)-coordinate atlas \( \{\varphi_i\} \) if necessary, we may suppose that \((\bigcup L_k^{(1)}) \cap (\bigcup \varphi_i(D^{n-1} \times \partial[0, 1])) = \emptyset \). Hence we can find \( U \), a component of \( M \setminus \bigcup L_k^{(1)} \) such that \( U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0, 1])) = \emptyset \). For any point \( x \) in \( \partial \hat{U} \), let \( \hat{\tau}_x \) be the leaf of \( F^1 \) which meets \( x \). Since \( U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0, 1])) = \emptyset \), \( \hat{\tau}_x \) is a proper subarc of \( \varphi_i(c \times [0, 1]) \) for some \( i \) and \( c \in D^{n-1} \). Hence \( \hat{\tau}_x \) is an arc properly embedded in \( \hat{U} \) with endpoints \( x \) and \( y \), say. Since \( F \) is transversely oriented, \( x \) and \( y \) are contained in different components of \( \partial \hat{U} \). Let \( F_x \) (\( F_y \) resp.) be the leaf of \( F^1 \) which meets \( i(x) \) (\( i(y) \) resp.). Then we immediately have the following.

**Claim 1** \( \hat{U} \) is homeomorphic to \( F_x \times [0, 1] \), where each \( \{p\} \times [0, 1] \) is contained in a leaf of \( F^\perp \).

Moreover we have the following.

**Claim 2** If \( F_x \neq F_y \), then \( F^1|_U \) is a product foliation with each leaf is at depth 0.
Proof of Claim 2. If \( F_x \) or \( F_y \) is a depth 1 leaf, then obviously \( F_x \) is equivalent to \( F_y \), this contradicts the assumption that each pair of elements of \( \{ L_k^{(1)} \} \) is not mutually equivalent. This together with Lemma 2.1.4, \( \partial U \) consists of depth 0 leaves. Since \( \mathcal{F}^1 \) is modified for all equivalence classes of the depth 0 leaves, \( \mathcal{F}^1|_U \) is a product foliation with each leaf is at depth 0. □

Suppose \( F_x = F_y \). Let \( \tilde{U} = U \cup F_x \).

Claim 3 Under the above conditions, we have the following:

1. \( F_x \) is a depth 1 leaf;
2. \( \partial \tilde{U} = F_x \setminus F_x \); and
3. \( F_x \) is the only element of \( \{ L_k^{(1)} \} \) which meets \( \tilde{U} \).

Remark 3.2.1 By 1 and 2 of the above Claim 3, we see that \( \partial \tilde{U} \) consists of depth 0 leaves.

Proof of Claim 3. By Lemma 2.1.5, it is obvious that 1 of the claim holds. We show that \( \partial \tilde{U} \supset F_x \setminus F_x \). Note that since \( F_x \) is a depth 1 leaf, \( F_x \setminus F_x \) is a union of depth 0 leaves. Let \( L_0 \) be a leaf contained in \( F_x \setminus F_x \). Since \( U \cong F_x \times (0, 1) \), \( F_x \) is noncompact, and each leaf of \( \mathcal{F}^1 \) is transverse to \( \mathcal{F}^1 \), every leaf of \( \mathcal{F}^1|_U \) is noncompact. Since \( L_0 \) is compact, this shows that \( L_0 \cap U = \emptyset \). Hence \( L_0 \cap \tilde{U} = \emptyset \). On the other hand, since \( L_0 \subset F_x \), we have \( L_0 \subset \tilde{U} \). These imply \( L_0 \subset \partial \tilde{U} \). Then, we show that \( \partial \tilde{U} \subset F_x \setminus F_x \). Let \( a \) be a point in \( \partial \tilde{U} \) and \( N_a \) a neighbourhood of \( a \). Then there exist points \( b_1 \) and \( b_2 \) of \( N_a \) such that \( b_1 \in \tilde{U} \) and \( b_2 \notin \tilde{U} \). Suppose \( b_1 \notin F_x \). Then take an arc \( b_1b_2 \) in \( N_a \) connecting \( b_1 \) and \( b_2 \). Then there is a point \( b \in b_1b_2 \) such that \( b \in F_x \). This shows that \( a \in F_x \). Since \( \delta U = F_x \), both sides of \( F_x \) are contained in \( U \). Hence \( F_x \subset \text{int} \, \tilde{U} \), and this shows that \( a \notin F_x \). These show \( \partial \tilde{U} \subset F_x \setminus F_x \), and 2 of the claim holds. We can show 3 of the claim immediately by the above Claim 1. □

We know that the number of the equivalence classes represented by the depth 0 leaves of \( \mathcal{F} \) is finite (Lemma 3.1.1). Since the modification does not change the number of the equivalence classes represented by the depth 0 leaves, the number of the equivalence classes represented by the depth 0 leaves of \( \mathcal{F}^1 \) is also finite. This fact together with the above Claim 2, 3 of the above Claim 3 and Remark 3.2.1 imply that \( \{ L_k^{(1)} \} \) consists of finitely many elements, a contradiction. □
Now, we modify the foliation $\mathcal{F}^1$. Since the number of the equivalence classes represented by the depth 0 leaves is finite (Lemma 3.1.1), $M \setminus \cup$ (depth 0 leaves) consists of finite number of components, say $U_1, U_2, \ldots, U_k$. Note that there is a depth 1 leaf in each $U_i$. Let $L(\subset U_1)$ be a depth 1 leaf. The situation is divided into the following two cases.

Case 1 There exist more than one leaves of $\mathcal{F}^1$ representing $[L]$.

Case 2 There exists exactly one leaf of $\mathcal{F}^1$ representing $[L]$.

In Case 1, let $\{\phi_\beta\}$ be the set of all the embeddings which give equivalence relations between $L$ and the leaves representing $[L]$. Let $\mathcal{U}^{(1)} = \cup \phi_\beta(L \times [0, 1])$. The situation is divided into the following two subcases.

Case 1.1 $\mathcal{U}^{(1)} = U_1$.

In this case, by applying the argument as in the proof of Claim 3 of Case 1.1 in Section 3.1, we can show that there exists a depth 1 leaf $L' \subset U$ such that for each side of $L$, $L$ is equivalent to $L'$ through the side. This implies that $U_1 \setminus L \cong L \times (0, 1)$ with each $x \times (0, 1)$ is contained in a leaf of $\mathcal{F}^\perp$. We modify $\mathcal{F}^1$ by replacing $\mathcal{F}^1|_{\mathcal{U}^{(1)}}$ with the image of the product foliation on $L \times [0, 1]$. Note that in this case, the modification on $U_1$ is completed.

Case 1.2 $\mathcal{U}^{(1)} \neq U_1$.

In this case, by applying the argument as in the proof of Claim 1 of Case 1.2 in Section 3.1, we can show that $\mathcal{U}^{(1)}$ is complete with respect to the induced Riemannian metric, which implies that $\mathcal{U}^{(1)} \cong L \times [0, 1]$ with each $x \times [0, 1]$ is contained in a leaf of $\mathcal{F}^\perp$. Let $\partial \mathcal{U}^{(1)} = L_- \cup L_+$. We modify $\mathcal{F}^1$ by replacing $\mathcal{F}^1|_{\mathcal{U}^{(1)}}$ with the image of the product foliation on $L_- \times [0, 1]$.

Case 2 is divided into the following two subcases.

Case 2.1 There exists an immersion $h : L \times [0, 1] \to U_1 \setminus L$ such that the image of each $x \times [0, 1]$ is contained in a leaf of $\mathcal{F}^\perp$.

In this case, we replace $\mathcal{F}^1|_{U_1 \setminus L}$ by the image of the product foliation on $L \times [0, 1]$. Note that in this case, the modification on $U_1$ is completed.

Case 2.2 There does not exist an immersion $h$ as in Case 2.1.

In this case $\mathcal{F}^1$ is unchanged by the modification.

In Cases 1.2 and 2.2, we further modify the foliation for another equivalence class of a depth 1 leaf in $U_1$, and repeat the procedure to modify the foliation in $U_1$. Then the desired foliation $\mathcal{F}^2$ is obtained by repeating the procedure for all $U_i$’s.
Note that this modification does not change the transverse foliation $\mathcal{F}^\perp$, i.e., $\mathcal{F}^{2\perp} = \mathcal{F}^\perp$.

### 3.3 Completion of modification

In this subsection, we apply the similar modifications for leaves at higher depth to give a means of completing the modification.

**Lemma 3.3.1** For the modified foliation $\mathcal{F}^2$, the number of equivalence classes represented by the depth 2 leaves is finite.

**Proof.** Let $\{L^{(2)}_i\}$ be a set of depth 2 leaves of $\mathcal{F}^2$ such that each pair of elements is not mutually equivalent. We assume that $\{L^{(2)}_i\}$ has infinitely many elements. By slightly modifying an $(\mathcal{F}, \mathcal{F}^\perp)$-coordinate atlas if necessary, we may suppose that $(\bigcup L^{(2)}_i) \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. Hence we can find $U$, a component of $M \setminus \bigcup L^{(2)}_i$ such that $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$. For any point $x$ in $\partial \tilde{U}$, let $\tilde{\tau}_x$ be the leaf of $\tilde{\mathcal{F}}^2$ which meets $x$. Since $U \cap (\bigcup_{i=1}^m \varphi_i(D^{n-1} \times \partial[0,1])) = \emptyset$, $\tilde{\tau}_x$ is a proper subarc of $\varphi_i(c \times [0,1])$ for some $i$ and $c \in D^{n-1}$. Hence $\tilde{\tau}_x$ is an arc properly embedded in $\tilde{U}$ with endpoints $x$ and $y$, say. Since $\mathcal{F}^2$ is transversely oriented, $x$ and $y$ are contained in different components of $\partial \tilde{U}$. Let $F_x$ ($F_y$ resp.) be a leaf of $\mathcal{F}^2$ which meets $x$ ($y$ resp.).

We immediately have the following.

**Claim 1** $\tilde{U}$ is homeomorphic to $F_x \times [0,1]$, where each $\{p\} \times [0,1]$ is contained in a leaf of $\mathcal{F}^\perp$.

By applying the arguments as in the proof of Claim 2 in Section 3.2, we have the following claim.

**Claim 2** If $F_x \neq F_y$, then $\mathcal{F}^2|_U$ is a product foliation with depth 0 leaves or depth 1 leaves.

Suppose $F_x = F_y$. Let $\tilde{U} = U \cup F_x$. By applying the arguments as in the proof of Claim 3 in Section 3.2, we have the following.

**Claim 3** Under the above conditions, we have the following:

1. $F_x$ is a depth 2 leaf;
2. $\partial \tilde{U} = \overline{F_x} \setminus F_x$; and
3. $F_x$ is the only element of $\{L^{(2)}_i\}$ which meets $\tilde{U}$.  

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**Remark 3.3.1** By 1 and 2 of the above Claim 3, we see that each component of \( \partial \tilde{U} \) is a depth 0 leaf or a depth 1 leaf.

We know that the number of the equivalence classes of the depth 0 leaves and depth 1 leaves of \( \mathcal{F}^1 \) are finite (Lemmas 3.1.1 and 3.2.1). Since the modification in Section 3.2 does not change the number of the equivalence classes represented by the depth 0 or depth 1 leaves, the number of the equivalence classes represented by the depth 0 or depth 1 leaves of \( \mathcal{F}^2 \) is also finite. This fact together with the above Claim 2, 3 of the above Claim 3, and Remark 3.3.1 imply that \( \{L^{(2)}_\ell\} \) consists of finitely many elements, a contradiction.

Then we can further apply the similar modifications for higher depth leaves to obtain a modified foliation \( \tilde{\mathcal{F}} \) that cannot be modified any more. Note that the modification for depth \( d \) leaves does not affect the leaves at depth less than \( d \). Note that if \( \mathcal{F}^\perp \) is fixed, then for each \( i \), \( \mathcal{F}^i \) is unique. Hence \( \tilde{\mathcal{F}} \) is unique. And by the construction, it is easy to show the following.

**Fact 3.3.1** \( \text{depth}(\tilde{\mathcal{F}}) \leq \text{depth}(\mathcal{F}) \).

Moreover, by applying the argument as in the proof of Lemma 3.3.1, we can show the following.

**Proposition 3.3.1** The number of the equivalence classes of the leaves of \( \tilde{\mathcal{F}} \) is finite.

Here, we note that the modified foliation \( \tilde{\mathcal{F}} \) satisfies the conditions soon after Remark 2.1.4 in Section 2.1.2.
4 Proof of Theorems

4.1 Structure of the manifold $M_{S_q}$

Let $K_q(= h_q(C))$ be a $q$-twisted double of $K'$, and $S_q(= h_q(S))$, $A_q(= h_q(A))$ as in Section 2.2.2. Recall that $(X_{S_q}, \gamma_{S_q})$ is a complementary sutured manifold of $S_q$. Let $M_{S_q}$ be the manifold obtained from $X_{S_q}$ by attaching a 2-handle along $\gamma_{S_q}$ (see Section 2.2.4) with the orientation inherited from $X_{S_q}$. We further suppose that each component of $\partial M_{S_q}$ is equipped with the orientation inherited from $R(\gamma_{S_q})$. Since $S_q$ is a genus one surface, the boundary of $M_{S_q}$ consists of two tori. Then $T^+$ denotes the component of $\partial M_{S_q}$ whose normal vectors point outward, and $T^-$ the other.

Let $L_q$ be the link $\partial A_q$ equipped with the orientation such that each component of $L_q$ is parallel to $K'$, hence the linking number of $L_q$ is $q$.

**Proposition 4.1.1** $M_{S_q}$ is homeomorphic to $E(L_q)$.

**Proof.** Recall that $S_q$ is obtained by plumbing $A_q$ and a Hopf annulus (see Section 2.2.2). Note that a part of $S_q$ near the Hopf annulus looks as in Figure 4.1(a). The part of $(X_{S_q}, \gamma_{S_q})$ corresponding to the part of $S_q$ in Figure 4.1(a),

looks as in Figure 4.1(b). Let $D'$ be the product disk for $(X_{S_q}, \gamma_{S_q})$ as in Figure 4.1(c). Recall that $M_{S_q}$ is obtained from $X_{S_q}$ and $D^2 \times [0,1]$ by identifying $\gamma_{S_q}$ and $\partial D^2 \times [0,1]$. Hence we may regard that $D'$ is a subspace of $M_{S_q}$. Since $D'$ is a product disk for $(X_{S_q}, \gamma_{S_q})$, $D' \cap \gamma_{S_q} (= D' \cap (\partial D^2 \times [0,1]))$ consists of two
arcs in $\partial D'$, say $a_1, a_2$. Let $D''$ be a product disk for $(D^2 \times [0, 1], \partial D^2 \times [0, 1])$ such that $D'' \cap (\partial D^2 \times [0, 1]) = a_1 \cup a_2$. Let $A_M = D' \cup D''$. Note that $A_M$ is an annulus properly embedded in $M_{S_q}$ such that a component of $\partial A_M$ is contained in $T^+$ and the other in $T^-$ (see Figure 4.2).

![Figure 4.2](image)

Now, let $N$ be the manifold which is obtained from $M_{S_q}$ by cutting along $A_M$ and $A_{M,1}, A_{M,2}$ the copies of $A_M$ in $\partial N$ (see Figure 4.3(a)).

![Figure 4.3](image)

Note that $(N, A_{M,1} \cup A_{M,2})$ naturally inherits a sutured manifold structure.
from \((X_{S_q}, \gamma_{S_q})\), i.e., \(R_{\pm}(A_{M,1} \cup A_{M,2})\) is the image of \(T^\pm\).

Let \((\tilde{X}_{S_q}, \tilde{\gamma}_{S_q})\) be the sutured manifold obtained from \((X_{S_q}, \gamma_{S_q})\) by the product decomposition along \(D'\) (see Figure 4.3(b)). By Figure 4.3, we have the following.

**Claim** The manifold pair \((N, A_{M,1} \cup A_{M,2})\) is homeomorphic to \((\tilde{X}_{S_q}, \tilde{\gamma}_{S_q})\).

Since \(M_{S_q}\) is retrieved from \(N\) by identifying \(A_{M,1}\) and \(A_{M,2}\), this claim implies that \(M_{S_q}\) is obtained from \(\tilde{X}_{S_q}\) by identifying the components of \(\tilde{\gamma}_{S_q}\) and that the image of \(\tilde{\gamma}_{S_q}\) is \(A_{M}\). It is directly observed from Figure 4.4 that \(\tilde{X}_{S_q}\) is homeomorphic to \(E(K')\), and the union of core curves of \(\tilde{\gamma}_{S_q}\) is equivalent to the link \(L_q\). Here we note that one component of \(\partial A_{M}\) is contained in \(T^+\) and the other is in \(T^-\), and that \(M_{S_q}\) is orientable. Hence the identification has to be as in Figure 4.5. These show that \(M_{S_q}\) is homeomorphic to \(E(L_q)\).

We describe two facts that will be required for the proof of Theorem 1.0.1. Let \(A_M\) be as in the proof of Proposition 4.1.1.

**Fact 4.1.1** Let \(T'\) be the component of \(\partial N(\partial M_{S_q} \cup A_M, M_{S_q})\) which is contained in \(\text{int}M_{S_q}\) (Figure 4.6). By Proposition 4.1.1, we may regard \(T'\) is contained in \(E(L_q)\). Moreover it is directly observed that the closure of a component of \(E(L_q) \setminus T'\) is homeomorphic to \(E(K')\) and the closure of the other, say \(Q\), is homeomorphic to \((\text{disk with two holes}) \times S^1\).

For the statement of the second fact, we prepare some notations. By Proposition 4.1.1, we can consider \(T^+, T^-\) as boundary components of \(E(L_q)\). Let
\( \ell^\pm = T^\pm \cap A_M \). Note that \( \ell^\pm \) is isotopic to a \( S^1 \)-fiber of \( Q \). Recall that \( M_{S_q} = X_{S_q} \cup (D^2 \times [0,1]) \) and \( T^+ = R^+(\gamma_{S_q}) \cup (D^2 \times \{1\}) \), \( T^- = R^-(\gamma_{S_q}) \cup (D^2 \times \{0\}) \).

By deforming \( \ell^\pm \) by an ambient isotopy, if necessary, we may suppose that \( \ell^\pm \subset R^\pm(\gamma_{S_q}) \). Recall that \( (X_{S_q}, \gamma_{S_q}) \) is the complementary sutured manifold of \( S_q \), hence \( R^\pm(\gamma_{S_q}) \) is homeomorphic to \( S_q \). Then,

\[
p^\pm : R^\pm(\gamma_{S_q}) \longrightarrow S_q
\]

denotes the natural homeomorphism. By Figure 4.5(b), \( \ell^+ \cup \ell^- \) looks as in Figure 4.6. By tracing the deformations of Figure 4.4 conversely together with \( \ell^+ \cup \ell^- \), we obtain Figure 4.7. This observation implies the following.

**Fact 4.1.2** Under the above notations, \( p^+(\ell^+) \) and \( p^-(\ell^-) \) are isotopic on \( S_q \) to loops which meet transversely in one point.
4.2 Foliations on $M^{(n)}$

Let $K$ be a $q$-twisted double of $K'$ (in this subsection, we basically follow the notations in Section 4.1, but we use $K, S, L, M_S$ for $K_q, S_q, L_q, M_{S_q}$ for simplicity). Let $S^3(K,0)$ be the manifold obtained from $S^3$ by performing 0-surgery on $K$. Note that $H_1(S^3(K,0);\mathbb{Z}) \cong \mathbb{Z}$. Let $\Sigma^{(n)}(K,0)$ be the $n$-fold cyclic covering space of $S^3(K,0)$ (see Section 1). Note that $\Sigma^{(n)}(K,0)$ admits a decomposition $\Sigma^{(n)}(K,0) = M_1 \cup \cdots \cup M_n$ where each $M_i$ ($i = 1, \ldots, n$) is homeomorphic to $M_S$, and $M_1, \ldots, M_n$ are arrayed cyclically i.e., $M_i \cap M_{i+1} = \partial M_i \cap \partial M_{i+1}$ consists of a torus $T_i$ (if $n > 2$) (subscript is taken in mod $n$) or $M_1 \cap M_2 = \partial M_1 = \partial M_2$ ($= T_1 \cup T_2$, say)(if $n = 2$). Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K,0)$ by cutting along $T_n$. In this subsection, for the proof of Theorem 1.0.1, we study depth of foliations on $M^{(n)}$ the union of the depth 0 leaves of each of which coincides with $\partial M^{(n)}$.

We may regard $M^{(n)} = M_1 \cup_{T_1} \cdots \cup_{T_{n-1}} M_n$. Then, we abuse notation $T_n$ for denoting the component of $\partial M^{(n)}$ such that $T_n \subset M_n$ and $T_0$ denotes the other component of $\partial M^{(n)}$. By Proposition 4.1.1, each $M_i$ is homeomorphic to $E(L)$. Let $T'_i$ be the torus in $M_i$ corresponding to $T'$ in Fact 4.1.1. Hence the closure of a component of $M_i \setminus T'_i$, say $E(K')_i$, is homeomorphic to $E(K')$ and the closure of the other component of $M_i \setminus T'_i$, say $Q_i$, is homeomorphic to $(\text{disk with two holes}) \times S^1$. Let $\ell^+_j$ ($\ell^-_j$ resp.) ($j = 1, \ldots, n-1$) be a simple closed curve in $T_j$ which is isotopic to an $S^1$-fiber of $Q_j$ ($Q_{j+1}$ resp.). By Fact 4.1.2, we obtain the following.

**Lemma 4.2.1** Under the above notations, $\ell^+_j$ and $\ell^-_j$ are isotopic to loops on $T_j$
which meet transversely in one point.

4.2.1 Foliation $\mathcal{F}'$ on $M^{(n)'}$

Let $\mathbb{T} = (\bigcup_{j=1}^{n-1} T_j) \cup (\bigcup_{i=1}^{n} T_i')$. Let $\mathcal{F}$ be a codimension one, transversely oriented taut $C^0$ foliation of finite depth on $M^{(n)}$ such that the union of the depth 0 leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$.

**Lemma 4.2.2** By deforming $\mathbb{T}$ by an ambient isotopy in $M^{(n)}$, if necessary, we may suppose $\mathbb{T}$ is transverse to $\mathcal{F}$.

**Proof.** Since $\mathcal{F}$ is taut, Theorem 4 of [17] implies that by ambient isotopy in $M^{(n)}$, we can deform $\mathbb{T}$ so that either $\mathbb{T}$ is transverse to $\mathcal{F}$ or there exists $\mathbb{T}$, a component of $\mathbb{T}$ such that $\mathbb{T}$ is a leaf of $\mathcal{F}$. However by the assumption that the union of the depth 0 leaves coincides with $\partial M^{(n)}$, we see that there does not exist such $\mathbb{T}$ as above. Hence we may suppose $\mathbb{T}$ is transverse to $\mathcal{F}$. This completes the proof of the lemma. □

Let $\hat{\mathbb{T}}_0, \hat{\mathbb{T}}_n$ be tori in int $M^{(n)}$ such that $\hat{\mathbb{T}}_0$ (resp.) is parallel to $T_0$ ($T_n$ resp.), and $\hat{\mathbb{T}}_0 \cap \mathbb{T} = \emptyset$ ($\hat{\mathbb{T}}_n \cap \mathbb{T} = \emptyset$ resp.). By applying the arguments as in the proof of Lemma 4.2.2, we can assume that $\hat{\mathbb{T}}_0 \cup \hat{\mathbb{T}}_n \cup \mathbb{T}$ is transverse to $\mathcal{F}$. Let $M^{(n)'}$ be the closure of the component of $M^{(n)} \setminus (\hat{\mathbb{T}}_0 \cup \hat{\mathbb{T}}_n)$ which does not meet $\partial M^{(n)}$. Note that $M^{(n)'}$ is homeomorphic to $M^{(n)}$. Let $\mathcal{F}' = \mathcal{F} |_{M^{(n)'}}$.

**Lemma 4.2.3** $\mathcal{F}'$ is of finite depth and for each leaf $L'$ of $\mathcal{F}'$ we have: if $L$ is the leaf of $\mathcal{F}$ such that $L \supseteq L'$, then depth($L$) > depth($L'$). In particular, depth($\mathcal{F}'$) ≤ depth($\mathcal{F}$) − 1.

For the proof of the lemma, for each $i$ ($0 \leq i \leq \text{depth}(\mathcal{F})$), we show the following inductively. Note that Assertion (depth($\mathcal{F}$)) gives the lemma.

**Assertion** ($i$) Let $L$ be a leaf of $\mathcal{F}$ such that depth($L$) = $i$. Then, either one of the following holds:

1. $L \cap M^{(n)'} = \emptyset$; or

2. for any component $F$ of $L \cap M^{(n)'}$, we can define depth($F$) (note that depth($F$) denotes the depth of $F$ as a leaf of $\mathcal{F}'$), and depth($F$) ≤ $i - 1$, i.e., $\mathcal{F} \setminus F$ is a union of leaves $F_\alpha$ such that depth($F_\alpha$) ≤ $i - 2$. 

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Proof of Assertion (i). Suppose \( i = 0 \). Then, clearly \( L \cap M^{(n)_i} = \emptyset \). Suppose for \( i \leq k \ (0 \leq k \leq \text{depth}(\mathcal{F}) - 1) \), Assertion (i) holds. Consider the case \( i = k + 1 \). If \( L \cap M^{(n)_i} = \emptyset \), then 1 holds. Suppose \( L \cap M^{(n)_i} \neq \emptyset \). Let \( F \) be a component of \( L \cap M^{(n)_i} \). Recall that \( \mathcal{L} \setminus L \) is a union of leaves \( L_\beta \) such that \( \text{depth}(L_\beta) \leq k \).

Claim 1 Suppose \( (\bigcup L_\beta) \cap M^{(n)_i} = \emptyset \). Then \( L \cap M^{(n)_i} \) is compact.

Proof of Claim 1. Assume \( L \cap M^{(n)_i} \) is not compact. Then there exists a sequence of points in \( L \cap M^{(n)_i} \), say \( \{x_i\}_{i=1,2,...} \) with an accumulating point \( x \) such that \( x \notin L \cap M^{(n)_i} \). Let \( L_x \) be the leaf of \( \mathcal{F}' \) which contains \( x \). Then we have \( L_x \cap (L \cap M^{(n)_i} \setminus (L \cap M^{(n)_i})) \neq \emptyset \), thus \( L_x \subset L \cap M^{(n)_i} \setminus (L \cap M^{(n)_i}) \), contradicting the assumption that \( (\bigcup L_\beta) \cap M^{(n)_i} = \emptyset \).

Claim 2 Suppose \( L \cap M^{(n)_i} \) is compact. Then each component of \( L \cap M^{(n)_i} \) is compact.

Proof of Claim 2. Assume there is a component \( F' \) of \( L \cap M^{(n)_i} \) which is not compact. Then there exists a sequence of points in \( F' \), say \( \{x_i\}_{i=1,2,...} \) with an accumulating point \( x \) such that \( x \notin F' \). On the other hand, since \( L \cap M^{(n)_i} \) is compact, \( x \in L \cap M^{(n)_i} \). This contradicts the fact that \( L \) is proper (Remark 2.1.3).

By the above Claims 1 and 2, we see if for each \( \beta \), \( L_\beta \cap M^{(n)_i} = \emptyset \), then \( \text{depth}(F) = 0 \leq (k + 1) - 1 \), that is, the conclusion 2 of Assertion \((k + 1)\) holds. Suppose \( (\bigcup L_\beta) \cap M^{(n)_i} \neq \emptyset \). Since Assertion (i) holds for \( i \leq k \), for each \( \beta \) either one of the following holds:

(a) \( L_\beta \cap M^{(n)_i} = \emptyset \); or
(b) for any component \( F_{\beta,\gamma} \) of \( L_\beta \cap M^{(n)_i} \), \( \text{depth}(F_{\beta,\gamma}) \leq k - 1 \).

Let \( \{L_\beta\} \) be the set of elements of \( \{L_\beta\} \) intersecting \( M^{(n)_i} \). Recall that \( \mathcal{L} \setminus L = \bigcup L_\beta \), and \( L_\beta \cap M^{(n)_i} = L_\gamma F_{\beta,\gamma} \). This implies that \( (\mathcal{L} \setminus L) \cap M^{(n)_i} = \bigcup_{\gamma} \bigcup_{\gamma} F_{\beta,\gamma} \).

Claim 3 \( \mathcal{F} \setminus F \subset (\mathcal{L} \setminus L) \).

Proof of Claim 3. Since \( \mathcal{F} \setminus F \subset \mathcal{L} \) is clear, we show that \( (\mathcal{F} \setminus F) \cap L = \emptyset \). Asssume that \( (\mathcal{F} \setminus F) \cap L \neq \emptyset \). Let \( x \) be a point in \( (\mathcal{F} \setminus F) \cap L \). Then there exists a sequence of points in \( F \), say \( \{x_i\}_{i=1,2,...} \) with an accumulating point \( x \) such that \( x \notin F \) and \( x \in L \). However this contradicts the fact that \( L \) is proper (Remark 2.1.3).
Hence $\overline{F} \setminus F \subset \bigcup_{\gamma} (\bigcup_{\gamma} F_{g_{\gamma}})$. Hence by (b), depth$(F) \leq k$, this completes the proof of Assertion $(k + 1)$.

Since each component of $M^{(n)} \setminus T$ naturally corresponds to each component of $M^{(n)} \setminus T$, in the remainder of this subsection, we use the notation that $M^{(n)} = M_1 \cup T_1 \cdots \cup T_{n−1} M_n = (E(K')_1 \cup T_1 Q_1) \cup T_1 \cdots \cup T_{n−1} (E(K')_n \cup T_n Q_n)$. Let $\overline{T}$ be a union of tori which is ambient isotopic to $T$ and transverse to $\mathcal{F}$. For such $\overline{T}$, let $\overline{T}_i$ be the component of $\overline{T}$ corresponding to $T_i$, and $\overline{M}_i$ the closure of the component of $M^{(n)} \setminus \bigcup_{i=1}^{n−1} \overline{T}_i$ corresponding to $M_i$ $(1 \leq i \leq n)$. Let $d_i(\overline{T})$ be the minimal value of the depths of the leaves of $\mathcal{F}'$ which meet $\overline{M}_i$.

Let $L_i$ be a leaf of $\mathcal{F}'$ which meets $M_i$ such that depth$(L_i) = d_i(\overline{T})$ and $L_i^* = L_i \cap M_i$.

Lemma 4.2.4 $L_i^*$ is compact and incompressible in $M_i$.

**Proof.** Assume $L_i^*$ is not compact. Then there exists a sequence of points in $L_i^*$, say $\{x_i\}_{i=1,2,\ldots}$ with an accumulating point $x$ such that $x \notin L_i^*$. Let $L_x$ be the leaf of $\mathcal{F}'$ which contains $x$. Then we have $L_x \cap (\overline{L}_i^*) \neq \emptyset$, thus $L_x \cap M_i \subset (\overline{L}_i^*)$. By the definition of depth of leaves, the depth of $L_x$ is less than $d_i(\overline{T})$ and this contradicts the definition of $d_i(\overline{T})$. Thus $L_i^*$ is compact. Since $\mathcal{F}'$ is transverse to $\overline{T}$, each component of $\partial L_i^*$ is essential in $\cup T_i$. Suppose there is a compression disk $D$ for $L_i^*$ in $M_i$. Since $\mathcal{F}$ is taut, the leaf of $\mathcal{F}$ containing $L_i$ (say, $\hat{L}_i$) is incompressible (Remark 2.2.1). Hence $\partial D$ is inessential in $\hat{L}_i$, i.e., there is a disk $D'$ in $\hat{L}_i$ such that $\partial D' = \partial D$. Since $D$ is a compression disk for $L_i^*$, $D'$ is not contained in $M_i$, hence $D \cap (\cup T_i) \neq \emptyset$. Let $\ell$ be a component of $D' \cap (\cup T_i)$ which is innermost in $D'$. Let $\Delta(\subset D')$ be the disk bounded by $\ell$. Since $\ell$ is essential in $\cup T_i$, $\Delta$ is a compression disk for $U T_i$, a contradiction. \qed

4.2.2 Putting $T$ in a nice position (with assuming $K'$ a non-cable knot)

In the remainder of this subsection, we suppose $K'$ is a non-cable knot. Let $\mathcal{F}$ be a codimension one, transversely oriented, taut $C^0$ foliation of finite depth on $M^{(n)}$ such that the union of the depth 0 leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$. Let $\overline{T}$, $\overline{T}_0$, $\overline{T}_n$ be as in Section 4.2.1. We suppose that $\overline{T}_0 \cup \overline{T}_n \cup \overline{T}$ is isotoped as in the paragraph preceding Lemma 4.2.3 (hence, $M^{(n)}, \mathcal{F}'$ are defined). Recall from Section 4.2.1 that we use the notation $M^{(n)} = M_1 \cup T_1 \cdots \cup T_{n−1} M_n = (E(K')_1 \cup T_1 Q_1) \cup T_1 \cdots \cup T_{n−1} (E(K')_n \cup T_n Q_n)$. 

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Lemma 4.2.5 By deforming $T'_k$ by an ambient isotopy, we may suppose that $\mathcal{F}'|_{E(K'_k)}$ has no annular leaves for each $i$ ($1 \leq i \leq n$).

Proof. We first note that the basic idea of the following proof is exactly the same as that of Lemma 2.3 of [2]. The key of our proof is to remove the assumption that the considered foliation is of class $C^2$ which is required in Lemma 2.3 of [2].

Assume there exists $k$ ($1 \leq k \leq n$) such that $\mathcal{F}'|_{E(K'_k)}$ has an annular leaf, say $A$. By the proof of Lemma 2.3 of [2], we see that $\partial A \subset T'_k$ is a pair of essential simple closed curves which devides $T'_k$ into two annuli, $A'$, $A''$, where either $A \cup A'$ or $A \cup A''$ is isotopic to $T'_k$ in $E(K'_k)$. Now, assume that $T_A = A \cup A'$ is isotopic to $T'_k$. Let $E_A$ be a component of $E(K'_k) \setminus A$. We say that $E_A$ is outside if $\partial E_A = T_A$ and that $E_A$ is inside otherwise. If there exists another annular leaf $B$ in $\mathcal{F}'|_{E(K'_k)}$, we say $A < B$ if $A$ is contained in the inside of $B$, or $A \leq B$ if we include the case $A = B$. Now, suppose $A < B$. Let $G$ be the closure of the component of $E(K'_k) \setminus (A \cup B)$ which is between $A$ and $B$. Then $G$ is homeomorphic to $A \times [0, 1]$, where $A \times \{0\} = A$, $A \times \{1\} = B$ and $\partial A \times [0, 1]$ corresponds to $G \cap T'_k$. In this case, we can think of $B$ as “concentric” with $A$ and pushed out from $A$. Now, we consider the set of annular leaves $B$ such that $A \leq B$. By using the above observation that $A$ and $B$ are concentric, it is easy to show that this set is linearly ordered by $<$ and the union of these leaves is a closed set with an outermost annular leaf $A^\infty$. (That is, $A^\infty$ is maximal with the property that $A \leq A^\infty$.) Let $A^\infty \subset T'_k$ be the annulus such that $T^\infty = A^\infty \cup A'^\infty$ is isotopic to $T'_k$.

Claim There exists a torus $\mathcal{T}$ in outside of $A^\infty$ which is transverse to $\mathcal{F}'$ and isotopic to $T^\infty$.

Proof of Claim. Remark that $A^\infty$ is an essential annulus in $T^\infty$ (Figure 4.8). Fix a point $p_0 \in A^\infty$. Let $\Sigma_0$ be a transverse section to $A^\infty$ at $p_0$, $\Sigma_0^p$ be the closure of the component of $\Sigma_0 \setminus p_0$ such that $\Sigma_0^p$ is contained in the outside of $A^\infty$. Let $\alpha' : [0, 1] \to A^\infty$ be a simple closed curve representing a generator of $\pi_1(A^\infty, p_0) \cong \mathbb{Z}$. By using the argument as in the proof of Lemma 2 of [15], we see that there is a map $F : [0, 1] \times [0, 1] \to E(K'_k)$ such that $F|_{[0,1] \times \{0,1\}}$ is an embedding, $F(\{0\} \times [0, 1]) \subset \Sigma_0^p$, $F(\{1\} \times [0, 1]) \subset \Sigma_0^p$, $F|_{[0,1] \times \{0,1\}} = \alpha'$, and $F$ is transverse to $\mathcal{F}'|_{E(K'_k)}$. This together with Theorem 2 of [15] shows that there is a homeomorphism $\Phi : \pi_1(A^\infty, p_0) \to G(\Sigma_0, p_0)$, where $G(\Sigma_0, p_0)$ is the group of germs of $C^0$-homeomorphism of $\Sigma_0$ which leaves $p_0$ fixed. Hence $\Phi(\pi_1(A^\infty, p_0))$ is the holonomy group of $A^\infty$ at $p_0$. 37
If \( \Phi(\pi_1(\mathcal{A}^\infty, p_0)) \) is trivial on \( \Sigma_0^o \), then there exist annular leaves outside of \( \mathcal{A}^\infty \), this contradicts the maximality of \( \mathcal{A}^\infty \). By taking \( \alpha'^{-1} \) instead of \( \alpha' \), if necessary, we may assume \( \Phi(\alpha') \) is contracting on \( \Sigma_0^o \), i.e., for each point \( x \in F([0,1] \times (0,1)) \), if \( x \) proceeds along the leaf of \( F([0,1] \times [0,1]) \cap \mathcal{F}' \) in the \( \alpha' \)-direction, then \( x \) approaches \( \mathcal{A}^\infty \) (Figure 4.9).

Let \( b_1 \) be an essential arc properly embedded in \( \mathcal{A}^\infty \) such that \( p_0 \in b_1 \). By using the argument of the proof of Lemma 2 of [15], we can show that there is an embedding \( F' : [0,1] \times [0,1] \to \mathcal{E}(K')_k \) such that \( F'([0] \times [0,1]) \subset T'_k \), \( F'([1] \times [0,1]) \subset T'_k \), \( F'([\frac{1}{2}] \times [0,1]) = \Sigma_0^o \), \( F'([0,1] \times [0]) = b_1 \), \( F'([0,1] \times \{1\}) \) is contained in a leaf, and \( F' \) is transverse to \( \mathcal{F}' \). Let \( R = F'([0,1] \times [0,1]) \).

Note that \( \mathcal{F}'|_R \) is isomorphic to the product foliation on \( [0,1] \times [0,1] \), i.e., the foliation on \( [0,1] \times [0,1] \) with each leaf being \( \{\ast\} \times [0,1] \). Let \( b_2 = F'([0,1] \times \{1\}) \), \( p_1 = F'([0] \times \{0\}) \), \( p_2 = F'([1] \times \{0\}) \), \( q_1 = F'([0] \times \{1\}) \), \( q_2 = F'([1] \times \{1\}) \), \( a_1 = F'([0] \times [0,1]) \), \( a_2 = F'([1] \times [0,1]) \). Note that \( p_i, q_i \) are points on \( T'_k \) (Figure 4.10).

Let \( L_{b_2} \) be the leaf of \( \mathcal{F}' \) which contains \( b_2 \). Let \( \ell_i \) be the arc in \( T^\infty \cap L_{b_2} \) with \( \partial \ell_i = \ell_i \cap a_i = q_i \cup r_i \) (where \( r_i \) is a point in \( \text{int } a_i \), \( b'_2 \) the component of \( L_{b_2} \cap R \) such that \( \partial b'_2 = r_1 \cup r_2 \) (Figure 4.11). Note that \( b'_2 \) is an arc properly embedded in \( R \) and by [15, Theorem 2], we see that \( b_2 \cup (\ell_1 \cup \ell_2) \cup b'_2 \) bounds a
rectangle $\tilde{A}_1$ in $L_{b_2}$. Let $a'_i$ be the subarc in $a_i$ with $\partial a'_i = p_i \cup r$, $a''_i$ the subarc in $a_i$ with $\partial a''_i = r_i \cup q_i$, and $R'$ the rectangle in $R$ with edges $a''_1, b'_2, a''_2, b_2$ (Figure 4.11). Then, $\ell_i \cup a''_i$ is a simple closed curve in $T^\infty$ disjoint from $\partial A^\infty$. Let $\ell_1^\infty, \ell_2^\infty$ be the components of $\partial A^\infty$ such that $p_i \in \ell_i^\infty$ (Figure 4.12). Let $B_1, B_2$ be mutually disjoint annuli in $T^\infty$ such that $\partial B_i = (\ell_i \cup a''_i) \cup \ell_i^\infty$ (Figure 4.12). Then $A^\infty \cup (B_1 \cup B_2) \cup \tilde{A}_1 \cup R'$ bounds a solid torus $\mathcal{V}$ in $E(K'_k)$ (Figure 4.11 and Figure 4.12).
Figure 4.12

Let $A^\infty = A^\infty \setminus b_1$. Let $\overline{A^\infty}$ be the metric completion of $A^\infty$. Note that $\overline{A^\infty}$ is obtained from $A^\infty$ by adding two edges, say $b_1^+, b_1^-$, each corresponding to $b_1$, where there is a simple closed curve in $A^\infty$ representing $\alpha'$ in $\pi_1(A^\infty, p_0)$ such that the image of the simple closed curve is an oriented arc in $A^\infty$ which goes from $b_1^+$ to $b_1^-$ (Figure 4.13).

Figure 4.13

Let $f : b_1^- \rightarrow b_1^+$ be the natural homeomorphism and $g : [0,1] \rightarrow [0,1]$ the local homeomorphism induced from the holonomy map corresponding to $\Phi([\alpha'])|_{x_0}$. Since $\overline{A^\infty}$ is simply connected, by [15, Theorem 2], we see that the foliated manifold $(\mathcal{V}, \mathcal{F}|_{x})$ is isomorphic to $(\overline{A^\infty} \times [0,1]/\sim, \mathcal{F}/\sim)$, where $\mathcal{F}$ is a product foliation on $\overline{A^\infty} \times [0,1]$ and the equivalence relation $\sim$ is defined as follows; for $x \in b_1^-, y \in [0,1]$, $(x, y) \sim (f(x), g(y))$. 40
Let \( \tilde{A}^* \) be the rectangle in \( \tilde{A}^\infty \times [0, 1] \) corresponding to the flat rectangle properly embedded in \( \tilde{A}^\infty \times [0, 1] \) (\( \cong [0, 1]^2 \times [0, 1] \)) such that \( ([0, 1] \times \{ \{0\} \} \times \{ g(1) \} \) \( (= b_1^i \times \{ g(1) \}) \) and \( ([0, 1] \times \{ \{1\} \} \times \{ \{1\} \} \) \( (= b_1^i \times \{ {1}\}) \) are edges of \( \tilde{A}^* \) (Figure 4.14). Note that \( \tilde{A}^* \) is transverse to \( \tilde{F}/\sim \). Let \( A^* \) be the annulus in \( V \) corresponding to \( \tilde{A}^*/\sim \). Note that one component of \( \partial A^* \) is contained in \( B_1 \), and the other is contained in \( B_2 \) and \( A^* \) is transverse to \( F' \). Let \( A'^* \) be the closure of the component of \( T^\infty \setminus \partial A^* \) which is contained in \( A^\infty \). Let \( \mathcal{T} = A^* \cup A'^* \). Then \( \mathcal{T} \) is transverse to \( F' \). Further, \( \mathcal{T} \) is ambient isotopic to \( T_k' \) since \( V \) is a solid torus and \( T^\infty \) is isotopic to \( T_k' \). This completes the proof of the claim. □

![Figure 4.14](image)

Let \( \mathcal{T} \) be as in the proof of Claim. By taking \( \mathcal{T} \) instead of \( T^\infty \), we can remove such annuli \( B(\geq A) \).

Assume there exist infinitely many pairs of annular leaves \( A^{\infty, 1}, A^{\infty, 2}, \ldots \) which are maximal. This implies that there exist infinitely many simple closed curves \( \partial A^{\infty, 1}, \partial A^{\infty, 2}, \ldots \). Then, the distances of each pair of simple closed curves will be close to 0, this contradicts the condition that leaves are transversely oriented. Thus, by repeating the operation finitely many times, we can remove all annular leaves. This completes the proof of the lemma. □

In what follows, we assume that the considered foliation satisfies the condition of Lemma 4.2.5.

**Lemma 4.2.6** For each \( i \), \( F'|_{E(K_i')}, \) is taut.

**Proof.** Let \( L \) be a leaf of \( F'|_{E(K_i')}, \). Let \( (2E(K_i'), 2\mathcal{F}|_{E(K_i')}) \) be a double of \( (E(K_i'), \mathcal{F}|_{E(K_i')}) \) along \( T_i' \), i.e., \( 2E(K_i') = E(K_i')^+ \cup E(K_i')^- \), where \( E(K_i')^\pm \) is a
copy of $E(K')_i$ and $2E(K')_i$ is obtained from $E(K')_i^+$ and $E(K')_i^-$ by identifying their boundaries by the natural homeomorphism. Then $2\mathcal{F}'|_{E(K')_i}$ is the foliation on $2E(K')_i$ which is the image of the foliations on $E(K')_i^+$ and $E(K')_i^-$ each of which corresponds to $\mathcal{F}'|_{E(K')_i}$.

By Lemma 4.2.5, we see that $2\mathcal{F}'|_{E(K')_i}$ has no toral leaves. Then, by using [7, Theorem 2.2], we see that there exists a closed transverse curve $\tau$ in $(2E(K')_i, 2\mathcal{F}'|_{E(K')_i})$ which meets $2L$ in one point, where $2L$ denotes the leaf of $2\mathcal{F}'|_{E(K')_i}$ which is the double of $L$. Without loss of generality, we may assume that the intersection of $2L$ and $\tau$ is contained in $\text{int}E(K')_i$. Let $\varphi$ be the natural involution on $2E(K')_i$, hence $\varphi(E(K')_i^+ \cup E(K')_i^-) = E(K')_i^\pm$. Then let $\tau' = (\tau \cap E(K')_i^+) \cup \varphi(\tau \cap E(K')_i^-)$. Then we deform $\tau'$ slightly in a small neighbourhood of $\partial E(K')_i^+ = \partial E(K')_i^- \subset 2E(K')_i$ so that $\tau' \subset \text{int}E(K')_i^+ \subset 2E(K')_i$ and $\tau'$ is transverse to $2\mathcal{F}'|_{E(K')_i}$. Note that $\tau'$ meets $2L$ in one point which is contained in $E(K')_i^+$. This immediately implies that $\mathcal{F}'|_{E(K')_i}$ admits a closed transverse curve which meets $L$ in one point. Hence $\mathcal{F}'|_{E(K')_i}$ is taut. □

**Lemma 4.2.7** Suppose there is a compact leaf $F_i$ of $\mathcal{F}'|_{E(K')_i}$. Then $F_i \cap \partial E(K')_i \neq \emptyset$ and each component of $\partial F_i (\subset T_i)$ is null-homologous in $E(K')_i$, i.e., corresponding to a longitude of $K'$.

**Proof.** Since the union of compact leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$, it is clear that $F_i \cap \partial E(K')_i \neq \emptyset$. We note that each component of $F_i \cap \partial E(K')_i$ is an essential simple closed curve on $\partial E(K')_i$ (otherwise, $\mathcal{F}'|_{\partial E(K')_i}$ has a singularity). We also note that $F_i$ is orientable and $\mathcal{F}$ is transversely oriented. This implies that for the homology boundary operator $\partial : H_2(E(K')_i, \partial E(K')_i) \rightarrow H_1(\partial E(K')_i)$, $\partial[F_i] = r\alpha$, where $\alpha$ is an indivisible element of $H_1(\partial E(K')_i)$. By Lemma 4.2.6, there exists a closed transverse curve $\sigma_i$ in $E(K')_i$ which meets $F_i$ in one point. Thus the intersection number of $\sigma_i$ and $F_i$ is $\pm 1$. This implies that $[F_i]$ is nonzero and is indivisible in $H_2(E(K')_i, \partial E(K')_i)$. Since $E(K')_i$ is an exterior of a knot, $H_2(E(K')_i, \partial E(K')_i) \cong \mathbb{Z}$ and any generator is the class represented by a Seifert surface for $K'$. Hence $[F_i]$ is a generator of $H_2(E(K')_i, \partial E(K')_i)$ and the homology class is represented by a Seifert surface for $K'$. Thus we see that $r = 1$ and $\alpha$ is the class represented by a longitude of $K'$. These imply that each component of $\partial F_i$ is null-homologous in $E(K')_i$. □
4.2.3 Putting $T$ in a better position with assuming $q = 0$

Recall that $K'$ is a non-cable knot. Let $\mathcal{F}$ be a codimension one, transversely oriented, taut $C^0$ foliation of finite depth on $M^{(n)}$ such that the union of the depth 0 leaves of $\mathcal{F}$ coincides with $\partial M^{(n)}$. Let $T, M^{(n)}, M_i, E(K'), Q_i, \mathcal{F}'$ are as in Section 4.2.2. That is, they satisfy the conditions of Lemmas 4.2.5–4.2.7. For each $i$ ($i = 1, \ldots, n$), let $d_i(T), L_i, L_i^*$ be as in Section 4.2.1. Recall that $Q_i \cong (\text{disk with two holes}) \times S^1$. We say that a surface $S$ in $Q_i$ is vertical if $S$ is ambient isotopic to a surface which is a union of $S^1$-fibers. Note that if $S$ is vertical, then each component of $S$ is either an annulus or a torus. We say that $S$ is horizontal if $S$ is ambient isotopic to a surface which is transverse to the $S^1$-fibers.

In the remainder of this subsection, we suppose that $q = 0$.

**Lemma 4.2.8** For each $i$ ($1 \leq i \leq n$), each component of $L_i^* \cap Q_i$ is a vertical annulus or a $\partial$-parallel annulus.

**Proof.** Let $\tilde{L}_i$ be a component of $L_i^* \cap Q_i$. Since $L_i^*$ is compact (Lemma 4.2.4) and $\mathcal{F}$ is proper (Remark 2.1.3), $\tilde{L}_i$ is compact (Claim 2 in Section 4.2.1). Since $L_i^*$ is incompressible in $M_i$ (Lemma 4.2.4) and each component of $L_i^* \cap T_i'$ is essential in $L_i^*$ (otherwise $T_i'$ is compressible), $\tilde{L}_i$ is incompressible in $Q_i$. Hence by [8, VI.34], $\tilde{L}_i$ is either vertical, horizontal or a $\partial$-parallel annulus. Hence for a proof of the lemma, it is enough to show that $\tilde{L}_i$ is not horizontal. Suppose $\tilde{L}_i$ is horizontal. Then it is clear that $\tilde{L}_i \cap T_i' \neq \emptyset$. Since $L_i^*$ is compact (Lemma 4.2.4) and $\mathcal{F}$ is proper (Remark 2.1.3), each component of $L_i^* \cap E(K')_i$ is compact. Hence by Lemma 4.2.7, each component of $\partial(L_i^* \cap E(K')_i)$ is a longitude of $K'$. Hence each component of $\tilde{L}_i \cap T_i'$ is a longitude of $K'$. On the other hand, since $q = 0$, Proposition 4.1.1 together with Fact 4.1.1 in Section 4.1 implies that each longitude of $K'$ is isotopic in $\partial E(K')$ to an $S^1$-fiber of $Q_i$. Hence each component of $\tilde{L}_i \cap T_i'$ is isotopic in $T_i'$ to an $S^1$-fiber of $Q_i$. However since $\tilde{L}_i$ is transverse to the $S^1$-fibers of $Q_i$, $\tilde{L}_i$ cannot have a boundary component which is isotopic in $\partial Q_i$ to an $S^1$-fiber of $Q_i$, a contradiction. □

Let $\tilde{T}$ be a union of tori which is ambient isotopic to $T$ and transverse to $\mathcal{F}'$. Now, we define a complexity of $\tilde{T}$, denoted by $c(\tilde{T})$, as follows.

$$c(\tilde{T}) = \sum_{i=1}^{n} d_i(\tilde{T}).$$

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In the remainder of this subsection, we discuss for $\mathcal{T}$ with $c(\mathcal{T})$ is maximal among all $\overline{\mathcal{T}}$ as above.

**Lemma 4.2.9** For each $i$ $(1 < i < n)$, there exists a depth $d_i(\mathcal{T})$ leaf $L$ such that there is a component of $L \cap Q_i$ which is a vertical annulus in $Q_i$ and meets $T_{i-1}$ or $T_i$.

**Proof.** By Lemma 4.2.8, for each $i$, we have; for each depth $d_i(\mathcal{T})$ leaf $L$ with $L \cap M_i \neq \emptyset$, each component of $L \cap Q_i$ is either a vertical annulus or a $\partial$-parallel annulus. Assume that for some $m$ $(1 < m < n)$, we have; for each depth $d_m(\mathcal{T})$ leaf $L$ with $L \cap M_m \neq \emptyset$, each component of $L \cap Q_m$ is a $\partial$-parallel annulus.

**Claim 1** For each depth $d_m(\mathcal{T})$ leaf $L$ with $L \cap M_m \neq \emptyset$, we have $L \cap T'_m = \emptyset$.

**Proof of Claim 1.** Assume that there exists a depth $d_m(\mathcal{T})$ leaf $L$ with $L \cap M_m \neq \emptyset$ and $L \cap T'_m \neq \emptyset$. Since $\mathcal{F}$ is a finite depth foliation and the union of the compact leaves of $\mathcal{F}$ is $\mathcal{T}_0 \cup \mathcal{T}_n$, we see by Lemma 2.1.1 that $L$ meets $\mathcal{T}_0$ or $\mathcal{T}_n$. This fact and the assumption that $L \cap T'_m \neq \emptyset$ imply that there is a component of $L \cap Q_m$, say $A'$, such that $A' \cap T'_m \neq \emptyset$ and $A' \cap T_j \neq \emptyset$ $(j = m - 1$ or $m)$. By the assumption, $A'$ is a $\partial$-parallel annulus, but any $\partial$-parallel annulus cannot meet both $T'_m$ and $T_j$, a contradiction. \hfill $\square$

By applying the arguments in the proof of Lemma 4.2.5 for the $\partial$-parallel annuli in $Q_i$, deform $\mathcal{T}$ by an ambient isotopy. Let $\mathcal{T}^*$ be the new union of tori. Let $M^*_i$ be the closure of the component of $M^{(n)} \setminus \bigcup_{i=1}^{n-1} T^*_i$ corresponding to $M_i$ with $T^*_i$ the component of $\mathcal{T}^*$ corresponding to $T_i$. Then by the construction, it is easy to see the following.

(i) for any depth $d_m(\mathcal{T})$ leaf $L$ with $L \cap M_m \neq \emptyset$, we have $L \cap M'_m = \emptyset$; and

(ii) $M^*_m \subset M_m$.

By (ii), we have $d_m(\mathcal{T}^*) \geq d_m(\mathcal{T})$. Moreover (i) implies $d_m(\mathcal{T}^*) \neq d_m(\mathcal{T})$. Hence the following claim is established.

**Claim 2** $d_m(\mathcal{T}^*) > d_m(\mathcal{T})$.

**Claim 3** If $j \neq m$, then $d_j(\mathcal{T}^*) = d_j(\mathcal{T})$.

**Proof of Claim 3.** By the proof of Claim 1, we see that the components of $\mathcal{T}$, other than $T_{m-1}$, $T_m$ are not changed by the deformation for obtaining $\mathcal{T}^*$. Hence we can immediately see that for $k$ $(1 \leq k \leq m - 2$ or $m + 2 \leq k \leq n)$,
$d_k(\mathbb{T}^*) = d_k(\mathbb{T})$. Thus we will prove for $j = m - 1, m + 1$. Since the situation is symmetric, it is enough to prove for the case $j = m - 1$. Suppose there does not exist a depth $d_m(\mathbb{T})$ leaf $L$ such that $L \cap T_{m-1} \neq \emptyset$. Then $T_{m-1} = T_{m-1}$ and we immediately have $d_{m-1}(\mathbb{T}^*) = d_{m-1}(\mathbb{T})$. Suppose there exists a depth $d_m(\mathbb{T})$ leaf $L$ with $L \cap T_{m-1} \neq \emptyset$. Note that the construction of $\mathbb{T}^*$ implies $M_{m-1}^* \supset M_{m-1}$ (cf. the above (ii)). Moreover by the deformation described in the proof of Lemma 4.2.5, we see that each component of $M_{m-1}^* \setminus M_{m-1}$ is a solid torus whose boundary is the union of an annulus in $T_{m-1}$ and an annulus in $T_{m-1}^*$. Since $M_{m-1} \supset M_{m-1}^*$, $d_{m-1}(\mathbb{T}^*) \leq d_{m-1}(\mathbb{T})$. Assume $d_{m-1}(\mathbb{T}^*) < d_{m-1}(\mathbb{T})$. This implies that there exists a leaf $L''$ of $\mathcal{F}'$ such that depth($L''$) < $d_{m-1}(\mathbb{T})$ and $L''$ intersects one of the solid tori. Note that $L'' \cap M_{m-1} \neq \emptyset$, which implies $d_{m-1}(\mathbb{T}) \leq$ depth($L''$), a contradiction. Hence we have $d_{m-1}(\mathbb{T}^*) = d_{m-1}(\mathbb{T})$. \hfill \Box

The above Claims 2 and 3 imply $c(\mathbb{T}^*) > c(\mathbb{T})$, this contradicts the assumption that $c(\mathbb{T})$ is maximal. \hfill \Box

Let $\mathcal{L}_i$ be the union of the leaves $L_i$ of $\mathcal{F}'$ such that $L_i \cap M_i \neq \emptyset$ and depth($L_i$) = $d_i(\mathbb{T})$.

**Lemma 4.2.10** By deforming $\mathbb{T}$ by an ambient isotopy, if necessary, in addition to the above conditions (i.e., $\mathbb{T}$ is transverse to $\mathcal{F}'$ and $c(\mathbb{T})$ is maximal), we may suppose that $\mathcal{L}_1 \subset M_1$ or $\mathcal{L}_n \subset M_n$.

**Proof.** Let $L$ be a depth 0 leaf. Then $L$ meets either $\partial T_0$ or $\partial T_n$. Suppose $L \cap \partial T_0 \neq \emptyset$. Then $d_1(\mathbb{T}) = 0$, and $\mathcal{L}_1$ is a union of depth 0 leaves. Assume that $L$ meets both $M_1$ and $M_2$. By Lemma 4.2.8, a component of $\mathcal{L}_1 \cap Q_2$ is either vertical or a $\partial$-parallel annulus. Assume there is a component, say $A_2$, of $\mathcal{L}_1 \cap Q_2$ which is vertical in $Q_2$. Since each component of $\mathcal{L}_1$ intersects $M_1$, by retaking $A_2$, if necessary, we may suppose that $A_2$ intersects $T_1$. On the other hand, since $L \cap \partial T_0 \neq \emptyset$ and $L \cap M_2 \neq \emptyset$, it is easy to see that there is a component of $L \cap Q_1$, say $A_1$, which is vertical in $Q_1$ and intersects $T_1$. Since $A_1$ ($A_2$ resp.) is a subset of $\mathcal{L}_1$, each component of $A_1 \cap T_1$ is either a component of $A_2 \cap T_2$ or disjoint from $A_2 \cap T_2$. However, this contradicts Lemma 4.2.1. Hence any component of $\mathcal{L}_1 \cap Q_2$ is $\partial$-parallel in $Q_2$. By the arguments of the proof of Lemma 4.2.5, we may suppose via an ambient isotopy that $\mathcal{L}_1 \subset M_1$. By applying the arguments for the case $L \cap \partial T_n \neq \emptyset$ and $L \cap M_{n-1} \neq \emptyset$, we may suppose that $\mathcal{L}_1 \subset M_1$ or $\mathcal{L}_n \subset M_n$. \hfill \Box
4.2.4 Behavior of $d_i$'s

In this subsection, we suppose $K, \mathcal{F}, T, M^{(n)'}, \mathcal{F}'$ are as in Section 4.2.3, particularly as in Lemma 4.2.10. In what follows, to simplify the notation, we use $d_i$ for $d_i(T)$. Let $k$ be an integer ($1 \leq k \leq n$) such that $d_k = \max\{d_1, \ldots, d_n\}$. The purpose of this subsection is to prove the following (note that $c(T)$ is maximal).

**Proposition 4.2.1** Suppose $k = \min\{\ell|d_\ell = \max\{d_1, \ldots, d_n\}\}$. Then, we have the following:

1. if $2 \leq \ell \leq k$ ($k + 1 \leq \ell \leq n - 1$ resp.), then $d_{\ell - 1} < d_\ell$ ($d_{\ell + 1} < d_\ell$ resp.), hence $d_1 < \cdots < d_k \geq d_{k+1} > \cdots > d_n$; and

2. suppose $d_k = d_{k+1} = d$, then there exists a depth $d + 1$ leaf of $\mathcal{F}'$.

For the proof of Proposition 4.2.1, we first prove the following lemmas (Lemmas 4.2.11 4.2.14).

**Lemma 4.2.11** For any $m$ ($1 \leq m \leq k$) ($j$ ($k \leq j \leq n$) resp.), we have the following.

\[(*) \text{ For any } m' \ (1 \leq m' \leq m) \ (j' \ (j \leq j' \leq n) \text{ resp.}), \ d_{m'} \leq d_m \ (d_{j'} \leq d_j \text{ resp.).}\]

(Hence we have either $d_1 \leq \cdots \leq d_k \geq \cdots \geq d_n$, $d_1 \leq \cdots \leq d_n = d_k$ or $d_k = d_1 \geq \cdots \geq d_n$.)

**Proof.** Let $L_k$ be a leaf of $\mathcal{F}'$ such that depth$(L_k) = d_k$ and $L_k \cap M_k \neq \emptyset$. Since $\mathcal{F}$ is a finite depth foliation and the union of the compact leaves of $\mathcal{F}$ is $T_0 \cup T_n$, we see by Lemma 2.1.1 that $L_k$ meets $\tilde{T}_0$ or $\tilde{T}_n$. Since the situation in symmetric, we may suppose without loss of generality that $L_k$ meets $\tilde{T}_0$. We consider $d_m$ for $m$ less than $k$. If $k = 1$, then $(*)$ is clear. Hence we may suppose $k > 1$. First, consider the case $m = k - 1$. If $d_{k-1} = d_k$, then $d_{m'} \leq d_{k-1}$ for any $m' \leq k - 2$ because $d_k$ is maximal. Hence we suppose $d_{k-1} < d_k$. In this case, we can show that any depth $d_{k-1}$ leaf does not meet $T_{k-1}$. In fact, if there is a depth $d_{k-1}$ leaf $L_{k-1}$ with $L_{k-1} \cap T_{k-1} \neq \emptyset$, then $L_{k-1} \cap M_k \neq \emptyset$. This implies that $d_k \leq$ depth$(L_{k-1}) = d_{k-1}$, a contradiction. Hence $L_{k-1} \cap T_{k-1} = \emptyset$. This implies that $L_{k-1} \cap \tilde{T}_0 \neq \emptyset$, and that for each $m'$ ($1 \leq m' \leq k - 2$), $L_{k-1} \cap M_{m'} \neq \emptyset$. This implies that $d_{m'} \leq d_{k-1}$, i.e., $(*)$ holds for $m = k - 1$. 46
Note that if \( k = 2 \), we have proved the lemma for \( m \leq k \). Hence we suppose \( k > 2 \) in the remainder of the proof. Next consider the case \( m = k - 2 \). If \( d_{k-2} = d_{k-1} \), then the above implies for any \( m' \leq k - 3 \), we have \( d_{m'} \leq d_{k-2} \). If \( d_{k-2} < d_{k-1} \), then we can apply the above arguments to show that \( L_{k-2} \cap T_0 \neq \emptyset \), and that for each \( m' (1 \leq m' \leq k - 3) \), \( L_{k-2} \cap M'_{m'} \neq \emptyset \). Hence \( d_{m'} \leq d_{k-2} \), i.e., (\( * \)) holds for \( m = k - 2 \). Then we apply the above arguments repeatedly to have the conclusion of the lemma for \( m \leq k \).

Next, we consider \( d_j \) for \( j \) greater than \( k \). If there exists \( L'_k \), a depth \( d_k \) leaf of \( F' \) with \( L'_k \cap M_k \neq \emptyset \) and \( L'_k \cap T_n \neq \emptyset \), we can apply the arguments for \( m (1 \leq m \leq k) \) to show that (\( * \)) holds.

Suppose there does not exist a leaf of \( F' \) satisfying the above conditions, i.e., for any depth \( d_k \) leaf \( L'_k \) of \( F' \) with \( L'_k \cap M_k \neq \emptyset \), we have \( L'_k \cap T_n = \emptyset \). First, consider the case \( j = k+1 \). If \( d_{k+1} = d_k \), then \( d_{j'} \leq d_{k+1} \) for any \( j' (k+2 \leq j' \leq n) \) because \( d_k \) is maximal. Hence we suppose \( d_{k+1} < d_k \). Then we can show that any depth \( d_{k+1} \) leaf intersecting \( M_{k+1} \) does not meet \( T_k \) by applying the above arguments. Thus \( L_{k+1} \cap T_n \neq \emptyset \), and this implies that for any \( j' \geq j \), \( d_{j'} \leq d_j \). Then, we can apply the arguments for the case \( m \leq k \) to show that (\( * \)) holds for \( j = k+1 \). Then we apply the above arguments repeatedly to have the conclusion of the lemma for \( j \geq k \). This completes the proof of the lemma. \( \square \)

Recall that \( d_k = \max \{d_1, \ldots, d_n \} \). In what follows we further suppose \( k = \min \{\ell \mid d_\ell = \max \{d_1, \ldots, d_n \} \} \).

**Lemma 4.2.12** Suppose \( d_{k+1} = d_k \). Then by deforming \( T \) by an ambient isotopy, we may suppose that \( L_k \cap T_k = \emptyset \) and \( L_{k+1} \cap T_k = \emptyset \).

**Proof.** Assume that \( L_k \cap T_k \neq \emptyset \) (hence \( L_{k+1} \cap T_k \neq \emptyset \)). By Lemma 4.2.8, each component of \( L_k \cap Q_{k+1} \) is either a vertical or a \( \partial \)-parallel annulus.

If any component of \( L_k \cap Q_{k+1} \) intersecting \( T_k \) is a \( \partial \)-parallel annulus, by applying the arguments in the proof of Lemma 4.2.5, we may suppose via an ambient isotopy that \( L_k \cap T_k = \emptyset \). Let \( T^*_i \) be the new tori, \( T^*_i \) the component of \( T^* \) corresponding to \( T_i \), and \( M^*_i \) the closure of the component of \( M^{(n)} \cap T^*_i \) corresponding to \( M_i \). Then we claim that there is a depth \( d_k \) leaf \( L_k \) of \( F \) such that \( L_k \cap M^*_k \neq \emptyset \). In fact, if there does not exist a depth \( d_k \) leaf intersecting \( M^*_k \), then we can show that \( c(T^*) > c(T) \) by applying the argument as in the proof of Lemma 4.2.9, a contradiction. Hence \( d_{k+1}(T^*) = d_{k+1} \), and \( L^*_{k+1} \cap T^*_k = \emptyset \), where \( L^*_{k+1} \) is the union of the leaves \( L' \) of \( F' \) such that \( L' \cap M^*_{k+1} \neq \emptyset \) and \( \text{depth}(L') = d_{k+1} \).
Assume there is a component $A_k$ of $\mathcal{L}_k \cap Q_{k+1}$ which is vertical in $Q_{k+1}$ and $A_k \cap T_k \neq \emptyset$. By Lemma 4.2.8, each component of $\mathcal{L}_{k+1} \cap Q_k$ is either vertical or a $\partial$-parallel annulus. If any component of $\mathcal{L}_{k+1} \cap Q_k$ intersecting $T_k$ is a $\partial$-parallel annulus, by applying the arguments as in the proof of Lemma 4.2.5, we may suppose via an ambient isotopy that $\partial A_k$ is vertical in $Q_k$ and $A_k \cap T_k \neq \emptyset$. Since $A_k$ (or disjoint from $A_k$), each component of $A_k \cap T_k$ is either a component of $A_{k+1} \cap T_k$ or disjoint from $A_{k+1} \cap T_k$, this contradicts Lemma 4.2.1. This completes the proof of the lemma. □

**Lemma 4.2.13** Suppose $d_{k+1} = d_k$. Then $d_1 = 0$ and $d_n = 0$. Moreover, by deforming $T$ by an ambient isotopy, we may suppose $\mathcal{L}_1 \subset M_1$ and $\mathcal{L}_n \subset M_n$.

**Proof.** Since $\mathcal{F}'$ is of finite depth (Lemma 4.2.3), there is a depth 0 leaf, say $F_k$, in $\overline{\mathcal{L}_k}$ and there is one, say $F_{k+1}$, in $\overline{\mathcal{L}_{k+1}}$ (see Lemma 2.1.1). By Lemma 4.2.12, we may suppose that $\mathcal{L}_k \cap T_k = \emptyset$ and $\mathcal{L}_{k+1} \cap T_k = \emptyset$. Since $\mathcal{F}'$ is transverse to $T$, these imply $\overline{\mathcal{L}_k} \cap T_k = \emptyset$ and $\overline{\mathcal{L}_{k+1}} \cap T_k = \emptyset$. Thus $F_k \subset M_1 \cup M_2 \cup \cdots \cup M_k$ and $F_{k+1} \subset M_{k+1} \cup \cdots \cup M_n$. These show $F_k \cap M_1 \neq \emptyset$, and $F_{k+1} \cap M_n \neq \emptyset$, which imply $d_1 = 0$ and $d_n = 0$.

By the arguments in the proof of Lemma 4.2.10, $F_k \subset M_1 \cup M_2 \cup \cdots \cup M_k$ implies via an ambient isotopy that $\mathcal{L}_1 \subset M_1$, and $F_{k+1} \subset M_{k+1} \cup \cdots \cup M_n$ implies via an ambient isotopy that $\mathcal{L}_n \subset M_n$. □

**Lemma 4.2.14** Suppose $n \geq 3$. If $d_{k+1} = d_k$, then $k + 2 \leq n$ and $d_{k+2} < d_{k+1}$.

**Proof.** By Lemma 4.2.13, for $i \neq 1, n$, we immediately have $d_i > 0$. Hence $d_{k+1}(= d_k) > 0$. Note that $d_n = 0$. Hence $k + 1 \leq n - 1$, i.e., $k + 2 \leq n$. Assume $d_{k+2} = d_{k+1}$. By Lemma 4.2.9, there is a component $A_{k+1}$ of $\mathcal{L}_{k+1} \cap Q_{k+1}$ which is vertical in $Q_{k+1}$ and satisfies $A_{k+1} \cap T_k \neq \emptyset$ or $A_{k+1} \cap T_k \neq \emptyset$. We have the following cases.

Case 1 Any component of $\mathcal{L}_{k+1} \cap Q_{k+1}$ does not intersect $T_k$.

In this case, there exists a component $A'_{k+1}$ of $\mathcal{L}_{k+1} \cap Q_{k+1}$ which is vertical and $A'_{k+1} \cap T_{k+1} \neq \emptyset$ (Lemma 4.2.9). If any component of $\mathcal{L}_{k+1} \cap Q_{k+2}$ intersecting $T_{k+1}$ is $\partial$-parallel in $Q_{k+2}$, by the arguments in the proof of Lemma 4.2.5, we can deform $T_{k+1}$ by an ambient isotopy so that (with abusing notations) $\mathcal{L}_{k+1} \cap$
Case 2 There is a component of $L_{k+1} \cap Q_{k+1}$ which intersects $T_k$.

Case 2.1 Any component of $L_{k+1} \cap Q_{k+1}$ intersecting $T_k$ is $\partial$-parallel in $Q_{k+1}$.

In this case, by applying the arguments in the proof of Lemma 4.2.5, we can deform by an ambient isotopy so that (with abusing notations) $L_{k+1} \cap T_k = \emptyset$. Let $T^*$ be the new union of tori corresponding to $T$, $M_j^*(Q_j^*, \text{ resp.})$ the manifold corresponding to $M_j(Q_j, \text{ resp.})$, $d_j^*$ the value corresponding to $d_j$. Note that $M_j^* \subset M_k \cup M_{k+1}$, $M_{k+1}^* \subset M_{k+1}$, $M_j^* = M_j$ ($j \neq k, k+1$).

Claim $c(T^*) = c(T)$.

Proof of Claim. The maximality of $c(T)$ implies that $c(T^*) \leq c(T)$. On the other hand, $M_{k+1}^* \subset M_{k+1}$ implies $d_{k+1}^* \geq d_{k+1}$. Since $M_k^* \subset M_k \cup M_{k+1}$ and $d_k = d_{k+1}$, we see that $d_j^* \geq d_k$. Since $M_j^* = M_j$ for $j \neq k, k+1$, we see that for $j \neq k$, $d_j^* = d_j$. Thus we obtain $c(T^*) \geq c(T)$, hence $c(T^*) = c(T)$. \hfill $\Box$

Let $L^*_i$ be the union of the leaves $L^*$ of $F^*$ such that $L^* \cap M_i^* \neq \emptyset$, and depth$(L^*) = d_i^*$. By Claim and Lemma 4.2.9, we see that there is a component $A_{k+1}^*$ of $L_{k+1}^* \cap Q_{k+1}^*$ which is vertical in $Q_{k+1}^*$ and satisfies $A_{k+1}^* \cap T_k^* \neq \emptyset$ or $A_{k+1}^* \cap T_{k+1}^* \neq \emptyset$. Recall that $L_{k+1}^* \cap T_k^* = \emptyset$. Hence $A_{k+1}^* \cap T_{k+1}^* \neq \emptyset$, we can apply the argument of Case 1 to have a contradiction.

Case 2.2 There is a component $A_{k+1}$ of $L_{k+1} \cap Q_{k+1}$ which meets $T_k$ and vertical.

Since $d_{k+1} = d_k$, each component of $L_{k+1} \cap Q_k$ is either vertical or a $\partial$-parallel annulus (see the proof of Lemma 4.2.8). If there exists a component of $L_{k+1} \cap Q_k$ which is vertical in $Q_k$ intersecting $T_k$, this contradicts Lemma 4.2.1. Suppose any component of $L_{k+1} \cap Q_k$ intersecting $T_k$ is $\partial$-parallel. Then we can apply the arguments of Case 2.1 to have a contradiction.

These contradictions completes the proof of the lemma. \hfill $\Box$

By Lemmas 4.2.11 and 4.2.14, we see that if $n \geq 3$, $d_1, d_2, \ldots, d_n$ has a unique maximum $d_k$, or two successive maxima $d_k, d_{k+1}$.
Proof of Proposition 4.2.1.

Proof of 1. We give a proof for the case $2 \leq \ell \leq k$. Assume there exists $\ell$ ($2 \leq \ell \leq k$) such that $d_{\ell-1} = d_\ell$. Note that by the definition of $k$, we have $d_{k-1} < d_k$. Hence $n \geq 3$ and by retaking $\ell$, if necessary, we may suppose $d_{\ell-1} = d_\ell < d_{\ell+1}$. Note that $d_\ell < d_{\ell+1}$ implies $\mathcal{L}_\ell \cap T_\ell = \emptyset$. Since $d_{\ell-1} = d_\ell$, we see that each component of $\mathcal{L}_\ell \cap M_{\ell-1}$ is compact (see the proof of Lemma 4.2.4), hence each component of $\mathcal{L}_\ell \cap Q_{\ell-1}$ is either vertical or a $\partial$-parallel annulus. If any component of $\mathcal{L}_\ell \cap Q_{\ell-1}$ intersecting $T_{\ell-1}$ is $\partial$-parallel in $Q_{\ell-1}$, then we can show that $T$ can be isotoped so that $\mathcal{L}_\ell \subset M_\ell$ as in the proof of Lemma 4.2.14, a contradiction. Hence there exists a component of $\mathcal{L}_\ell \cap Q_{\ell-1}$ which is vertical in $Q_{\ell-1}$ intersecting $T_{\ell-1}$. However as in the proof of Lemma 4.2.14, we can show that there is a component of $\mathcal{L}_\ell \cap Q_{\ell}$ which is vertical and intersects $T_\ell$, contradicting Lemma 4.2.1. This completes the proof of the lemma for $\ell$ ($2 \leq \ell \leq k$). We can prove for $\ell$ ($k + 1 \leq \ell \leq n - 1$) by the same way as above.

Proof of 2. Let $L$ be a leaf of $\mathcal{F}'$ intersecting $T_k$. Since $d_k = d_{k+1}$, we have $\mathcal{L}_k \cap T_k = \emptyset$, $\mathcal{L}_{k+1} \cap T_k = \emptyset$ (Lemma 4.2.12). This implies that $\text{depth}(L) \neq d$. On the other hand, by the definition of $d_i$, we have $\text{depth}(L) \geq d_k = d$. These imply $\text{depth}(L) > d$. \hfill \Box

Corollary 4.2.1  The depth of $\mathcal{F}'$ is greater than or equal to $[\frac{n}{2}]$.

Proof. Suppose $n = 1$. Then clearly we have $\text{depth}(\mathcal{F}') \geq [\frac{1}{2}] = 0$. Suppose $n = 2$. By Lemma 4.2.10, without loss of generality we may suppose that $d_1 = 0$. If $d_2 = 0$, by 2 of Proposition 4.2.1, there exists a depth 1 leaf of $\mathcal{F}'$. Thus $\text{depth}(\mathcal{F}') \geq 1 = [\frac{2}{2}]$. If $d_2 > 0$, then it is clear that $\text{depth}(\mathcal{F}') \geq d_2 \geq 1 = [\frac{2}{2}]$. Suppose $n \geq 3$. If $n$ is odd, 1 of Proposition 4.2.1 implies that $d_k \geq \frac{n-1}{2}$, i.e., $\text{depth}(\mathcal{F}') \geq \frac{n-1}{2} = [\frac{n}{2}]$. If $n$ is even, 1 of Proposition 4.2.1 implies that $d_k \geq \frac{n}{2} - 1$. Note that $d_k = \frac{n}{2} - 1$ holds if and only if $d_k = d_{\frac{n}{2}} = d_{\frac{n}{2} + 1} = \frac{n}{2} - 1$. Hence in this case, by 2 of Proposition 4.2.1, we have $\text{depth}(\mathcal{F}') \geq \frac{n}{2} = [\frac{n}{2}]$. Hence, for any $n$ ($n \geq 1$), we have $\text{depth}(\mathcal{F}') \geq [\frac{n}{2}]$. \hfill \Box

4.3 Proof of Theorem 1.0.1

Let $K$, $\Sigma^{(n)}(K,0)$, $\alpha$ be as in Theorem ???. Let $\mathcal{F}$ be a codimension one, transversely oriented, taut $C^0$ foliation on $\Sigma^{(n)}(K,0)$ of finite depth which has exactly one depth 0 leaf $\hat{T}$ representing $\alpha$ such that $\text{depth}(\mathcal{F}) = \text{depth}_{0,\alpha}^{(0)}(\Sigma^{(n)}(K,0))$. 50
Since \( \hat{T} \) is the compact leaf of the taut foliation \( \hat{\mathcal{F}} \), \( \hat{T} \) is taut, i.e., incompressible and norm minimizing. Note that \([\hat{T}] = \alpha = \pm [T_i] \) \((1 \leq i \leq n)\) and each \( T_i \) is a torus. Hence \( \hat{T} \) is a torus or a 2-sphere. Assume that \( \hat{T} \) is a 2-sphere. Theorem 3 of [15] implies that \( \Sigma^{(n)}(K,0) \cong S^2 \times S^1 \). By Theorem 7 of [9], \( \Sigma^{(n)}(K,0) \cong S^2 \times S^1 \) implies that \( S^3(K,0) \cong S^2 \times S^1 \). However since \( K \) is a non-trivial knot, Property R [6] implies that \( S^3(K,0) \not\cong S^2 \times S^1 \). Hence \( \tilde{\mathcal{F}} \) is an incompressible torus.

**Claim** The compact leaf \( \hat{T} \) is isotopic to some \( T_i \) \((1 \leq i \leq n)\).

**Proof.** Let \( \hat{T} = (\bigcup_{i=1}^{n} T_i) \cup (\bigcup_{i=1}^{n} T'_i) \) \((\subset \Sigma^{(n)}(K,0))\). Since \( \mathcal{F} \) is taut, Theorem 4 of [17] implies that by an ambient isotopy in \( \Sigma^{(n)}(K,0) \), we can deform \( \hat{T} \) so that either \( \hat{T} \) is transverse to \( \mathcal{F} \) or there exists \( T^{(n)} \), a component of \( \hat{T} \) which coincides with \( \hat{T} \). Suppose \( \hat{T} \) is transverse to \( \mathcal{F} \). By applying the arguments as in the proof of Lemma 4.2.5, we may suppose for any \( i \) \((1 \leq i \leq n)\), \( \mathcal{F} |_{E(K')_i} \) has no annular leaves, particularly \( \hat{T} \cap (\partial T'_i) = \emptyset \). We suppose \( |\hat{T} \cap \hat{T}'| \) is minimal among all union of tori which is isotopic to \( \hat{T} \) and transverse to \( \mathcal{F} \). If \( |\hat{T} \cap \hat{T}'| > 0 \), i.e., there is a torus \( T_j \) with \( \hat{T} \cap T_j \neq \emptyset \), there exist \( A_j \) a component of \( \hat{T} \cap Q_j \), and \( A_{j+1} \) a component of \( \hat{T} \cap Q_{j+1} \) such that \( A_j \cap A_{j+1} \neq \emptyset \). Since \( \hat{T} \) is transverse to \( \mathcal{F} \), each component of \( \hat{T} \cap \hat{T} \) is an essential simple closed curve in \( \hat{T} \). This shows that \( \hat{T} \cap Q_i \), if exists, is incompressible in \( Q_i \). Since \( \hat{T} \cap T'_i = \emptyset \), \( \hat{T} \) is not horizontal in \( Q_i \). Hence by [8, VI.34], each component of \( \hat{T} \cap Q_i \) is either vertical or a \( \partial \)-parallel annulus. Then we note that the arguments in the proof of Lemma 4.2.5 for removing \( \partial \)-parallel annuli work for \( \partial \)-parallel components of \( \hat{T} \cap Q_i \). This together with the minimality of \( |\hat{T} \cap \hat{T}'| \) shows that each component of \( \hat{T} \cap Q_i \) is vertical. Particularly \( A_j \) and \( A_{j+1} \) are vertical, contradicting Lemma 4.2.1. Hence \( \hat{T} \cap \hat{T} = \emptyset \). This shows either \( \hat{T} \subset E(K')_i \) or \( \hat{T} \subset Q_i \). If \( \hat{T} \subset E(K')_i \), then \( \hat{T} \) is separating in \( E(K')_i \) hence separating in \( \Sigma^{(n)}(K,0) \). Thus \( \hat{T} \) can not be a leaf of the taut foliation \( \mathcal{F} \), a contradiction. Hence \( \hat{T} \subset Q_i \). Since \( Q_i \) is homeomorphic to \((\text{disk with two holes}) \times S^1 \), \( \hat{T} \) is isotopic to either \( T'_i \), \( T_{i-1} \) or \( T_i \). Since \( T'_i \) is separating, \( \hat{T} \) can not isotopic to \( T'_i \), so \( \hat{T} \) is isotopic to \( T_{i-1} \) or \( T_i \).

We may suppose without loss of generality that \( \hat{T} \) is isotopic to \( T_n \). Then we can obtain a foliation \( \mathcal{F'} \) on \( M^{(n)} \) by cutting \( \mathcal{F} \) along \( \hat{T} \). By Corollary 4.2.1 and Lemma 4.2.3, we have

\[
\text{depth}(\mathcal{F}) = \text{depth}(\mathcal{F'}) \geq 1 + \left[\frac{n}{2}\right].
\]

This completes the proof of Theorem 1.0.1.
4.4 Proof of Theorem 1.0.2

Let $\mathcal{F}$ be a codimension one, transversely oriented, taut $C^0$ foliation of finite depth, with $C^\infty$ leaves on $\Sigma^{(n)}(K,0)$ with exactly one depth 0 leaf representing $\alpha$, where $\alpha$ is corresponding to a generator of $H_1(S^3(K,0)) \cong \mathbb{Z}$. Let $\widetilde{\mathcal{F}}$ be the foliation obtained by modifying $\mathcal{F}$ as in Section 3, hence $\widetilde{\mathcal{F}}$ satisfies the conditions given soon after Remark 2.1.4 in Section 2.1.2. Then, by applying the argument as in the proof of Claim in Section 4.3, we may assume that the compact leaf of $\widetilde{\mathcal{F}}$ is isotopic to $T_n$. Let $M^{(n)}$ be the manifold obtained from $\Sigma^{(n)}(K,0)$ by cutting along $T_n$ (in this subsection, we basically use the same three dimensional manifolds and surfaces as in Section 4.2 and adopt the same notations for denoting the manifolds, e.g., $M^{(n)}$, $M_i$, $T_j$, $T'_j$, etc.) Then, $M^{(n)}$ clearly corresponds to $\hat{M}$ which appears in the paragraph preceding Lemma 2.1.6 in Section 2.1.2. Recall that $\partial M^{(n)} = T_0 \cup T_n$. Let $\tilde{\mathcal{F}}$ be the foliation on $M^{(n)}$ induced from $\mathcal{F}$. Then, for the above foliations $\tilde{\mathcal{F}}$ and $\mathcal{F}$, we can consider the graph $\hat{G}(\tilde{\mathcal{F}})$ and $\hat{G}(\mathcal{F})$ in Definitions 2.1.10 and 2.1.11. Let $g = \text{gap}(\tilde{\mathcal{F}})$. Since $M^{(n)}$ is not homeomorphic to $(\text{torus}) \times [0,1]$, we see that $\tilde{\mathcal{F}}$ is not a foliation given by a surface bundle structure over $S^1$. Thus $G(\tilde{\mathcal{F}})$ has an edge, hence $g \geq 1$. By the construction of $\tilde{\mathcal{F}}$ described in Section 3, we see that $\tilde{\mathcal{F}}$ contains exactly one depth 0 leaf. These imply that $\hat{G}(\tilde{\mathcal{F}})$ contains exactly two vertices at depth 0.

**Lemma 4.4.1** $\hat{G}(\tilde{\mathcal{F}})$ is connected.

**Proof.** Assume that $\hat{G}(\tilde{\mathcal{F}})$ is not connected. Since the union of the compact leaves of $\tilde{\mathcal{F}}$ is $T_0 \cup T_n$, there are exactly two vertices at depth 0. By Remark 2.1.5, any component of $\hat{G}(\tilde{\mathcal{F}})$ must have a depth 0 vertex. Hence $\hat{G}(\tilde{\mathcal{F}})$ consists of two components, say $G_1$ and $G_2$. Let $\{u_j\}$ ($\{v_k\}$ resp.) be the vertices of $G_1$ ($G_2$ resp.) Let $U_j$ ($V_k$ resp.) be the union of the leaves representing $u_j$ ($v_k$ resp.). Then we show that $\bigcup_j U_j$ is closed. Let $\{x_i\}_{i=1,2,...}$ be a Cauchy sequence in $\hat{M}$ such that each $x_i$ is contained in $\bigcup_j U_j$ and converges to $x_\infty$. We show that $x_\infty \in \bigcup_j U_j$. Let $L_\infty$ be the leaf of $\tilde{\mathcal{F}}$ which contains $x_\infty$. Let $P$ be a plaque of $\mathcal{F}^\perp$ through $x_\infty$. We may suppose each $x_i$ is contained in $P$. Let $P_+, P_-$ be the components of $P \setminus x_\infty$. Then, by retaking $x_i$ if necessary, we may suppose that each $x_i$ is contained in $P_+$. If there is a leaf $L$ of $\tilde{\mathcal{F}}$ which contains infinitely many $x_i$, this implies that $\overline{\tilde{L}} \supset L \supset L_\infty$. Thus there exists a path connecting $u_j$ and the vertex representing $L_\infty$. Hence we have $x_\infty \in \bigcup_j U_j$. Suppose there does not exist such $L$. Let $L_i$ be the leaf of $\tilde{\mathcal{F}}$ which contains $x_i$. Since $L_i$ intersects $P$ finitely many times, we may suppose that $x_i$ is the nearest to $x_\infty$ among all the
points of $L_i \cap P$. By applying the arguments as in the proof of Claim given soon after the proof of Lemma 3.1.1, we can show that $L_\infty$ is equivalent to $L_i$. This implies that $x_\infty \in \bigcup_j U_j$. Hence $\bigcup_j U_j$ is closed. By applying the arguments as above, we can show that $\bigcup_k V_k$ is also closed. Note that $M^{(n)} = (\bigcup_j U_j) \cup (\bigcup_k V_k)$ and that $(\bigcup_j U_j) \cap (\bigcup_k V_k) = \emptyset$, contradicting the fact that $M^{(n)}$ is connected. □

We say that graph $\Gamma$ is a tree if $\Gamma$ is connected and $\Gamma$ does not contain a cycle.

**Lemma 4.4.2** The following two conditions are equivalent to each other.

1. $\hat{G}(\hat{F})$ is a tree.

2. The number of the cycles of $G(\tilde{F})$ is one.

**Proof.** We first show that 1 implies 2. Suppose 1 holds. Recall that the number of the depth 0 vertices of $\hat{G}(\hat{F})$ is two, and $G(\tilde{F})$ is obtained from $\hat{G}(\hat{F})$ by identifying them. Since $\hat{G}(\hat{F})$ is a tree, $\hat{G}(\hat{F})$ does not have a cycle and there is a unique path in $\hat{G}(\hat{F})$ connecting the depth 0 vertices, the path becomes a cycle in $G(\tilde{F})$ and this is the only cycle in $G(\tilde{F})$.

Suppose 2 holds. Since $\hat{G}(\hat{F})$ is connected (Lemma 4.4.1), we only need to prove that $\hat{G}(\hat{F})$ does not have a cycle. Assume that $\hat{G}(\hat{F})$ has a cycle. By applying the argument as above, the path in $\hat{G}(\hat{F})$ connecting the depth 0 vertices become a cycle in $G(\tilde{F})$ and since the operation obtaining $G(\tilde{F})$ does not remove a cycle, this implies that the number of the cycles of $G(\tilde{F})$ is two, a contradiction.

□

In the following, we suppose that $\hat{G}(\hat{F})$ is a tree. Let $\Gamma$ be the path connecting the depth 0 vertices of $\hat{G}(\hat{F})$. Then, clearly gap($\hat{F}$) = $g$. Suppose $g = 1$. Note that Theorem 1.0.1 implies that depth($\tilde{F}$) $\geq 1 + \left\lfloor \frac{n}{2} \right\rfloor \geq \frac{1+n}{2}$. Thus Theorem 1.0.2 holds.

Hence in the remainder of this proof, we suppose $g > 1$.

**Lemma 4.4.3** There exists exactly one edge, say $e$, of $\hat{G}(\hat{F})$ with length($e$) > 1. (Hence we have length($e$) = $g$.)

**Proof.** Let $e$ be an edge of $\hat{G}(\hat{F})$ such that length($e$) > 1. Let $v, v'$ be the endpoints of $e$ such that depth($v$) < depth($v'$). Let $\Gamma_1$ (if $\Gamma_2$ resp.) be a directed path from $v$ ($v'$ resp.) to a vertex at depth 0 such that each edge of $\Gamma_1$ (if $\Gamma_2$ resp.) has length one (Remark 2.1.5). Here we regard the vertex $v$ as $\Gamma_1$ if depth($v$) = 0. Then, since $\hat{G}(\hat{F})$ is a tree and the number of the vertices at depth 0 is two, it
is clear that $\Gamma = \Gamma_1 \cup e \cup \Gamma_2$. Take any edge $e'$ of $\hat{G}(\hat{F})$ with $\text{length}(e') > 1$. By applying the argument as above, we can show that there exist directed paths $\Gamma_1', \Gamma_2'$ from the endpoints of $e'$ to the depth 0 vertices, each edge of which has length one. Moreover we have $\Gamma = \Gamma_1' \cup e' \cup \Gamma_2'$. Since each edge of $\Gamma_1, \Gamma_2, \Gamma_1', \Gamma_2'$ has length one, this shows that $e' = e$. Hence $e$ is the only edge of length greater than one, thus we have $\text{length}(e) = g$. □

Let $v, v', \Gamma_1, \Gamma_2$ be as in the proof of Lemma 4.4.3. Since the situation is symmetric, we may suppose $\Gamma_1$ ($\Gamma_2$ resp.) contains the vertex representing $T_0$ ($T_n$ resp.). Let $m$ be the number of edges of $\Gamma_1$. Since $\text{gap}(\hat{F}) = g$, the number of edges of $\Gamma_2$ is $m + g$. Rename the vertices in $\Gamma_1 \cup \Gamma_2$ by $v_0, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m+g+1}$ so that $v_i$ ($0 \leq i \leq 2m + g + 1$) are on $\Gamma$ in this order, and that $v_0 = [T_0]$, $v_{2m+g+1} = [T_n]$.

Let $L_k$ be a leaf representing $v_k$. Let $\mathbb{T} = (\bigcup_{j=1}^{n-1} T_j) \cup (\bigcup_{i=1}^{n} T'_i)$ be as in Section 4.2.1. Let $M_i$ be as in Section 4.2. Let $M^{(n)}_i$ be as in Section 4.2.1. Let $M'_i$ be the closure of the component of $M^{(n)} \setminus \mathbb{T}$ corresponding to $M_i$. Note that for $i \neq 1, n$, we have $M'_i = M_i$.

Claim $\text{depth}(\mathbb{F}) \geq m + g$.

Proof of Claim. It is clear that $\text{depth}(\hat{F}) \geq \max\{\text{depth}(v_i)\}$. Note that $v_{m+1}$ corresponds to $v'$ in the proof of Lemma 4.4.3. Hence $\max\{\text{depth}(v_i)\} = \text{depth}(v_{m+1})$. Note that $\text{depth}(v_{m+1}) = \sum_{\text{edges of } \Gamma_1} \text{length}(e) + \text{length}(e)$ (see Figure 4.15). Since the length of each edge of $\Gamma_1$ is one, this implies that $\text{depth}(v_{m+1}) = (\text{number of edges of } \Gamma_1) + g = m + g$. Hence we have $\text{depth}(\hat{F}) \geq m + g$. By Lemma 2.1.6 and Fact 3.3.1, we see that $\text{depth}(\mathbb{F}) \geq \text{depth}(\hat{F}) \geq m + g$. □

Now we estimate the value $m + g$. If $m \geq n$, we have $m + g \geq n + g > \frac{n + g}{2}$. By the above Claim, this shows that Theorem 1.0.2 holds. Hence in the remainder of this subsection, we suppose $m < n$.

Lemma 4.4.4 There is an ambient isotopy $f_t$ ($0 \leq t \leq 1$) of $M^{(n)}$ whose support is contained in $\bigcup_{i=1}^{m+1} M_i$ satisfying the following two conditions:

1. $f_1(\mathbb{T})$ is transverse to $\hat{F}$;
2. for $k$ ($1 \leq k \leq m$), $L_k \subset \bigcup_{i=1}^{k} \hat{M}_i$, where $\hat{M}_i$ is the closure of the component of $M^{(n)} \setminus f_1(\bigcup_{j=1}^{n-1} T_j)$ corresponding to $M_i$. 

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Proof. We consider for $k = 1$. By applying the argument as in the proof of Assertion(i) in the proof of Lemma 4.2.3, we see that $L_1 \cap M^{(n)_1} = \emptyset$ or $L_1 \cap M^{(n)_1}$ is compact. Suppose $L_1 \cap M_2 \neq \emptyset$. Then, by applying the argument as in the proof of Lemma 4.2.10, we can show that there is an ambient isotopy $f_1^t$ ($0 \leq t \leq 1$) whose support is contained in $M_1 \cup M_2$ such that $L_1 \subset \tilde{M}_1$, where $\tilde{M}_i$ is the closure of the component of $M^{(n)} \setminus f_1^t(\cup_{j=1}^{n-1} T_j)$ corresponding to $M_i$, and that $f_1^t(T)$ is transverse to $\hat{F}$. Suppose $L_1 \cap M_2 = \emptyset$. Then let $f_1^t = id_{M^{(n)}}$ ($0 \leq t \leq 1$). Then, we consider for $k = 2$. Suppose $L_2 \cap M^{(n)} \setminus \tilde{M}_1 \neq \emptyset$. If $L_2 \cap M^{(n)} \setminus \tilde{M}_1$ is noncompact, then there exists a depth 1 leaf $L'_1$ such that $L'_1 \subset \overline{T}_2$ and $L'_1 \cap M^{(n)} \setminus \tilde{M}_1 \neq \emptyset$. Now, since $L_1 \subset \tilde{M}_1$, $L'_1 \neq L_1$. Let $v'_1$ be the vertex representing $L'_1$. We claim that $v'_1 \neq v_1$. In fact, if $v'_1 = v_1$, then there is an embedding $\phi : L_1 \times [0,1] \to M^{(n)}$ giving equivalence relation between $L$ and $L'$. Note that $\hat{F}|_{\phi(L_1 \times [0,1])}$ is a product foliation, and $L_1 \cap T_1 = \emptyset$. These imply that there is a point $x$ in $T_1 \cap \phi(L_1 \times [0,1])$ such that $\hat{F}$ and $T_1$ are not transverse at $x$, a contradiction. By Remark 2.1.5, there exists a directed path $\Gamma'_1$ from $v'_1$ to $v_0$ or $v_{2m+2+1}$. This contradicts the assumption that $\hat{G}(\hat{F})$ is a tree. Hence $L_2 \cap M^{(n)} \setminus \tilde{M}_1$ is compact. Suppose $L_2 \cap \tilde{M}_3 \neq \emptyset$. Then by applying the argument as in the proof of Lemma 4.2.10, we can show that there is an ambient isotopy $f_2^t$ whose support is contained in $\tilde{M}_2 \cup \tilde{M}_3$ such that $L_2 \subset \tilde{M}_2$, where $\tilde{M}_i$ is the closure of the component of $M^{(n)} \setminus f_2^t(\cup_{j=1}^{n-1} T_j)$ corresponding to $M_i$, and $f_2^t(T)$ is transverse to $\hat{F}$. Suppose $L_2 \cap \tilde{M}_3 = \emptyset$. Then we let
\( f_t^2 = \text{id}_{M^{(n)}} \) (0 \( \leq t \leq 1 \)). By applying the argument as above, we can obtain a sequence of ambient isotopies \( f_t^1, f_t^2, \ldots, f_t^{m-1}, f_t^m \). Then, the desired ambient isotopy \( f_t \) is obtained by applying \( f_t^1, f_t^2, \ldots, f_t^m \) successively in this order (with reparametrizing the parameter \( t \)). \( \square \)

In the following, we abuse notation \( T \) for denoting \( f_1(T) \) for simplicity, hence for \( k \) (1 \( \leq k \leq m \)), \( L_k \subset \bigcup_{i=1}^k M_i \) holds. For \( k \) (1 \( \leq k \leq m \)), let \( j_k \) be the integer which satisfies \( L_k \cap M_{j_k} \neq \emptyset \) and \( L_k \cap M_{j_k+1} = \emptyset \). We extend the definition of \( j_k \) by putting \( j_0 = 0 \). Since \( L_k \subset \bigcup_{i=1}^k M_i \), we immediately have the following.

Lemma 4.4.5 For \( k \) (1 \( \leq k \leq m \)), we have \( j_k \leq k \).

Suppose \( j_m \geq n - m - g + 1 \). By applying Lemma 4.4.5 for the case \( k = m \), we have \( j_m \leq m \). These inequalities imply \( m + g \geq \frac{n + g + 1}{2} > \frac{n + g}{2} \). This together with the claim in this subsection shows that Theorem 1.0.2 holds. Hence in the remainder of this subsection, we may suppose \( j_m < n - m - g + 1 \). Note that \( \Gamma_2 \) contains \( m + g + 1 \) vertices, \( v_n+1, v_{n+2}, \ldots, v_{n+g+1} \). By applying the argument as in the proof of Lemma 4.4.4 to leaves corresponding to the \( m + g - 1 \) vertices \( v_{n+g}, v_{n+g+1}, \ldots, v_{n+2} \), we can obtain the following lemma. (Note that \( L_{2n+g+1-k'}(M_{n+1-k'} \text{ resp.}) \) in Lemma 4.4.6 corresponds to \( L_{k'}(M_k \text{ resp.}) \) in Lemma 4.4.4.)

Lemma 4.4.6 There is an ambient isotopy \( f_t' \) (0 \( \leq t \leq 1 \)) of \( M^{(n)} \) whose support is contained in \( \bigcup_{i=1}^{m+g} M_{n+1-i} \) satisfying the following two conditions:

1. \( f_t' \) is transverse to \( \tilde{\mathcal{F}} \);

2. for \( k' \) (1 \( \leq k' \leq m + g - 1 \)), \( L_{2n+g+1-k'} \subset \bigcup_{i=1}^{k'} \tilde{M}_{n+1-i} \), where \( \tilde{M}_{n+1-i} \) is the closure of the component of \( M^{(n)} \setminus \bigcup_{i=1}^{m-1} T_i \) corresponding to \( M_{n+1-i} \).

Note that since \( j_m < n - m - g + 1 = n + 1 - (m + g) \), \( f_t' \) does not change \( \bigcup_{i=0}^{m} L_i \). In the following, we abuse notation \( T \) for denoting \( f_1(T) \) for simplicity, i.e., for \( k' \) (1 \( \leq k' \leq m + g - 1 \)), \( L_{2n+g+1-k'} \subset \bigcup_{i=1}^{k'} M_{n+1-i} \). For \( k' \) (1 \( \leq k' \leq m + g - 1 \)), let \( j'_{k'} \) be the integer which satisfies \( L_{2n+g+1-k'} \cap M_{n-j'_{k'}+1} \neq \emptyset \), and \( L_{2n+g+1-k'} \cap M_{n-j'_{k'}} = \emptyset \). Since \( L_{2n+g+1-k'} \subset \bigcup_{i=1}^{k'} M_{n+1-i} \), we immediately have the following.

Lemma 4.4.7 For 1 \( \leq k' \leq m + g - 1 \), we have \( j'_{k'} \leq k' \).

Then, we have the following.
Lemma 4.4.8 \( n - j'_{m+g-1} \leq j_m + 1 \).

**Proof.** Assume that \( n - j'_{m+g-1} \geq j_m + 2 \). By the definition of \( j_k \), we see that \((M_{j_m+1} \cup M_{j_m+2}) \cap L_m = \emptyset\). On the other hand, the above inequality implies that \((M_{j_m+1} \cup M_{j_m+2}) \cap L_{m+2} = \emptyset\). Since \( L_{m+1} \) approaches both \( L_m \) and \( L_{m+2} \), \( L_{m+1} \) intersects both \( M_{j_m+1} \) and \( M_{j_m+2} \). By applying the argument as in the proof of Lemma 4.4.4, we can show that \( L_{m+1} \cap M_{j_m+1} \) and \( L_{m+1} \cap M_{j_m+2} \) are compact. By applying the argument as in the proof of Lemma 4.2.8, we can show that each component of \( L_{m+1} \cap Q_{j_m+1} \) (\( L_{m+1} \cap Q_{j_m+2} \) resp.) is a vertical or boundary parallel annulus in \( Q_{j_m+1} \) (\( Q_{j_m+2} \) resp.), where \( Q_i \) is as in Section 4.2. By applying the argument as in the proof of Lemma 4.2.9, there exists a vertical annulus \( A \subset L_{m+1} \cap Q_{j_m+1} \) such that \( A \cap T_{j_m} \neq \emptyset \) or \( A \cap T_{j_m+1} \neq \emptyset \), and there exists a vertical annulus \( A' \subset L_{m+1} \cap Q_{j_m+2} \) such that \( A \cap T_{j_m+1} \neq \emptyset \) or \( A \cap T_{j_m+2} \neq \emptyset \). Since \( \overline{L_{m+1}} \supset T_n \), we may suppose that \( A \cap T_{j_m+1} \neq \emptyset \), and since \( \overline{L_{m+1}} \supset T_0 \), we may suppose that \( A' \cap T_{j_m+1} \neq \emptyset \). However these contradict Lemma 4.2.1.

By Lemma 4.4.5, we see that \( j_m \leq m \) and by Lemma 4.4.7, we see that \( j'_{m+g-1} \leq m + g - 1 \). These together with Lemma 4.4.8 imply that

\[
n - m - g + 1 \leq m + 1.
\]

Thus we obtain

\[
m + g \geq \frac{n + g}{2}.
\]

This together with the claim of this subsection, we can show that Theorem 1.0.2 holds. \( \square \)

This completes the proof of Theorem 1.0.2.
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References


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