The role of the round spheres

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Outline

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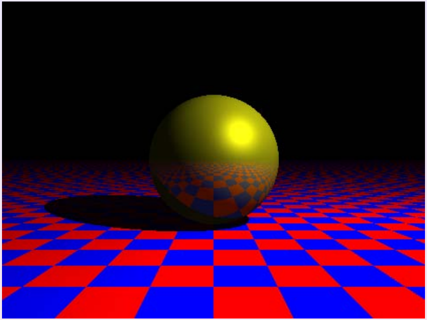
A round sphere



$$\mathbb{S}^2(r) = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2\}$$

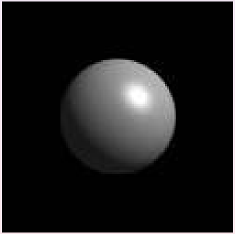
Remark: In this talk, "surfaces" are all connected and without boundary.

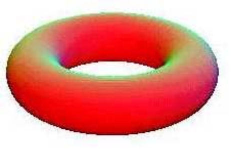
Start from linear algebra

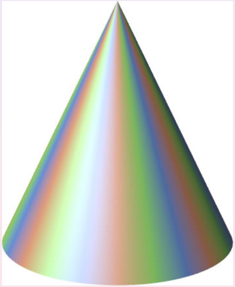
- Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a 2×2 matrix. If $\exists \lambda \in \mathbb{R}, \xi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, s.t. $A\xi = \lambda\xi$, then λ is called the eigenvalue of A and ξ is called the eigenvector of A w.r.t. the eigenvalue λ.
- If A is symmetric, i.e., $a_{12} = a_{21}$, then the eigenvalues of A are real and A is similar to a diagonal matrix.
- Each symmetric matrix relates to a quadratic form.

Definition of surfaces in $\mathbb{R}^3$

- **Goal**: Use calculus to study properties of surfaces.
- **Question**: how to define surfaces?







Definition of surfaces in $\mathbb{R}^3$

- **(Intuitive definition of surfaces)** A surface is a subset of $\mathbb{R}^3$ s.t. each of its points has a neighborhood similar to a piece of a plane which blends smoothly and without self-intersections when bent in $\mathbb{R}^3$.
- **(Definition)** A smooth surface in $\mathbb{R}^3$ is a subset $\Sigma \subset \mathbb{R}^3$ such that each point has an open neighborhood $U \subset \Sigma$ and a map $\mathrm{X} : V \to \mathbb{R}^3$ from an open set $V \subset \mathbb{R}^2$ such that
  - $\mathrm{X} : V \to U$ is a homeomorphism
  - $\mathrm{X}(u, v) = (x(u, v), y(u, v), z(u, v))$ has derivative of all orders
  - $(d\mathrm{X})_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective for all $q \in V$

Two important quadratic forms on surfaces in $\mathbb{R}^3$

- The first fundamental form:

$$\begin{aligned} \mathrm{I} &= d\mathrm{X} \cdot d\mathrm{X} \\ &= \mathrm{X}_u \cdot \mathrm{X}_u du^2 + 2\mathrm{X}_u \cdot \mathrm{X}_v dudv + \mathrm{X}_v \cdot \mathrm{X}_v dv^2 \end{aligned}$$

- The second fundamental form :

$$\begin{aligned} \mathrm{II} &= d^2\mathrm{X} \cdot \mathrm{N} = -d\mathrm{X} \cdot d\mathrm{N} \\ &= -\mathrm{X}_u \cdot \mathrm{N}_u du^2 - 2\mathrm{X}_u \cdot \mathrm{N}_v dudv - \mathrm{X}_v \cdot \mathrm{N}_v dv^2 \end{aligned}$$

$$N := \frac{X_u \times X_v}{|X_u \times X_v|}, \quad \text{a unit normal vector field on} \quad \Sigma$$

Definition of curvatures

N, a unit normal field on the surface Σ, can be thought of as a differentiable map $\mathrm{N} : \Sigma \to \mathbb{S}^2$, the so-called Gauss map.

- The endomorphism $-d\mathrm{N}_p : T_p\Sigma \to T_{\mathrm{N}(p)}\mathbb{S}^2 = T_p\Sigma$ is self-adjoint.
- Its eigenvalues $k_1(p), k_2(p)$ are called principal curvatures of Σ at p.
- $K(p) = k_1(p)k_2(p)$, $H(p) = \frac{k_1(p)+k_2(p)}{2}$ are called the Gauss curvature and mean curvature, respectively.

$$\boxed{K(p) = \det(dN)_p, \quad H(p) = -\frac{1}{2}\mathrm{tr}(dN)_p, \quad p \in \Sigma.}$$

Totally umbilical surfaces

- (**Planes**) If P is a plane of $\mathbb{R}^3$ with unit normal vector a, then $(d\mathrm{N})_p = 0$ and so $h_p = 0$ for each $p \in P$. Hence, $k_1 = k_2 \equiv 0$.
- (**Round Sphere**) The inner unit normal N of $\mathbb{S}^2(r)$ is $-\frac{1}{r}\mathrm{X}$. Then $-d\mathrm{N} = \frac{1}{r}d\mathrm{X}$. So $k_1 = k_2 \equiv \frac{1}{r}$.

Proposition (Classification of totally umbilical surfaces)

A connected surface in $\mathbb{R}^3$ satisfies $k_1 = k_2$ everywhere, i.e. totally umbilical, if and only if it is a plane or a round sphere.

Gauss map and the second fundamental form

For the endomorphism $d\mathrm{N}_p$ of $T_p\Sigma$, $p \in \Sigma$, we can associate a quadratic form h_p:

$$h_p : T_p\Sigma \times T_p\Sigma \longrightarrow \mathbb{R}, \quad p \in \Sigma,$$
$$h_p(v, w) = -\langle d\mathrm{N}_p(v), w\rangle, \quad v, w \in T_p\Sigma.$$

This is nothing else but the second fundamental form of the surface Σ at the point p. In terms of it,

$$\boxed{K(p) = \det h_p, \quad H(p) = \frac{1}{2}\mathrm{tr}\, h_p, \quad p \in \Sigma.}$$

Outline

Fights between the Gauss curvature and mean curvature

- The Gauss curvature and mean curvature were born in the 18th century.
- Since then, they have fought to prevail over each other.
- The initial battle was won by the Gauss curvature because of the famous Gauss's Theorema Egregium.
- Sophie Germain (1776-1831) argued against Gauss by preferring mean curvature function during her study on the vibration of elastic surfaces.


What can curvatures say about the shape of surfaces?

- It is one of the most favorite questions in modern differential geometry – skip from local to global property
- To determine compact surfaces with one or several simplest curvature behavior

Which are the compact surfaces with constant Gauss curvature?

- Hilbert(1901)-Liebmann (1899): The only compact surfaces with constant Gauss curvature are round spheres.
- Hadamard (1897): Any compact surface with positive Gauss curvature is convex.

** The global problems on the mean curvature proved to be more complicated and so, more interesting.

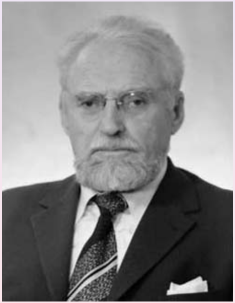
Is a CMC surface necessarily a round sphere?

- Liebmann (1900): A closed strictly convex CMC surface in $\mathbb{R}^3$ must be a round sphere.
- Hopf (1951): A CMC topological sphere in $\mathbb{R}^3$ must be a round sphere.
- Alexandrov (1956): A compact embedded CMC surface in $\mathbb{R}^3$ must be a round sphere.
  - Alexandrov (1956, 1962) reflection
  - Reilly (1976) a purely analytic proof
  - Montiel and Ros (1991) a relatively elementary proof
  - Hijazi, Montiel and Zhang (2001) a proof based on boundary problem of Dirac operator

Alexandrov's theorem

A compact embedded CMC surface in $\mathbb{R}^3$ must be a round sphere.

— One of the most beautiful theorems in classical differential geometry.



A. D. Alexandrov
(1912-1999)

Outline

Step 1: Heintze-Karcher's inequality

- Any compact embedded surface Σ in $\mathbb{R}^3$ can determine a compact connected domain, s.t. $\partial\Omega = \Sigma$.
- Study the square of the distance function $f(p) = |p - p_0|^2$ from points of Σ to a fixed point $p_0 \in \mathbb{R}^3$.

Proposition

Take $q \in \Omega$. If $p \in \Sigma$ is the point of Ω closest to q, then $q = p + t\mathrm{N}(p)$, where $\mathrm{N}(p)$ is the inner normal and $0 \leq t \leq \frac{1}{k_{\max(p)}}$.

Hence, $\Omega \subset F(A)$, where $F(p,t) = p + t\mathrm{N}$,
$A = \{(p,t) \in \Sigma \times \mathbb{R} | 0 \leq t \leq \frac{1}{k_{\max}}\}$.

$\mathrm{Vol}(\Omega)$ and $\mathrm{Area}(\Sigma)$

$$
\begin{aligned}
\mathrm{Vol}(\Omega) &= \int_\Sigma \int_0^{c(p)} d\mathrm{Vol}(\mathrm{X} + t\mathrm{N}) \\
&= \int_\Sigma \int_0^{c(p)} (1 - tk_1)(1 - tk_2) dt dA \\
&\leq \int_\Sigma \int_0^{\frac{1}{k_{\max}}} (1 - tH)^2 dt dA \\
&\leq \int_\Sigma \int_0^{1/H} (1 - tH)^2 dt dA \\
&= \frac{1}{3} \int_\Sigma \frac{1}{H} dA,
\end{aligned}
$$

Heintze-Karcher's inequality

Let $\mathrm{X} : \Sigma \to \mathbb{R}^3$ be a compact embedded surface whose mean curvature H w.r.t. the inner normal is everywhere positive, then

$$\boxed{\mathrm{Vol}(\Omega) \leq \frac{1}{3} \int_\Sigma \frac{1}{H} dA,}$$

where Ω is the inner domain determined by Σ. Moreover, equality holds $\Leftrightarrow$ M is totally umbilical $\Leftrightarrow$ Σ is a round sphere.

Step 2: Minkowski formula

The divergence theorem gives

$$3\mathrm{Vol}(\Omega) = -\int_{\Sigma} \langle \mathrm{X}, \mathrm{N} \rangle dA,$$

$$\int_{\Sigma} (1 + H\langle \mathrm{X}, \mathrm{N} \rangle) dA = 0$$

$$(\because \triangle |\mathrm{X}|^2 = 4(1 + H\langle \mathrm{X}, \mathrm{N} \rangle).)$$

Final step

For CMC case, the Heintze-Karcher's inequality implies

$$\boxed{3H\mathrm{Vol}(\Omega) \leq \mathrm{Area}(\Sigma).}$$

The Minkowski formula implies

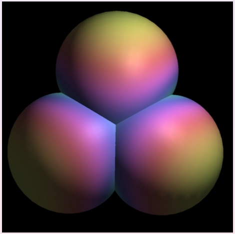
$$\int_\Sigma dA - 3H\,\mathrm{Vol}(\Omega) = \int_\Sigma (1 + H\langle \mathrm{X}, \mathrm{N}\rangle) dA = 0,$$

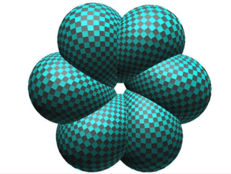
i.e. "$=$" attached in Heintze-Karcher type inequality, then Σ is a round sphere.

The story does not end

Is a CMC surface necessarily a round sphere?

No !





Thanks for Your Attention!