From Brain Waves to Mathematics of Fractals

- The beauty of numbers behind irregular functions -

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Motivation

There are a lot of people who are suffering from depression.

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- Mathematician? How can I help?
- Brain scientist?

Motivation

There are a lot of people who are suffering from depression.

What can I do for them?

- Medical doctor? Sorry....no patience..
- Mathematician? How can I help?
- Brain scientist? It might be interesting!?

Brain waves

- Why is it difficult to analyze functions like brain waves?

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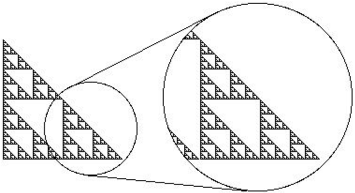
My research dream

Find new techniques to analyze irregular functions like brain waves!

Fractals



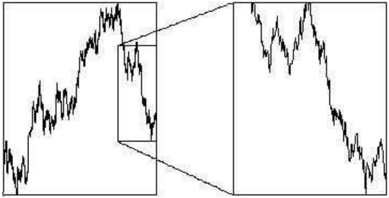
Self-similarity

The geometric characterization of the simplest fractal is self-similarity: the shape is made of smaller copies of itself. The copies are similar to the whole.

Examples of fractals

- The simplest fractals are constructed by iteration. For example, start with a filled-in triangle and iterate this process:
- For every filled-in triangle, connect the midpoints of the sides and remove the middle triangle. Iterating this process produces, in the limit, the Sierpinski Gasket.
- The gasket is self-similar.

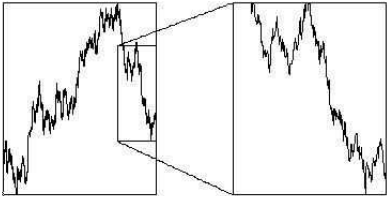
Examples of fractals (continued)



Extention of self-similarity

- Self-similarity can be extended to allow the pieces to look like the whole in some sense.
- The right window is a rescaling of the x-axis by a factor of 4, and the y-axis by a factor of 2. The right picture has about the same distribution of jumps as the left.

Examples of fractals (continued)



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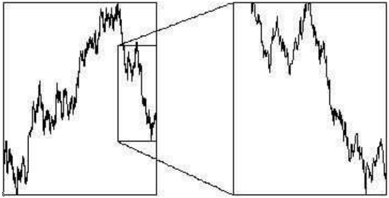
Examples

Stock price movement?

Examples of fractals (continued)



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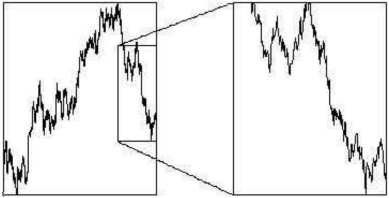
Examples

Stock price movement? Coastline?

Examples of fractals (continued)



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Examples

Stock price movement? Coastline? Mountain?

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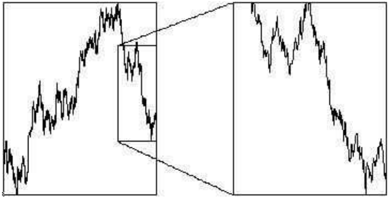

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Examples

Stock price movement? Coastline? Mountain? Tree?

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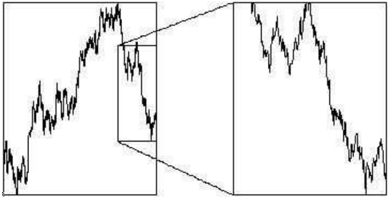

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Examples

Stock price movement? Coastline? Mountain? Tree? Female!

Preparation

Notation

Let the binary expansion of $x \in [0,1]$ be

$$x = \sum_{k=1}^{\infty} \epsilon_k 2^{-k}, \qquad \epsilon_k = \epsilon_k(x) \in \{0,1\}.$$

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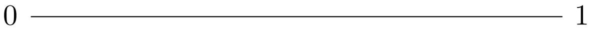

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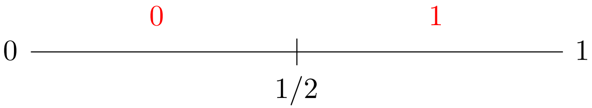

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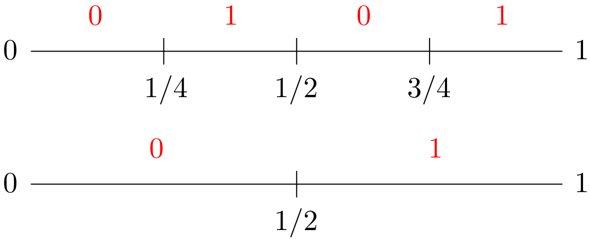


Preparation(continued)

Examples

- If $x \in [0,1]$ is a dyadic point, there are two ways to express:

$$\frac{1}{2} = .011111\cdots = .100000\cdots .$$

- If $x \in [0,1]$ is a rational number, the binary expansion is either terminating or repeating:

$$\frac{3}{8} = .011, \qquad \frac{1}{3} = .0101010101\cdots = .\overline{01}.$$

Lebesgue's singular function: unfair coin tossing

- Imagine flipping a coin.
- Suppose this coin has probability a of landing heads and probability $1-a$ of landing tails. $(a \neq 1/2)$
- Determine a number $t \in [0,1]$ by flipping the coin infinitely many times:

$$t = 0.\epsilon_1\epsilon_2\cdots = \sum_{k=1}^{\infty} \epsilon_k 2^{-k},$$

  where ϵ_k is 0 if the kth flip is heads, or 1 if it is tails.
- Define Lebesgue's singular function as the probability distribution

$$L_a(x) := Prob(t \leq x).$$

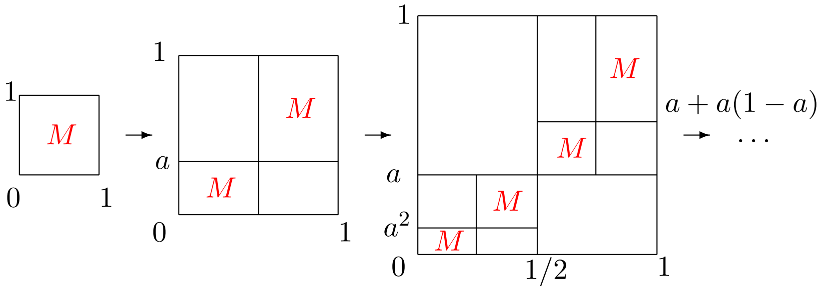
Functional equation

Theorem (De Rham, 1957)

$L_a(x)$ is the unique continuous solution of the functional equation

$$L_a(x) = \begin{cases} aL_a(2x), & 0 \leq x \leq \frac{1}{2}, \\ (1-a)L_a(2x-1)+a, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

where $0 < a < 1$, and $a \neq 1/2$.



Property of Lebesgue's singular function

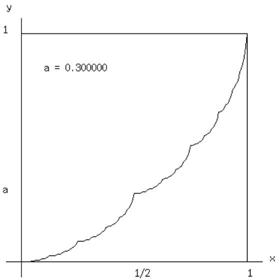


Figure: Lebesgue's singular function ($a = 0.3$)

Theorem (Salem, 1943)

$L_a(x)$ is strictly increasing and $L'_a(x) = 0$ almost everywhere.

Review

Definition (Differentiability)

A function f if differentiable at $x = a$, if there exist finite numbers M and N such that

$$f'_{+}(a) := \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h} = M,$$
$$f'_{-}(a) := \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h} = N,$$

and $M = N$.

Question 1

- Can I give a concrete example of $x \in [0,1]$ where $L_a'(x) = 0$?
- What set of $x \in [0,1]$ have $L_a'(x) = 0$?

Question

Question 1

- Can I give a concrete example of $x \in [0,1]$ where $L_a'(x) = 0$?
- What set of $x \in [0,1]$ have $L_a'(x) = 0$?

For instance,

- $x = 1/2$?
- $x = 1/3$?
- $x = 3/7$?

Density of 0 and 1 in the binary expansion of x

Definition

For $x = \sum_{k=1}^{\infty} \epsilon_k(x) 2^{-k}$, define

$$D_1(x) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \epsilon_k(x) = \lim_{n\to\infty} \frac{|\{1 \le k \le n : \epsilon_k = 1\}|}{n},$$

provided the limit exists. Put

$$D_0(x) := 1 - D_1(x) = \lim_{n\to\infty} \frac{|\{1 \le k \le n : \epsilon_k = 0\}|}{n}.$$

$D_i(x)$ is the density of the digit i in the binary expansion of x.

Examples

- $x = 1/3$?

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Examples

- $x = 1/3? \to x = .\overline{01}$

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Examples

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Examples

- $x = 1/3? \to x = .\overline{01} \to D_0(x) = D_1(x) = 1/2.$
- $x = 3/7? \to x = .\overline{011}$

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Examples

- $x = 1/3? \to x = .\overline{01} \to D_0(x) = D_1(x) = 1/2.$
- $x = 3/7? \to x = .\overline{011} \to D_0(x) = 1/3, D_1(x) = 2/3.$

Answer to Question 1

Theorem (Kawamura, 2009)

1. *If* $x \in [0,1]$ *is* dyadic, *then* $L_a'(x)$ *does not exist, since*

$$L_{a+}'(x) \neq L_{a-}'(x).$$

2. *If* $x \in [0,1]$ *is not dyadic and*
   - $D_0(x) = D_1(x)$, *then* $L_a'(x) = 0$.
   - $D_0(x) \neq D_1(x)$, *then*

$$L_a'(x) = \begin{cases} 0, & \text{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} < 1/2, \\ +\infty, & \text{if} \quad a^{D_0(x)}(1-a)^{D_1(x)} > 1/2. \end{cases}$$

Answer for Question 1 (cont.)

Examples

- $x = 1/2$?

Answer for Question 1 (cont.)

Examples

- $x = 1/2? \to 1/2$ is a dyadic point!

Answer for Question 1 (cont.)

Examples

- $x = 1/2?$ $\rightarrow$ $1/2$ is a dyadic point!
  $\rightarrow$ $L_a'(1/2)$ does not exist.
- $x = 1/3?$

Answer for Question 1 (cont.)

Examples

- $x = 1/2$? $\rightarrow$ $1/2$ is a dyadic point!
  $\rightarrow L_a'(1/2)$ does not exist.
- $x = 1/3$? $\rightarrow D_0(x) = D_1(x) = 1/2$.

Answer for Question 1 (cont.)

Examples

- $x = 1/2$? $\to$ $1/2$ is a dyadic point!
  $\to L_a'(1/2)$ does not exist.

- $x = 1/3$? $\to D_0(x) = D_1(x) = 1/2$.
  $\to L_a'(1/3) = 0$.

- $x = 3/7$?

Answer for Question 1 (cont.)

Examples

- $x = 1/2? \to 1/2$ is a dyadic point!
  $\to L'_a(1/2)$ does not exist.

- $x = 1/3? \to D_0(x) = D_1(x) = 1/2.$
  $\to L'_a(1/3) = 0.$

- $x = 3/7? \to D_0(x) = 1/3, D_1(x) = 2/3.$

Answer for Question 1 (cont.)

Examples

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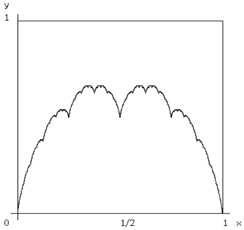
- $x = 3/7? \to D_0(x) = 1/3, D_1(x) = 2/3.$

$$\to L'_a(3/7) = \begin{cases} 0, & \text{if} \quad a^{1/3}(1-a)^{2/3} < 1/2, \\ +\infty, & \text{if} \quad a^{1/3}(1-a)^{2/3} > 1/2. \end{cases}$$

Takagi's nowhere differentiable function



Definition (Takagi's function, 1903)

$$T(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \phi^{(n)}(x), \quad 0 \leq x \leq 1,$$

where

$$\phi(x) := \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2 - 2x, & 1/2 \leq x \leq 1. \end{cases}$$

ϕ is a typical chaotic dynamical system on $[0, 1]$.

Question 2

We know that Takagi's function is nowhere differentiable.
But, at which $x \in [0,1]$ does $T(x)$ have an infinite derivative?

Note!

Recall that if $T'(x) = \pm\infty$, the graph of T has a vertical tangent line at x.

Answer to Question 2

- Let I_n and O_n be the number of 1's and 0's, respectively in the first n binary digits of x.
- Let $D_n := O_n - I_n$.

Theorem (Begle & Ayres, 1937)

1. *If $x \in [0,1]$ is dyadic, then $T'(x)$ does not exist, since*

$$T'_+(x) \neq T'_-(x).$$

2. *If $x \in [0,1]$ is not dyadic, then*

$$T'(x) = \begin{cases} +\infty, & \text{if } \lim_{n\to\infty} D_n = +\infty, \\ -\infty, & \text{if } \lim_{n\to\infty} D_n = -\infty, \\ \text{does not exist}, & \text{if } \lim_{n\to\infty} D_n \text{ does not exist.} \end{cases}$$

Composition of functions

Recall that in classical calculus, the chain rule is used to compute the derivative of the composition of two differentiable functions: If

$$h(x) = f(g(x)),$$

then

$$h'(x) = f'(g(x))g'(x).$$

Composition of functions

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Question 3

But what about a function such as $T(L_a^{-1}(x))$: the composition of Takagi's function and the inverse of Lebesgue's singular function?

Question

Put

$$f(x) = T(x), \qquad g(x) = L_a^{-1}(x),$$

and

$$h(x) = f(g(x)) = T(L_a^{-1}(x)).$$

If we try to use the chain rule to compute $h'(x)$, we may run into one of the indeterminate products

$$+\infty \cdot 0, \qquad \text{or} \qquad -\infty \cdot 0.$$

How can we overcome this problem?

Question

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Answer

Wait for my next paper, please!

Maximum points of $T(x)$

Question 4

Takagi's function $T(x)$ is nowhere differentiable. Then how can we find the maximum points?

Maximum points of $T(x)$

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Takagi's function $T(x)$ is nowhere differentiable. Then how can we find the maximum points?

Theorem (Kahane, 1959)

- *The maximum value of $T(x)$ is $2/3$.*
- *The binary expansion of any maximum point of $T(x)$ is*

$$x = .\begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} \cdots .$$

Theorem (Hata-Yamaguti, 1984)

Takagi's function and Lebesgue's singular function are related by

$$T(x) = \frac{1}{2} \left.\frac{\partial L_a(x)}{\partial a}\right|_{a=\frac{1}{2}}$$

Why not keep differentiating?

Relationship among two irregular functions

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Why not keep differentiating?

Definition (n-th partial derivatives of $L_a(x)$)

$$T_n(x) := \frac{1}{n!} \left. \frac{\partial^n L_a(x)}{\partial a^n} \right|_{a=\frac{1}{2}}, \qquad n = 1, 2, 3, \ldots.$$

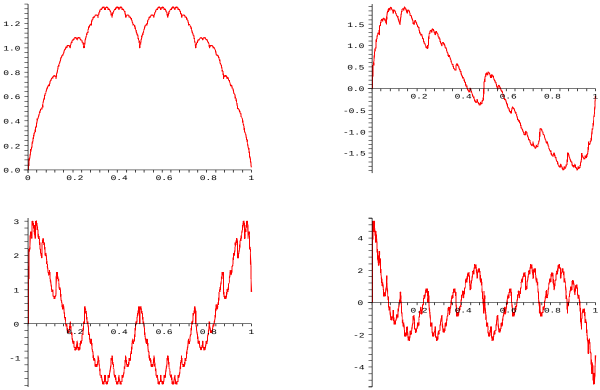


Figure: Graphs of T_1 (top left), T_2 (top right), T_3 (bottom left) and T_4 (bottom right).

Question

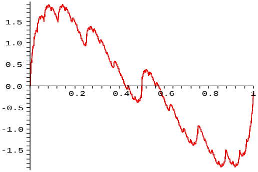
Theorem (Allaart-Kawamura, 2006)

For each $n \in \mathbf{N}$, $T_n(x)$ is continuous but nowhere differentiable.

Question 5

- What are the maximum values of T_n?
- Where are they attained?

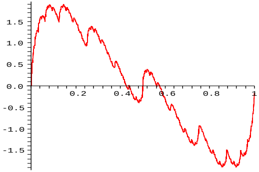
Maximum of T_2



Theorem (A-K, 2006)

The binary expansion of any maximum point of $T_2(x)$ has the form:

Maximum of T_2



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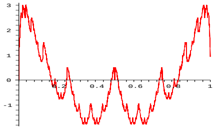
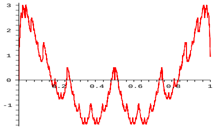
$$x = .00 \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} 010 \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} 01010 \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} 0101010 \begin{cases} 01 \\ \textit{or} \\ 10 \end{cases} 010101010 \cdots .$$

Maximum of T_3



Theorem (A-K, 2006)

The binary expansion of any maximum point of $T_3(x)$ *in* $[0, \frac{1}{2}]$ *has the form:*



Theorem (A-K, 2006)

The binary expansion of any maximum point of $T_3(x)$ *in* $[0, \frac{1}{2}]$ *has the form:*

$$x = .0000 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} 0 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{010}\,\underbrace{010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{01010}\,\underbrace{01010}$$

$$\begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{01010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{0101010}\,\underbrace{0101010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{0101010} \cdots.$$

Mysterious conjecture for T_n

Extensive numerical computation suggested that

Conjecture

For each $n \geq 4$, T_n has only finitely many maximum points!

Self-similar sets in the complex plane

Consider a unique bounded continous solution $G_\alpha(x)$ of

$$G_\alpha(x) = \begin{cases} \alpha G_\alpha(2x), & 0 \leq x < 1/2, \\ (1-\alpha)G_\alpha(2x-1)+\alpha, & 1/2 \leq x \leq 1, \end{cases}$$

where $\alpha \in \mathbf{C}$ such that $|\alpha| < 1$.

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Remark

If $\alpha = a$, a real number, then

$$G_a(x) = L_a(x), \qquad 0 < a < 1, \quad a \ne 1/2.$$

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Properties of G_α

- $G_\alpha : [0,1] \to \mathbf{C}$,
- $\overline{G_\alpha([0,1])}$: self-similar set on $\mathbf{C}$.

Example: if $\alpha = 1/2 + 1/2i$, $\overline{G_\alpha([0,1])}$ is Levy's dragon curve.

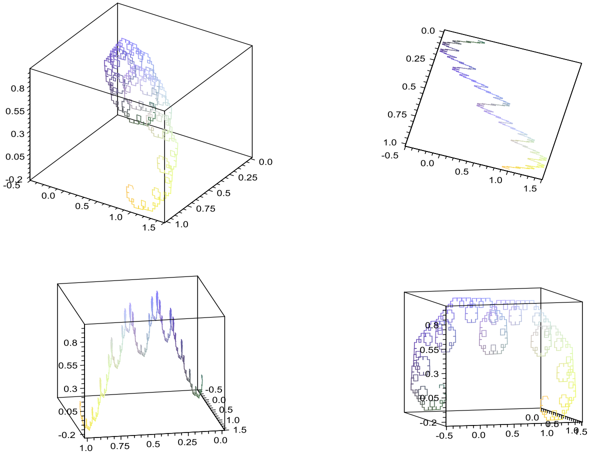

Figure: Graphs of G_α (top left), view from the top (top right), from a different angle (bottom left), Levy dragon curve (bottom right).

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Each coordinate function is somewhat similar to Takagi's function!

This is no surprise, because...

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Theorem (K, 2003)

$$\mathrm{Im}\left.\frac{\partial G_\alpha(x)}{\partial \alpha_I}\right|_{\alpha_R=1/2} = 2T(x).$$

What about my dream?!?!

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Brain science (Tsuda and Yamaguchi, 2006)

Self-similar functions whose graphs look remarkably like $G_\alpha(x)$ appear in neural systems!

Message to future female scientists


We Can Do It!