# Carleman factorization of layer potentials on smooth domains: A novel derivation of geometric formulas

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Papers:

- [1] K. Ando, H. Kang, Y. Miyansihi and M. Putinar, CARLEMAN FACTORIZATION OF LAYER POTENTIALS ON SMOOTH DOMAINS, Archive for Rational Mechanics and Analysis (2025)
- [2] Y. Miyanishi, Circle foliations by geodesics revisited: Periods of flows whose orbits are all closed, International Journal of Geometric Methods in Modern Physics (2025)

# §1. An inequality (A nontrivial inequality)

**Theorem (A nontrival inequality (K. Ando, H. Kang, M. Putinar and Y. M., ARMA, (2025))**

*Let* $\Omega \subset \mathbb{R}^3$ *be a smooth convex bounded region.*

$$\kappa_- \leq \left(\frac{3W(\partial\Omega) - 4\pi}{2\mathrm{Area}(\partial\Omega)}\right)^{1/2} \leq \kappa_+ \tag{1}$$

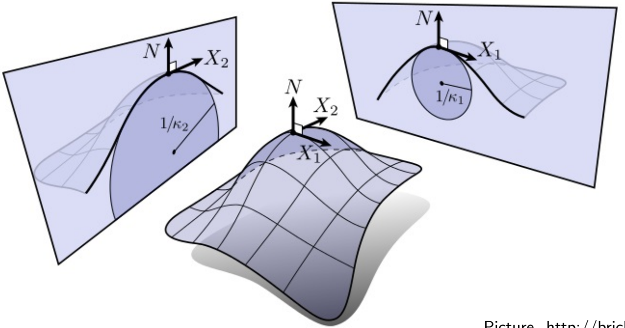
(*The equality holds iff* $\partial\Omega = S^2$ ). *Here* $\kappa_-$ *and* $\kappa_+$ *resp. denote the minimum and the maximum of principal curvatures. The notation* $W(\partial\Omega) := \int_{\partial\Omega} H^2(x)\, dS_x$ *denotes Willmore energy with* $H(x) = \frac{\kappa_1(x)+\kappa_2(x)}{2}$ *(mean curvarture).* $\mathrm{Area}(\partial\Omega)$ *is the area of the surface.*

This theorem is proved uniquely by Spectral Theory!?

# §1. Mean curvature and Willmore energy (Supplements)

Mean curvature is the average of principal curvatures

$$\boldsymbol{H(x) = \frac{\kappa_1(x) + \kappa_2(x)}{2}}$$




Picture. http://brickisland.net

The Willmore energy $\boldsymbol{W(\partial\Omega) = \int_{\partial\Omega} H^2(x)\, dS_x}$

is invariant under Möbius transforms. W. Blaschke (1929) etc.

# §1. A fundamental proof of $\kappa_- \leq \left(\frac{3W(\partial\Omega)-4\pi}{2\text{Area}(\partial\Omega)}\right)^{1/2}$

Inequality of arithmetic and geometric mean relationships

$$\sqrt{K(x)} = \sqrt{\kappa_-(x)\kappa_+(x)} \leq \frac{\kappa_+(x)+\kappa_-(x)}{2}.$$

$$\begin{aligned}
&2\kappa_-^2\,\text{Area}(\Gamma) \leq 2\int_\Gamma \kappa_-(x)^2 dS_x \\
&\leq \int_\Gamma 3\left(\frac{\kappa_+(x)+\kappa_-(x)}{2}\right)^2 dS_x - \int_\Gamma \left(\frac{\kappa_+(x)+\kappa_-(x)}{2}\right)^2 dS_x \\
&\leq \int_\Gamma 3\left(\frac{\kappa_+(x)+\kappa_-(x)}{2}\right)^2 dS_x - \int_\Gamma K(x) dS_x \\
&= 3W(\Gamma) - 4\pi \ \cdots (\text{Gauss-Bonnet Theorem}).
\end{aligned}$$

Thus $\kappa_- \leq \left(\frac{3W(\partial\Omega)-4\pi}{2\text{Area}(\partial\Omega)}\right)^{1/2}$. But $\left(\frac{3W(\partial\Omega)-4\pi}{2\text{Area}(\partial\Omega)}\right)^{1/2} \leq \kappa_+$ Easy??

# §2. A nontrivial inequality $\left(\frac{3W(\partial\Omega)-4\pi}{2\mathrm{Area}(\partial\Omega)}\right)^{1/2} \leq \kappa_+$ ?

[(Outline of the proof. (Step1) )]

Single layer potential: $S[\psi](x) \equiv \int_{\partial\Omega} E(x-y)\psi(y)dS_y$
Double layer potential: $\mathcal{K}_{\partial\Omega}[\psi](x) \equiv \int_{\partial\Omega} \psi(y)\frac{\partial}{\partial n_y}E(x,y)\ dS_y$,
where $\frac{\partial}{\partial n_y}$ denotes the outer normal derivative on $\partial\Omega$,

$$E(x,y) = \begin{cases} \frac{1}{2\pi}\log|x-y|, & \text{if } n=2, \\ \frac{-1}{4\pi}\frac{1}{|x-y|}, & \text{if } n=3 \end{cases}$$

$\mathcal{S}_{\partial\Omega},\ \mathcal{K}_{\partial\Omega} \in \Psi^{-1}(\partial\Omega)$. $A = \mathcal{S}_{\partial\Omega}^{-1}\mathcal{K}_{\partial\Omega}$ is self-adjoint in $L^2$. Thus $\mathcal{K}_{\partial\Omega} = \mathcal{S}_{\partial\Omega}A$ (Carleman Factorization (Product of self-adjoint op.)).

# §2. A nontrivial inequality $\left(\frac{3W(\partial\Omega)-4\pi}{2\mathrm{Area}(\partial\Omega)}\right)^{1/2} \leq \kappa_+$ ?

Denoting $\sigma_{ess}(A) = [\tau_-, \tau_+]$, we have

$$\tau_- \leq \liminf_{n\to\infty} \frac{\sigma_n(\mathcal{K}_{\partial\Omega})}{\sigma_n(\mathcal{S}_{\partial\Omega})} \leq \limsup_{n\to\infty} \frac{\sigma_n(\mathcal{K}_{\partial\Omega})}{\sigma_n(\mathcal{S}_{\partial\Omega})} \leq \tau_+. \qquad (2)$$

Here, $\sigma_n(\mathcal{K}_{\partial\Omega})$ and $\sigma_n(\mathcal{S}_{\partial\Omega})$ denote the $n$-th singular value., namely, the $n$-th eigenvalue of $\sqrt{\mathcal{K}^*_{\partial\Omega}\mathcal{K}_{\partial\Omega}}$. It is know that

$\sigma_n(\mathcal{K}_{\partial\Omega}) \sim \left\{\frac{3W(\partial\Omega)-2\pi\chi(\partial\Omega)}{128\pi}\right\}^{1/2} n^{-1/2}$ and

$\sigma_n(\mathcal{S}_{\partial\Omega}) \sim \left\{\frac{\mathrm{Area}\partial\Omega}{64\pi}\right\}^{1/2} n^{-1/2}.$

Hence

$$\tau_- \leq \left(\frac{3W(\partial\Omega)-4\pi}{2\mathrm{Area}(\partial\Omega)}\right)^{1/2} \leq \tau_+.$$

[(Outline of the proof. (Step 2))] $A = \mathcal{S}_{\partial\Omega}^{-1}\mathcal{K}_{\partial\Omega} \in \Psi^0(\partial\Omega)$ and the principal symbol $\sigma_0(A)$ is

$$\sigma_0(A)(x,\xi) = \frac{L\xi_1^2 - 2M\xi_1\xi_2 + N\xi_2^2}{E\xi_1^2 + 2F\xi_1\xi_2 + G\xi_2^2}$$

I$= Edu^2 + 2Fdudv + Gdv^2$ and II$= Ldu^2 + 2Mdudv + Ndv^2$ resp. denote the first - and the second fundamental form.
It follows from M. Adams's Theorem (JFA (1983)) that

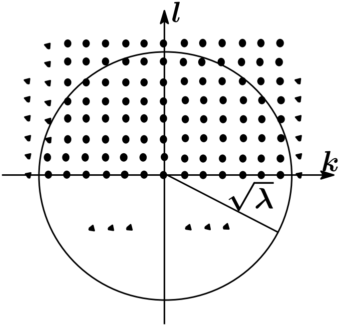
$$\begin{aligned}[\tau_-, \tau_+] &= \sigma_{ess}(A) = \mathrm{Ran}(\sigma_0(A)) \\ &= \{\sigma_0(A)(x,\xi) |\ (x,\xi) \in \partial\Omega \times S^2\} \\ &= [\kappa_-, \kappa_+]. \qquad \square\end{aligned}$$

# §3. What I intend to do

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2 = [0, 2\pi]_x \times [0, 2\pi]_y$ be a flat torus and $-\triangle = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}\right)$. Then the set of eigenvalues is denoted as

$$\sigma_p(-\triangle) = \{\lambda_{k,l} = k^2 + l^2 |\ k, l \in \mathbb{Z}\}.$$

The counting number $N(\lambda) := \sharp\{(k, l) \in \mathbb{Z}^2 |\ \lambda_{k,l} = k^2 + l^2 < \lambda\}$.




Gauss circle problem
(Lattice point problem)

**Conjecture**

$N(\lambda) \sim \pi\lambda + O(\lambda^{1/4+\varepsilon})$ as $\lambda \to \infty$?

**Theorem (Spectral theory)**

$N(\lambda) \sim \pi\lambda + o(\lambda^{1/2})$.

**Theorem (Number theory (Huxley))**

$N(\lambda) \sim \pi\lambda + O(\lambda^{131/416+\varepsilon})$.

# §3 What I intend to do

(1) For $S \subset \mathbb{C}$, find a suitable linear operator whose spectrum is $S$.

(2) Spectrum analysis yields the relieved explanation of the behavior of $S$.

**Theorem (Wedderburn (Fundamental theorem of algebra)(Matrix))**

*Let $K$ be a field and $f$ be a square-free irreducible polynomial. Then $\exists A \in M_n(K)$ s.t. $f(A) = O$.*

**Example**

(Ex.1) $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in M_2(\mathcal{Q})$. Then $A^2 - 2I = O$.

(Ex.2) $\nexists A \in M_2(\mathcal{Q})$, but $\exists A \in M_3(\mathcal{Q})$ s.t. $A^3 - 2I = O$.

(Ex.3) Find $A \in M_5(\mathcal{Q})$ s.t. $A^5 - 5A + I = O$

**Theorem (due esseintially to G. W. Mackey)**

*$S \subset \mathbb{C}$. Let $V$ be an infinite dimensional Banach space over $\mathbb{R}$. $\exists A(\textbf{a linear operator on } V)$ s.t. $\sigma_p(A) = S$.*

(Ex.3)

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

satisfies $A^5 - 5A + I = O$.

N.B. Gal$(\mathcal{Q}(\{\alpha_i |\ \alpha_i^5 - 5\alpha_i + 1 = 0 (i = 1, \ldots, 5)\})/\mathcal{Q}) = S_5$

**Theorem**

*A concrete solution formula of quintic equation with rational coefficients can be constructed via matrices (Linear operators).*

Ralations ?

- V. F. Klein, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vomfünften Grade (正 20 面体と 5 次方程式)
- Modular forms (C. Hermite)

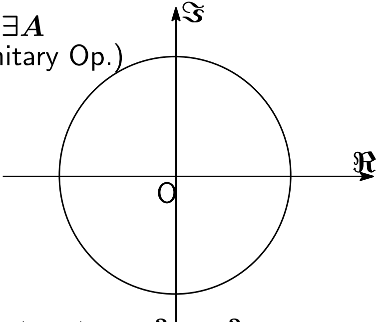
# §3. What I intend to do (Supplement)

**Theorem (Burchnall, Chaundy, Krichever, etc.)**

*Let* $V = L^2(\mathbb{R})$ *and* $f(x, y)$ *be a suitable real polynomial with 2-variables. Then*
$\exists A$(**a concrete linear operator on** $V$) *s.t.*

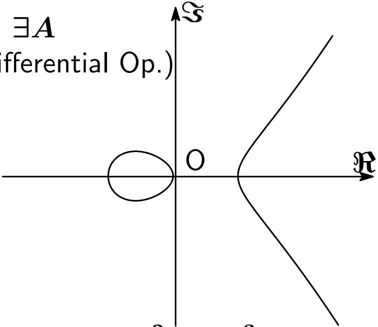
$$\sigma(A) = \{x + iy \in \mathbb{C} \mid f(x, y) = 0\}.$$

$\exists A$
(Unitary Op.)




$f(x, y) = x^2 + y^2 - 1$

$\exists A$
(Differential Op.)




$f(x, y) = y^2 - 4x^3 + g_2 x + g_3$

# §3. What I intend to do (Supplement)

As for (2), for the metric $g(x) = x^{-2}(1 + |\log x|^2)^{\delta}$ (the volume form is $\sqrt{g(x)}dx$),

$$\mathcal{Q}(xp)f(x) = \frac{1}{i}xf'(x) + \frac{\delta}{2i}\frac{\log x}{(1 + |\log x|^2)}f(x).$$

The operator $\mathcal{Q}(xp)$ is self-adjoint on $L^2_\delta = L^2(\mathbb{R}_+, \sqrt{g(x)}dx)$.

## Definition (Theta sum)

Let $\varphi \in C^2(\mathbb{R}_+)$ be a function such that

$$\int_0^\infty \varphi(x)\, dx = 0, \quad \varphi(0) = 0, \quad |\varphi(x)| \le \frac{c}{1 + |x|^2} \quad \text{as } |x| \to \infty.$$

Define the theta sum of $\varphi$ by

$$\theta(\varphi)(x) := x^{1/2} \sum_{n \neq 0} \varphi(nx). \tag{3}$$

Especially, for the metric $g(x) = x^{-2}(1 + |\log x|^2)^{\delta}$ (the volume form is $\sqrt{g(x)}dx$),

**Theorem (G. Lachaud (2003) due essentially to A. Connes)**

$$\mathcal{Q}(xp)f(x) = \frac{1}{i}xf'(x) + \frac{\delta}{2i}\frac{\log x}{(1 + |\log x|^2)}f(x).$$

*and*

$$\Theta := \overline{\{\theta(\varphi)(x) \mid \varphi(x) \in \mathcal{S}_0(\mathbb{R}_+), \int_{\mathbf{R}_+} \varphi = 0, \text{and } \varphi(0) = 0)\ \}}^{L^2_\delta}.$$

*Then*

$$\sigma_p(\mathcal{Q}(xp)) = \mathcal{F} \quad \text{on } \mathcal{H}_\delta := L^2_\delta/\Theta \quad (0 < \delta < 3) \tag{5}$$

*where*

$$\mathcal{F} := \{\gamma \in \mathbb{R} \mid \zeta(\frac{1}{2} + i\gamma) = 0\}.$$

# §3. What I intend to do (A Short Summary)

The obtained results in Spectral theory are weaker than those in Number theory.

# §4 Weyl's formula (n=3)

(Notation)

$$\boldsymbol{H(x)} \text{ : Mean curvature of } \boldsymbol{\partial\Omega}$$

$$\boldsymbol{\chi(\partial\Omega)} \text{ : Euler characteristics of } \boldsymbol{\partial\Omega}$$

Under these notations, we have

**Theorem (Weyl's law for $n = 3$ (Y. M., Adv. Math. (2022)) )**

*Let $\Omega$ be a $C^{k,\alpha}$ bounded domain in $\mathbb{R}^3$ with $k \geq 2$ and $\alpha > 0$. Then*

$$|\lambda_j(\mathcal{K}_{\partial\Omega})| \sim \left\{\frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{128\pi}\right\}^{1/2} j^{-1/2} \quad \text{as } j \to \infty.$$

*Here $W(\partial\Omega) := \int_{\partial\Omega} H^2(x)\, dS_x$ (Willmore energy).*

Thus the principal part of decay rate is independent of the boundary smoothness.

# §4. $1/2$ conjecture (Numerical computations)

(The meaning)
Consider the surface $\partial\Omega := \{(x, y, z) \mid x^4 + y^4 + 2z^2 = 1\} \subset \mathbb{R}^3$.
Then the eigenvalues of $\mathcal{K}_{\partial\Omega}$ are numerically given as

$$
\begin{aligned}
&0.5\\
&0.332\cdots, 0.211\cdots, -0.043\cdots\\
&0.30\cdots, 0.22\cdots, 0.12\cdots, -0.11\cdots, -0.03\cdots\\
&\quad\cdots
\end{aligned}
$$

The sum of each subsequence is

$$\frac{1}{2}, \ \frac{1}{2}, \ \frac{1}{2}, \cdots$$

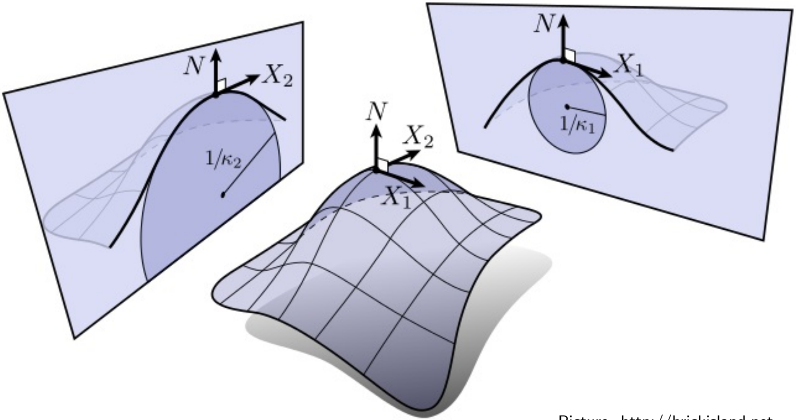
Can one prove this? (only for ellipsoids?)
Furthermore the sequence consisting of order absolute values $\{a_j\}$ is

$$0.5, \ 0.322\cdots, \ 0.30\cdots, \ 0.22\cdots, \ 0.211\cdots, \cdots$$

The decay rate: $a_j \succeq \frac{1}{4}j^{-1/2}$. (Partitioning problem)

Mean curvature is the average of principal curvatures

$$\kappa_1(x) + \kappa_2(x)$$




Picture. http://brickisland.net

The Willmore energy $\boldsymbol{W(\partial\Omega) = \int_{\partial\Omega} H^2(x)\, dS_x}$
is invariant under Möbius transforms !! W. Blaschke (1929) etc.

# §4. Meanings of Weyl's law (supplements)

(Example 3) ($n = 3$, Sphere)

$$\partial\Omega = S^2 \iff W(S^2) = 4\pi \text{ and } \chi(S^2) = 2.$$

Thus

$$|\lambda_j(\mathcal{K}_{S^2})| \sim \left\{ \frac{3W(S^2) - 2\pi\chi(S^2)}{128\pi} \right\}^{1/2} j^{-1/2} = \frac{1}{4} j^{-1/2}$$

Conversely if we know the asymptotics of eigenvalues,

## Corollary

*Let $\Omega \subset \mathbb{R}^3$ be a bounded region of class $C^{2,\alpha}$.*

$$|\lambda_j(\mathcal{K}_{\partial\Omega})| \sim \frac{1}{4} j^{-1/2} \iff \partial\Omega = S^2.$$

*Especially*

$$\sigma_p(\partial\Omega) = \sigma_p(S^2) \iff \partial\Omega = S^2.$$

# §5. Algebra (Proved uniquely by Spectral Theory?)

## Definition (Odd Partitions (e.g. V. Drobot, Trans. AMS (1981).))

For $n \in \mathrm{N}_{\geq 0}$, let $\{a_{n,k}\}_{k=1,2,\ldots,2n+1}$ be a set of positive numbers s.t.

$$\sum_{k=1}^{2n+1} a_{n,k} = 1/2.$$

Union of such sequences is a countable set
$S = \{a_{n,k} |\ n \in \mathrm{N}_{\geq 0}, k = 1, \ldots, 2n+1, \sum_{k=1}^{2n+1} a_{n,k} = 1/2\}$.
$S$ is called odd partitions of an interval $1/2$.

## Theorem (An existence of partitions (explained in Spectral Theory))

*For* $\forall C \geq 1/4$. *there exists an odd partition of an interval* $1/2$ *s.t. the enumeratied (decreasing ordered) sequence satisfies*

$$a_j \sim C j^{-1/2} \quad \text{as } j \to \infty$$

$C = 1/4$ *iff the partition is almost surely equi-partition.*

Thank you for your attention!