空間3次元における双曲型方程式のストリッカーツ評価式について

第3回 奈良女大・埼大 関数不等式研究

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§1 Introduction

$\boxed{\text{Wave equation}}$

$$\begin{cases} \square u(t,x) = F(t,x) & ; \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0,x) = f(x) & ; \quad x \in \mathbb{R}^d \\ u_t(0,x) = g(x) & ; \quad x \in \mathbb{R}^d \end{cases}$$

$\Longleftrightarrow$

$$u = E'(t) f + E(t) g + \int_0^t E(t-s) F(s)\, ds$$

where $E(t) = \dfrac{\sin t|D_x|}{|D_x|}$, $E'(t) = \cos t|D_x|$

Strichartz estimate (with $F=0$, $g=0$)

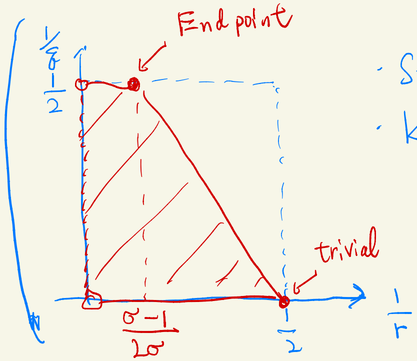
$$\| u(t,x) \|_{L^q_t L^r_x} \lesssim \| f \|_{\dot{H}^\alpha}$$

if

- (q,r) is wave adimissible

$$\Longleftrightarrow \quad q, r \geq 2 \quad r \neq \infty$$

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}, \text{ where } \sigma = \frac{d-1}{2}$$



- Strichartz 1977
- Keel-Tao 1999

(wave-admissible)

- Gap condition $\quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \alpha$

Rem · It is fundamental in nonlinear analysis!

Strichartz est. > Energy est.

↖ it contains more information

· Schrödinger eq. has similar estimate

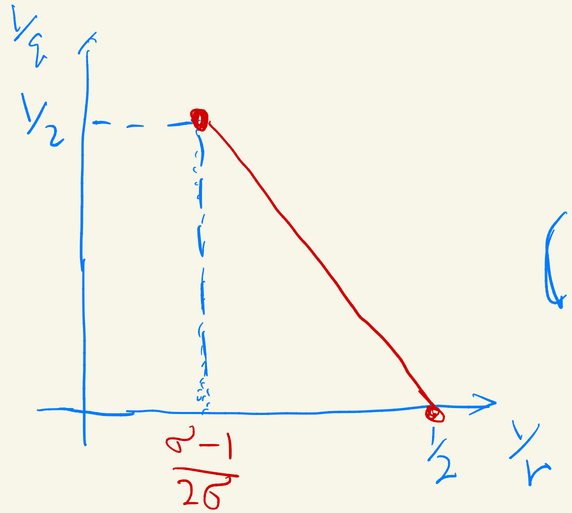
$$E(t) = e^{it\Delta}$$

$$\Rightarrow \|E(t)g\|_{L^q_t L^r_x} \lesssim \|g\|_{L^2} \qquad \frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}$$

$$\sigma = \frac{d}{2}$$

if (q, r) is Schrödinger admissible

$$\Leftrightarrow \frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}, \quad \sigma = \frac{d}{2}, \quad (q, r, \sigma) \neq (2, \infty, 1)$$



(Schrödinger admissible)

$\boxed{\text{Question}}$ Does Strichartz estimate hold for general hyperbolic equations ?

System (2nd order)

matrix

$$(D_t^2 - A_2(D_x))U(t,x) = 0$$

ex Elastic wave equation $(n=3)$

$$3\times 3 \rightarrow A_2(D_x) = \left(A_{ij}(D_x)\right) : A_{ij}(D_x) = \sum_{p,q=1}^{3} C_{ijpq} D_{x_p} D_{x_q}$$

$$C_{ijpq} = C_{jipq} = C_{ijqp} = C_{pqij}$$

System (1st order)

$$(D_t - A_1(D_x))\, u(t,x) = 0$$

ex **Maxwell equations**

6×6

$$A_1(D_x) = \frac{1}{i}\begin{pmatrix} 0 & \varepsilon^{-1}\,\mathrm{curl} \\ -\mu^{-1}\,\mathrm{curl} & 0 \end{pmatrix}$$

$$\varepsilon = \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix}, \quad \mu = \begin{pmatrix} \nu & & \\ & \nu & \\ & & \nu \end{pmatrix}$$

Single (higher order)

$$P(D_t, D_x)\, u(t,x) = 0$$

$$P(D_t, D_x) = (D_t - \lambda_1(D_x)) \cdots (D_t - \lambda_m(D_x))$$

real symbol

(In an ideal situation)

These equations are reduced to single 1-st order eq.:

Wave: $\lambda(\xi)=|\xi|$

$$\boxed{\begin{cases} \left(D_t \mp \lambda(D_x)\right) u(t,x)=0 \\ u(0,x)=f(x) \end{cases}}$$

$\lambda(\xi)>0$, homogeneous of order 1

$$\Longleftrightarrow \quad u(t,x)=E(t)f, \quad \underline{E(t)=e^{\pm it\lambda(D_x)}}$$

<u>Rem</u> Stricharts estimate for Wave equation

$$\Longleftrightarrow \boxed{\|E(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^\alpha} \qquad \lambda(\xi)=|\xi|}$$

$$\begin{cases} \circ\ (q,r) \text{ is } \underline{\text{wave adimissible}} \\ \iff \quad q, r \geq 2 \quad r \neq \infty \\ \qquad \frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}, \text{ where } \sigma = \frac{d-1}{2} \\ \circ\ \underline{\text{Gap condition}} \quad \frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \alpha \end{cases}$$

Answer to Question

Yes, but for different σ depending on a geometry of

$$\boxed{\Sigma_\lambda := \{ \xi \mid \lambda(\xi) = 1 \}}$$

§2 Reduction to L^p-$L^{p'}$ estimates

Let $E(t) = e^{it\lambda(D_x)}$

L^p-$L^{p'}$-estimate $\quad \left(1 = \frac{1}{p} + \frac{1}{p'},\ 1 < p \leq 2\right)$

$$\| |D_x|^{-\alpha} E(1) \|_{L^p \to L^{p'}} \lesssim 1 \quad \cdots \text{☆}$$

for $\alpha \geq \alpha(p) \left(= (2d - \rho)\left(\frac{1}{p} - \frac{1}{2}\right) \right)$

def of ρ

$\Longrightarrow$ Strichartz estimate with $\sigma = \frac{\rho}{2}$

(Scalling + TT^*)

Known results

$\left(\text{e.g. } \lambda(\xi) = |\xi| \leftrightarrow \Sigma = S^{n-1}\right)$

① $\rho = d-1$ when Gaussian curvature of Σ never vanishes ※ (Brenner 1975)

② $\rho = \min\limits_{\xi \neq 0} \operatorname{rank} \lambda''(\xi)$ (Brenner 1977)

(Note! ※ $\Rightarrow \rho = d-1$)

③ $\rho = \dfrac{2(d-1)}{\gamma(\Sigma)}$ when Σ is convex (S. 1994)

$\gamma(\Sigma)$: maximal order of contact to its tangent hyperplane

(Note! ※ $\Rightarrow \gamma(\Sigma) = 2$)

example

$$\lambda(\xi) = \left(\xi_1^{2N} + \cdots + \xi_d^{2N}\right)^{1/2N} \qquad N = 1, 2, \cdots$$

$\Rightarrow$

- $\rho = \text{min rank}\, \phi''(\xi) = \begin{cases} d-1 & (N=1) \\ 0 & (N \geq 2) \end{cases}$

- $\rho = \dfrac{2(d-1)}{\underbrace{\gamma(\Sigma)}_{= 2N}} = \dfrac{d-1}{N}$ $\gg$ better result

Note: . This example attains the optimal case in 3

- Σ is convex

§3 Main result

- We consider the case when Σ is non-convex
- In this case, ρ depends on the geometry of Σ more delicately

① Assume $d=3$. Let Σ have a local coordinate

$$\boxed{\{(y, h(y)) \in \mathbb{R}^3 \mid y=(y_1, y_2) \in U : \text{nbd of } (0,0)\}}$$

near $(0, 0, h(0)) \in \Sigma$

Assume also $\underline{\operatorname{rank} h''(0,0) \neq 0}$

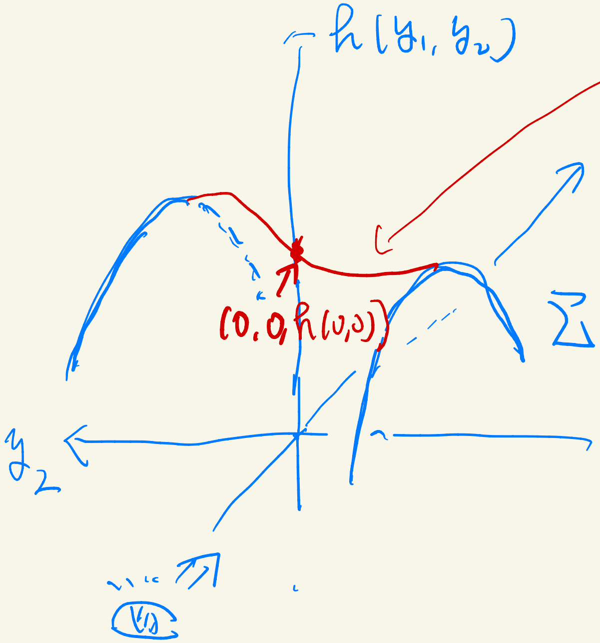
<u>Rem</u> Gaussian curvature of $\Sigma \neq 0$

$\Longleftrightarrow \det h''(0,0) \neq 0$

$\Longrightarrow \operatorname{rank} h''(0,0) \neq 0$

$\not\Longleftarrow$

For simplicity, we assume $\underline{h_{11}''(0,0) \neq 0}$



"ridge" when we see Σ in the direction parallel to y_1-axis

$$\boxed{\{(a_1(y_2), y_2, a_0(y_2)) \mid y_2 \approx 0\}}$$

where $\begin{cases} h_1'(a_1(y_2), y_2) = 0, \\ a_0(y_2) = h(a_1(y_2), y_2) \end{cases}$

Let $m :=$ smallest $k \geq 2$ s.t. $a_0^{(k)}(0) \neq 0$

$n :=$ smallest $k \geq 2$ s.t. $a_1^{(k)}(0) \neq 0$

$\chi \in C^\infty(\mathbb{R}^n \setminus 0)$: hom. of order 0

s.t. $\operatorname{supp} \chi \subset$ small conic n.b.d of $(0, 0, h(0,0))$

Better than Brenner's result

Th (S.-I. Ikromov) $n = 3$, $1 < p \leq 2$, $1 = \frac{1}{p} + \frac{1}{p'}$

$$\| \, |D_x|^{-\alpha} \chi(D_x) E(1) \|_{L^p \to L^{p'}} \lesssim 1 \iff \alpha \geq \alpha(p)$$

where

$$\alpha(p) = \begin{cases} \left(5 - \frac{2}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) & 2m \geq n \\ \left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right) \vee \left(6 - \frac{2(m+1)}{n}\right)\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} + \frac{m}{n} & \text{for } 2m < n \quad (m \geq 3) \\ \left(5 - \frac{1}{m}\right)\left(\frac{1}{p} - \frac{1}{2}\right) \vee 6\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2} & \text{for } 2m < n \quad (m = 2) \end{cases}$$