多強度点渦系平均場方程式の解の存在と一意性

2026. 06. 14

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Keller-Segel system

1970

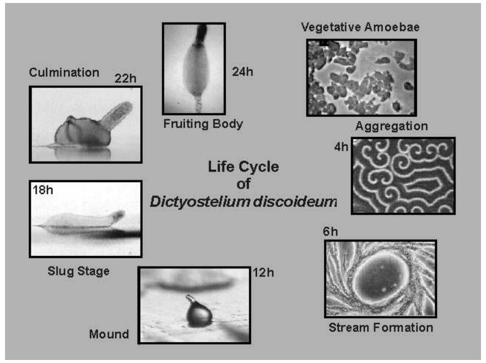
$$u_t = \nabla\cdot(d_1(u,v)\nabla u) - \nabla\cdot(d_2(u,v)\nabla v)$$
$$v_t = d_v\Delta v - k_1 vw + k_{-1}p + f(v)u$$
$$w_t = d_w\Delta w - k_1 vw + (k_{-1}+k_2)p + g(v,w)u$$
$$p_t = d_p\Delta p + k_1 vw - (k_{-1}+k_2)p$$

$u = u(x,t)$ cellular slime molds

$v = v(x,t)$ attractant

$w = w(x,t)$ enzyme

$p = p(x,t)$ complex



1. transport, gradient
   (a) diffusion u, v, w, p
   (b) chemotaxis $v \to u$
2. production $u \to (v, w)$
3. chemical reaction v, w, p

$$V + W \ \underset{\leftarrow (k_{-1})}{\overset{(k_1)\to}{}} \ P \ \underset{\to}{\overset{(k_2)}{}} \ W + A$$

$$v_t = -k_1 vw + k_{-1}p$$ mass action
$$w_t = -k_1 vw + (k_{-1}+k_2)p$$
$$p_t = k_1 vw - (k_{-1}+k_2)p$$

Reductions

Nanjundiah 73

Michaelis-Menten

$$k_1 vw - (k_{-1}+k_2)p = 0$$ quasi-static
$$w + p = c$$ total mass conservation

$$u_t = \nabla\cdot(d_1(u,v)\nabla u) - \nabla\cdot(d_2(u,v)\nabla v)$$
$$v_t = d_v\Delta v - k(v)v + f(v)u$$

parabolic-parabolic system of chemotaxis

$$k(v) = \frac{ck_1k_2}{(k_{-1}+k_2)+k_1 v}$$

Childress-Percus 81

$d_1(u,v),\ k(v),\ f(v)$ constant

$d_2(u,v) = u\chi'(v)$ constant sensitivity

sensitivity

Jager-Luckhaus 92

short term approximation

$$u_t = \nabla\cdot(\nabla u - u\nabla v), \quad \left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

Smoluchowski

Poisson

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \quad \int_\Omega v = 0, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

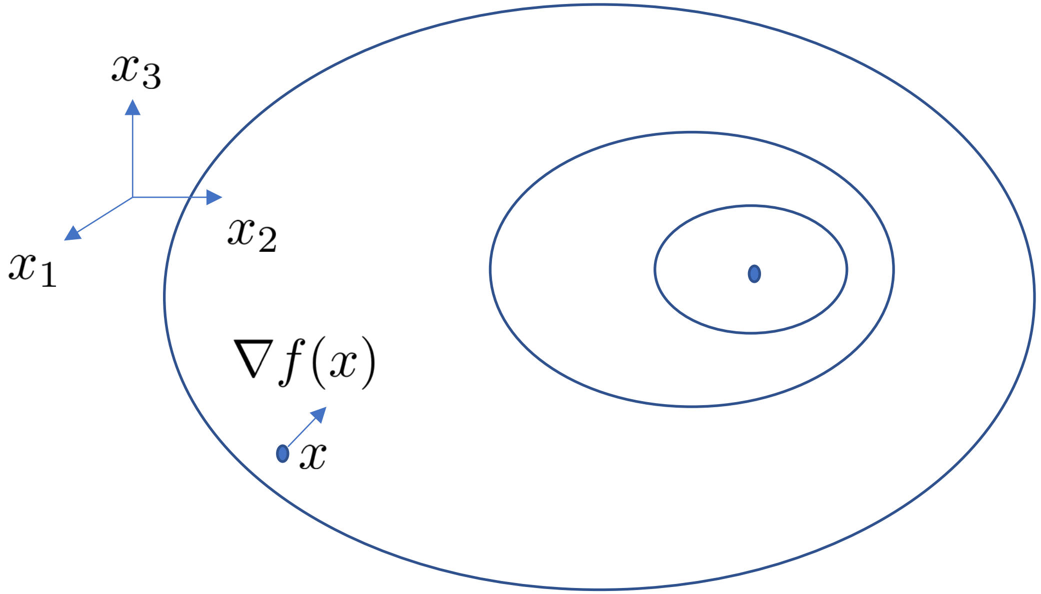
parabolic-elliptic system of chemotaxis

gradient

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)^T$$

$$\left.\frac{d}{ds} f(x+se)\right|_{s=0} = \nabla f(x) \cdot e$$

$f(x)$ scalar field (chemical concentration)



maximize $\nabla f(x) \cdot e$ $\quad e \in \mathbf{R}^3,\ |e| = 1$

$\longrightarrow$ $e = \dfrac{\nabla f(x)}{|\nabla f(x)|}$ $\quad \nabla f(x) \cdot e = |\nabla f(x)|$

$$|\nabla f(x)| e = \nabla f(x)$$

flux

particle density $\quad \dfrac{d}{dt}\displaystyle\int_\Omega \rho = -\int_{\partial\Omega} \nu \cdot j = -\int_\Omega \nabla \cdot j$ $\longrightarrow$ $\rho_t = -\nabla \cdot j$

(Euler coordinate; Gauss; flux; mass conservation; **material transport**)

velocity $\quad v = v(x) \in \mathbf{R}^3,\ x \in \mathbf{R}^3$

flow

$$\frac{d}{dt} T_t x = v(T_t x)$$

$$T_t x|_{t=0} = x$$

Liouville $\longrightarrow$ Jacobian $\quad \left.\dfrac{d}{dt} \det DT_t\right|_{t=0} = \nabla \cdot v$

$$\left.\frac{d}{dt}\int_{\Omega_t} \rho\right|_{t=0} = \left.\int_\Omega \rho_t + v \cdot \nabla\rho + \rho\nabla \cdot v \, dx\right|_{t=0}$$

Lagrange coordinate

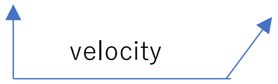
$\Omega_t = T_t \Omega$ $\quad = \left.\displaystyle\int_\Omega \rho_t + \nabla \cdot \rho v \, dx\right|_{t=0} = 0$

$\longrightarrow$ $j = \rho v$ $\quad$ flux =mass×velocity (momentum)

$j = -d(u)\nabla u$ diffusion

$j = d(u, f) u \nabla f$ chemotaxis

concentration gradient



mean waiting time $\quad \tau = \dfrac{(\Delta x)^2}{2ND}$ $\quad$ **Einstein formula**

(mean jump length; space dimension; diffusion coefficient)

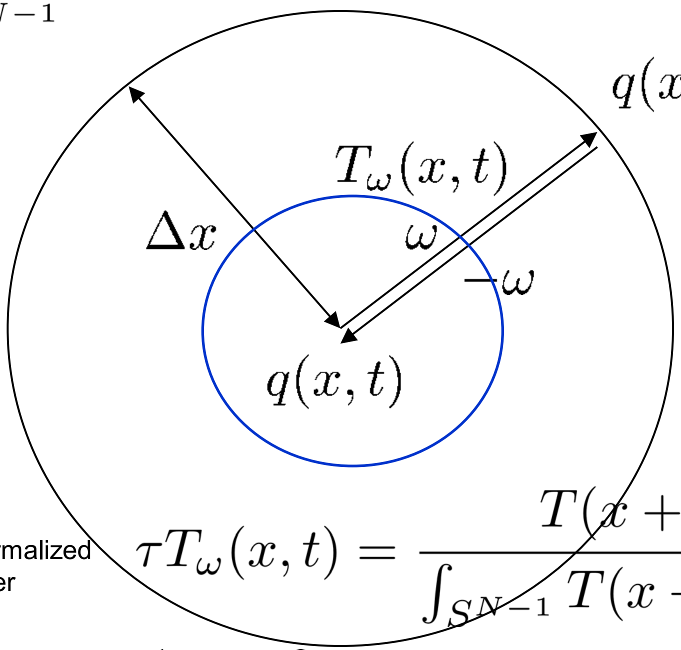
Diffusion $S^{N-1}=\{\omega\in\mathbf{R}^N \mid |\omega|=1\}$

$q(x,t)$ particle density $T_\omega(x,t)$ transient probability

$$\frac{q(x,t+\Delta t)-q(x,t)}{\Delta t}=\int_{S^{N-1}}T_{-\omega}(x+\omega\Delta x,t)q(x+\omega\Delta x,t)d\omega-\int_{S^{N-1}}T_\omega(x,t)d\omega\cdot q(x,t)$$

τ mean waiting time master equation

$$\int_{S^{N-1}}T_\omega(x,t)d\omega=\tau^{-1}$$



Othmer-Stevens 97
Ichikwa-Rouzi-S. 12

$$T=T(x,t)$$

renormalized barrier $$\tau T_\omega(x,t)=\frac{T(x+\omega\frac{\Delta x}{2},t)}{\int_{S^{N-1}}T(x+\omega'\frac{\Delta x}{2},t)d\omega'}$$

Einstein $$\tau^{-1}(\Delta x)^2=2ND$$

→ Smoluchowski equation

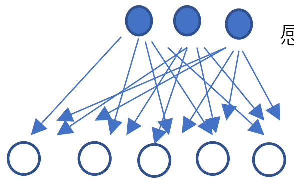
$$\frac{\partial q}{\partial t}=D\nabla\cdot(\nabla q-q\nabla\log T)$$

Mass Action $S+I\to I\ (\beta),\ I\to R\ (\gamma)$ $\rho=\frac{\gamma}{\beta}$

$$\frac{dS}{dt}=-\beta SI,\quad \frac{dI}{dt}=\beta SI-\gamma I,\quad \frac{dR}{dt}=\gamma I$$

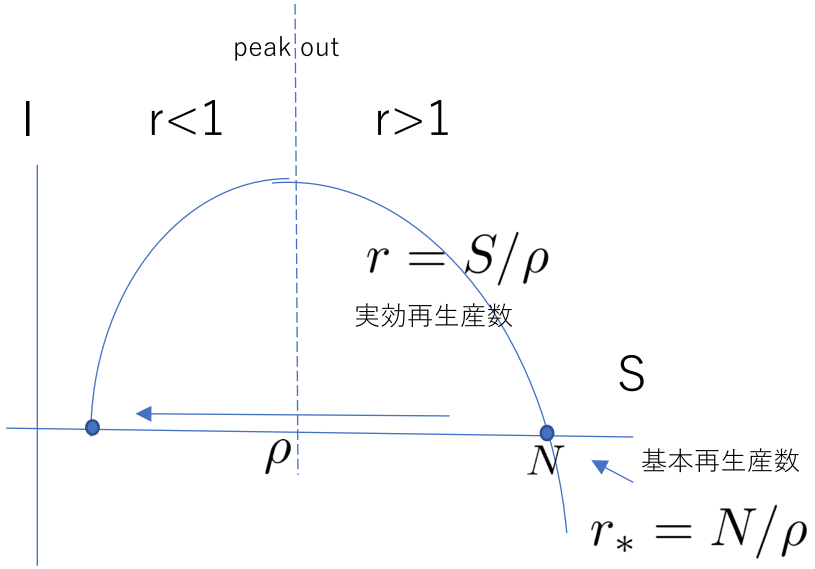
Kermack-McKendric model → $$\frac{dI}{dS}=-1+\frac{\rho}{S}$$

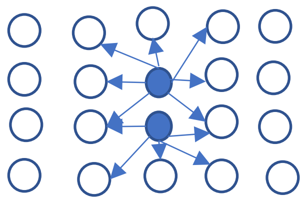
3×5＝15



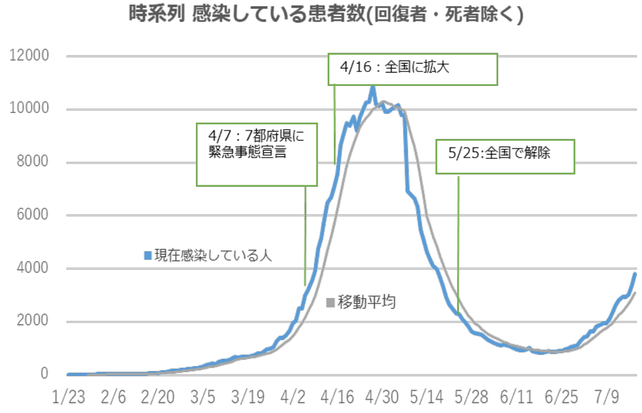
場合の数

S 未感染者
I 感染者
R 除外者





chemical reaction



Models in Biology and Geometry Driven by the Boltzmann-Poisson Equation

Full system of chemotaxis (Keller-Segel 1970)

$$\varepsilon u_t = \nabla \cdot (\nabla u - u \nabla v)$$
$$\tau v_t - \mu \Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial u} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T)$$

diffusion
chemotaxis
reaction

$\mu = 0$

Rascle's Smolchowski-ODE quation 1978

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$
$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$
$$v_t = u - \frac{1}{|\Omega|}\int_\Omega u$$

Nonlocal ODE equation

$$v_t = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) \quad \text{in } \Omega \times (0,T)$$

$\varepsilon = 0$

Nonlocal parabolic equation (Wolansky 1997)

$$v_t = \Delta v + \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right)$$

$\tau = 0$

simplified system of chemotaxis

Smoluchowski-Poisson equation (Jäger-Luckhaus 1992)

$$u_t = \nabla \cdot (\nabla u - u \nabla v)$$
$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0,T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T)$$
$$\int_\Omega v = 0$$

Hamilton's normalized Ricci flow 1988

$$u_t = \Delta \log u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0,T)$$

Statistic Ensemble and Non-equilibrium Thermodynamics

system	consistency	dynamics	ensemble
isolated	energy	entropy	micro-canonical
closed	temperature	Helmholtz free energy	canonical
open	pressure	Gibbs free energy	grand-canonical

particle density — **duality** — **field potential**

Smoluchowski ⟷ Poisson

$$v=(-\Delta)^{-1}u=\int_\Omega G(\cdot,x')u(x')dx'$$

symmetry

$$u_t=\nabla\cdot(\nabla u-u\nabla v)$$

$$\left.\frac{\partial u}{\partial\nu}-u\frac{\partial v}{\partial\nu}\right|_{\partial\Omega}=0$$

$$-\Delta v=u,\ \ v|_{\partial\Omega}=0$$

Helmholtz fee energy $A=U-TS$

$$\mathcal{F}(u)=\int_\Omega u(\log u-1)-\frac{1}{2}\langle(-\Delta)^{-1}u,u\rangle$$

$$\delta\mathcal{F}(u)=\log u-(-\Delta)^{-1}u$$

Hohenberg

Model (B) equation

$$u_t=\nabla u\cdot\nabla\delta\mathcal{F}(u),\ \left.\frac{\partial}{\partial\nu}\delta\mathcal{F}(u)\right|_{\partial\Omega}=0$$

total mass conservation, free energy decreasing

closed system

$$\Rightarrow\ \frac{d}{dt}\int_\Omega u=0,\ \frac{d\mathcal{F}}{dt}=-\int_\Omega u|\nabla\delta\mathcal{F}(u)|^2\leq 0$$

Field-Particle Duality

X Banach space $/\mathbf{R}$

$F: X \to (-\infty, +\infty]$ prop. c'x, lsc

$\longrightarrow$

$F^*: X^* \to (-\infty, +\infty]$ prop. c'x, lsc

$$F^*(p) = \sup_{x \in X} \{\langle x, p\rangle - F(x)\}$$ **Legendre transformation**

Proposition (Fenchel-Moreau) $F^{**} = F$

$$F^{**}(x) = \sup_{p \in X^*} \{\langle x, p\rangle - F^*(p)\}$$ second Legendre transformation

$F, G: X \to (-\infty, +\infty]$ **prop. c'x, lsc**

$$J(x) = G(x) - F(x)$$ **field variation**

$$J^*(p) = F^*(p) - G^*(p)$$ **free energy**

$$L(x,p) = F^*(p) + G(x) - \langle x, p\rangle$$ **Lagrangian**

Proposition (Toland)

$$\inf_{X \times X^*} L = \inf_X J = \inf_{X^*} J^*$$ **unfolding-minimality**

$$G(v) = \frac{1}{2}\|\nabla v\|_2^2, \ v \in H_0^1(\Omega)$$

$$F(v) = \lambda \log \int_\Omega e^v - \lambda(\log \lambda - 1)$$

$\longrightarrow$

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \langle v, u\rangle$$

$$u > 0, \ \|u\|_1 = \lambda, \ v \in H_0^1$$

model B – model A equation

$$u_t = \nabla \cdot u L_u(u,v), \ \tau v_t = -L_v(u,v)$$

parabolic-parabolic system of chemotaxis

$$\left.\left(u\frac{\partial}{\partial \nu} L_u(u,v), v\right)\right|_{\partial\Omega} = 0$$

Smoluchowski $u_t = \nabla \cdot (\nabla u - u\nabla v)$

$$\left.\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0$$

Poisson $-\Delta v = u, \ \ v|_{\partial\Omega} = 0$ **action in distance**

Sire-Chavanis 02

Scaling $u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \ \mu > 0$

$u_\mu(x) = \mu^2 u(\mu x), \ \mu > 0$ stationray

critical dimension $\|u\|_1 = \|u_\mu\|_1 \equiv \lambda \ \Leftrightarrow \ n = 2$

$$\mathcal{F}(u) = \int_{\mathbf{R}^2} u(\log u - 1) - \frac{1}{2}\langle \Gamma * u, u\rangle, \ \Gamma(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$$

$$\mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u) \qquad \lambda = 8\pi$$ *critical mass*

Stationary State

thermally closed system (canonical ensemble)

total mass conservation $\int_\Omega u = \lambda, \ u > 0$

free energy decreasing $\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \le 0$ ($\log u - v$: $\delta\mathcal{F}(u)$)

stationary $\Longrightarrow$ $u = \frac{\lambda e^v}{\int_\Omega e^v dx}$ Boltzmann

Boltzmann-Poisson equation

$\Longrightarrow$ $-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v}, \ \ v|_{\partial\Omega} = 0$

Trudinger-Moser inequality

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle \quad \overset{\text{Toland duality}}{\longleftrightarrow} \quad J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\int_\Omega e^v + \lambda(\log\lambda - 1)$$

$u \ge 0, \ \|u\|_1 = \lambda \ \longleftrightarrow \ J^*$ free energy

$v \in H_0^1(\Omega) \ \longleftrightarrow \ J$ field variational

Nonlinear Spectral Mechanics

$$u_t = \nabla \cdot (\nabla u - u \nabla v), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u$$

$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)\Big|_{\partial\Omega} = 0, \ \int_\Omega v = 0$$

Jager-Luckhaus

Theorem A

$n = 2, \ T = T_{\max} < +\infty$

critical dimension

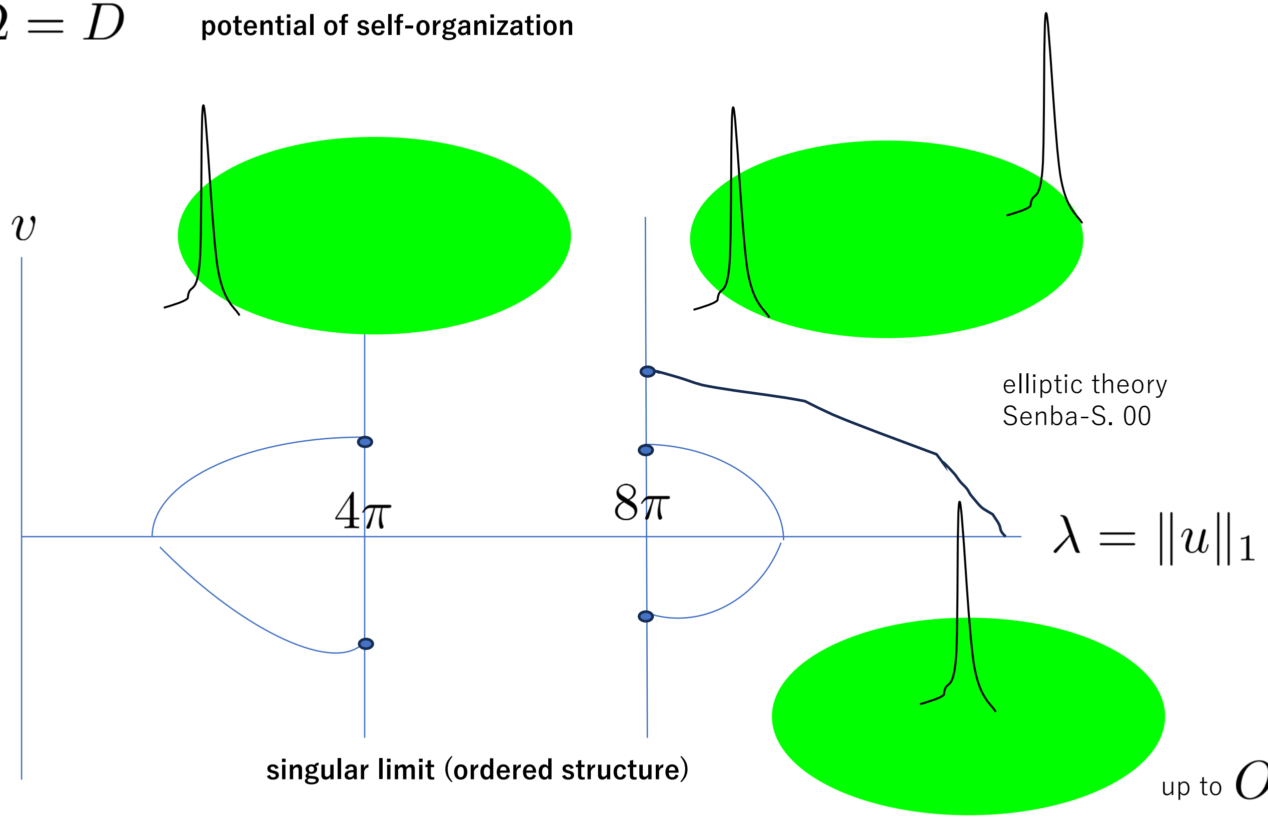
$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

critical mass

$$m(x_0) \in m_*(x_0)\mathbf{N}, \ m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases}$$

$$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S})$$

$\Omega = D$



quantized blowup mechanism

existence of the boundary blowup

blowup in finite and infinite time

stationary state

$$-\Delta v = \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right), \ \int_\Omega v = 0, \ \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0$$

nonlinear eigenvalue problem with nonlocal term

closed system in thermodynamics

Point Vortices

equilibrium statistical mechanics

Euler's equation of motion

$$v_t + (v\cdot\nabla)v = -\nabla p, \ \nabla\cdot v = 0, \ \nu\cdot v|_{\partial\Omega} = 0$$

2D $\omega = \nabla^\perp \cdot v \ \rightarrow \ \omega_t + \nabla\cdot(v\omega) = 0, \ \nabla\cdot v = 0$

$v = \nabla^\perp \psi$ stream function (simply-connected)

$$\omega_t + \nabla\cdot(\omega\nabla^\perp\psi) = 0, \ -\Delta\psi = \omega$$ **vorticity equation**

boundary condition $\rightarrow$ $\psi|_{\partial\Omega} = 0$

$$\omega(dx,t) = \sum_{i=1}^{\ell} \alpha_i \delta_{x_i(t)}(dx)$$

intensity; point vortex system

local second moment; weak form p.v. $\rightarrow$ **Kirchhoff equation** $\dfrac{dx_i}{dt} = \nabla^\perp_{x_i} H_\ell$

point vortex Hamiltonian

$$H_\ell(x_1,\ldots,x_\ell) = \sum_{i=1}^{\ell} \frac{\alpha_i^2}{2} R(x_j) + \sum_{1\le i<j\le\ell} \alpha_i\alpha_j G(x_i,x_j)$$

Green function

Robin function $R(x) = \left[G(x,x') + \dfrac{1}{2\pi}\log|x-x'| \right]_{x'=x}$

canonical statistics (thermal equilibrium)

$\mathbf{R}^{6\ell}/\{T\}, \ \beta = 1/(kT)$ inverse temperature

$$d\mu^{\beta,\ell} = \frac{e^{-\beta H}dx}{Z(\beta,N)}, \ Z(\beta,\ell) = \int_{\mathbf{R}^{6\ell}} e^{-\beta H}dx$$

canonical measure; Boltzmann constant; weight factor

equal a priori probability to k point reduced pdf of the micro-canonical measure

single intensity $\alpha_i = \hat\alpha, \ \hat\alpha\ell = 1, \ \hat H_\ell = H, \ \hat\alpha^2\ell\hat\beta = \beta$

$\ell \uparrow +\infty$ propagation of chaos

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}$$ vorticity (limit of the one-point pdf)

Joyce-Montgomery 73
Pointin-Ludgren 76
Caglioti-Lions-Marchoro-Pulvirenti 92
Kiessling 93

$$\psi = \int_\Omega G(\cdot,x')\rho(x')dx'$$ stream function

negative inverse temperature

$\lambda = -\beta$ **Boltzmann-Poisson equation**

$$\Longleftrightarrow \ -\Delta v = u, \ v|_{\partial\Omega} = 0$$

$$u = \frac{\lambda e^v}{\int_\Omega e^v dx}, \ \lambda = \|u\|_1$$

Boltzmann Poisson Equation

$\Omega \subset \mathbf{R}^2$ bounded domain $\quad \partial\Omega$ smooth

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \text{ in } \Omega, \ \ v|_{\partial\Omega} = 0$$

ordered structure

order structure in negative temperature

L. Onsager 49

Theorem B (Nagasaki-S. 90) $\{(\lambda_k, v_k)\}$ solution sequence

$$\lambda_k \to \lambda_0 \in [0,\infty), \ \ \|v_k\|_\infty \to \infty \ \Longrightarrow \ \lambda_0 = 8\pi\ell$$

quantized blowup mechanism

sub-sequence $\mathcal{S} \subset \Omega, \ \sharp\mathcal{S} = \ell$

blowup set $\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0, \ v_k(x_k) \to +\infty\}$

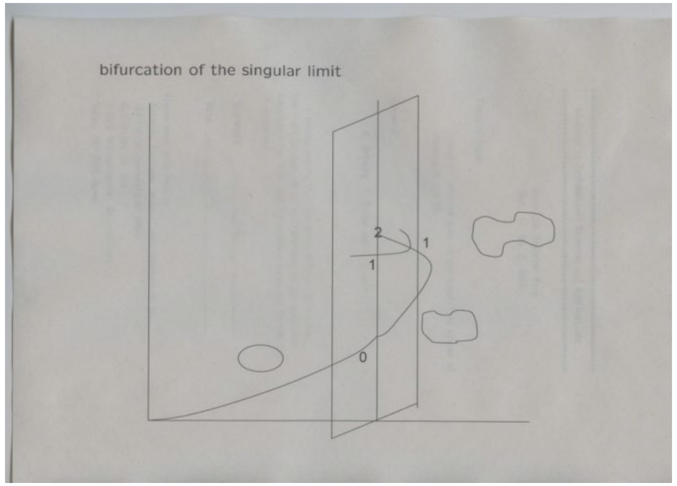
$v_k \to v_0$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$ $\qquad v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$

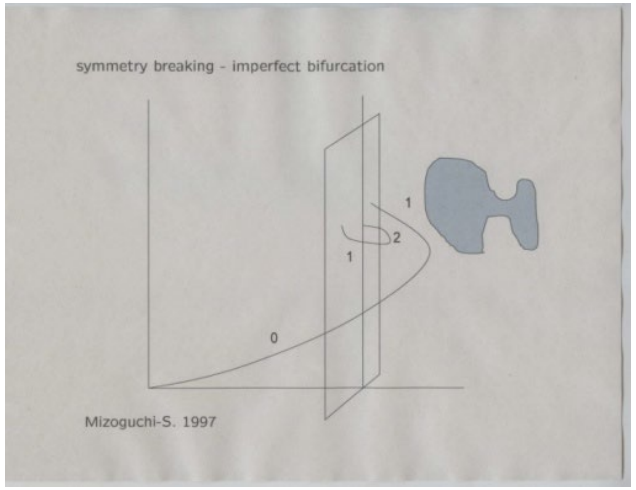
$\nabla_{x_i} H_\ell(x_1^*, \cdots, x_\ell^*) = 0, \ 1 \leq i \leq \ell$ **recursive hierarchy** $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$

$G = G(x, x')$ Green function

$$H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j) \quad \text{Hamiltonian}$$

$$R(x) = \left[G(x, x') + \frac{1}{2\pi}\log|x - x'|\right]_{x=x'} \quad \text{Robin function}$$





Theorem C (S. 92) (Ω simply-connected)

$0 < \lambda < 8\pi \ \Longrightarrow \ \exists 1$ solution

$\Longrightarrow$ **propagation of chaos (Caglioti-Lions-Marchiro-Privlenti 92, 95)**

Liouville Integral

Ω simply-connected $\Longrightarrow$ $F(z)$ single-valued

$$-\Delta v = \sigma e^v \text{ in } \Omega \subset \mathbf{R}^2$$

$$\Longleftrightarrow \exists F = F(z), \ z \in \Omega \quad \text{meromorphic} \quad \mathbf{R}^2 \cong \mathbf{C}$$

$$\rho(F) = \left(\frac{\sigma}{8}\right)^{1/2} e^{v/2} = \frac{|F'|}{1+|F|^2} \quad \text{spherical derivative}$$

conformal

$$F : \Omega \to S^2, \quad \left.\frac{d\Sigma}{ds}\right|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$$

round sphere

$$(S^2, d\Sigma), \ |S^2| = \pi$$

$$\int_{\partial\Omega} \frac{d\Sigma}{ds} ds = |\partial\Omega| \left(\frac{\sigma}{8}\right)^{1/2} \quad \text{immersed length of } F(\partial\Omega)$$

$$\int_\Omega \left(\frac{d\Sigma}{ds}\right)^2 dx = \frac{1}{8}\int_\Omega \rho(F)^2 dx = \frac{1}{8}\int_\Omega \sigma e^v \quad \text{immersed area of } F(\Omega)$$

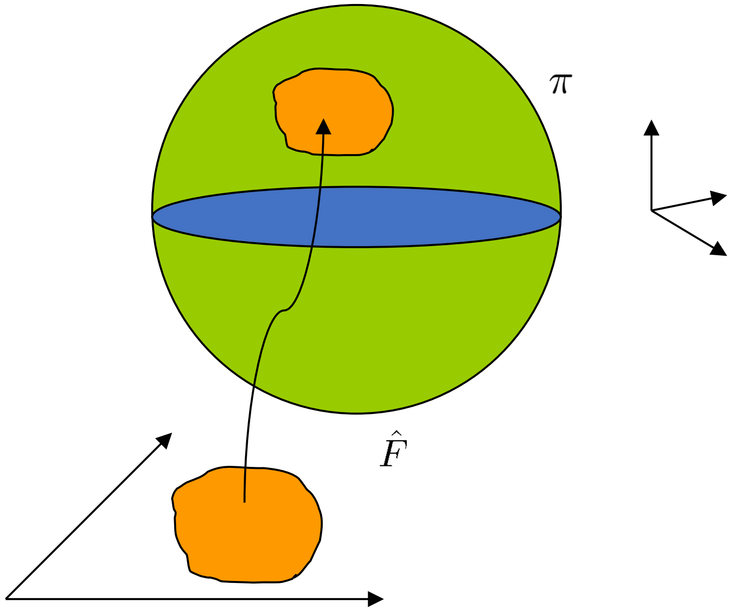
$$-\Delta v = \sigma e^v, \ v|_{\partial\Omega} = 0$$

$$\Longleftrightarrow \rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$$

$$\lambda = \int_\Omega \sigma e^v \to 8\pi\ell$$

total mass quantization due to

ℓ covering



Proof of Theorem B

1. Liouville integral
2. Boundary reflection
3. Elliptic regularity
4. Complex function theory
4.1. maximum principle
4.2. Montel's theorem
4.3. theorem of coincidence
4.4. residue analysis

Blowup analysis

$v \mapsto v + \log\lambda, \ \lambda \downarrow 0$

$\Omega \subset \mathbf{R}^2$ open set $V \in C(\overline{\Omega})$ variable coefficients without boundary condition

$$-\Delta v = V(x)e^v, \ 0 \leq V(x) \leq b \text{ in } \Omega, \quad \int_\Omega e^v \leq C$$

bounded sequence

Theorem (Brezis-Merle 91)

$\{(V_k, v_k)\}$ solution sequence $\Longrightarrow$ alternatives for sub-solutions

1. $\{v_k\}$ locally uniformly bounded in Ω

2. $\exists \mathcal{S} \subset \Omega, \ \sharp\mathcal{S} < +\infty$

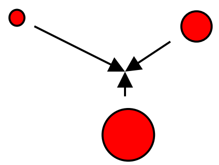
$v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$

$$\mathcal{S} = \{x_0 \in \Omega \mid \exists x_k \to x_0, \ v_k(x_k) \to +\infty\}$$

$$V_k(x)e^{v_k}dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) \text{ in } \mathcal{M}(\Omega)$$

$\boxed{m(x_0) \geq 4\pi}$

3. $v_k \to -\infty$ locally uniformly in Ω



collision of bubbles
+ residual vanishing

↑

sup+inf inequality

compact sequence

Theorem (Li-Shafrir 94)

$V_k \to V$ locally uniformly in Ω $\Longrightarrow$ $\boxed{m(x_0) \in 8\pi\mathbf{N}}$

Lemma (Brezis-Merle) $\quad -\Delta v = f, \ v|_{\partial\Omega} = 0$

$$\Longrightarrow \int_\Omega \exp\left(\frac{4\pi - \delta}{\|f\|_1}|v(x)|\right) \leq \frac{4\pi^2}{\delta}(\text{diam } \Omega)^2$$

scaling $\quad \tilde{v}_k(x) = v_k(\sigma_k x + x_0) + 2\log\sigma_k$

Lemma (Chen-Li 91) **Liouville property**

$$-\Delta v = e^v \text{ in } \mathbf{R}^2, \quad \int_{\mathbf{R}^2} e^v < +\infty$$

$$\Longrightarrow v(x) = \log\left\{\frac{8\mu^2}{(1 + \mu^2|x - x_0|^2)^2}\right\}$$

Lemma (localization) $\quad B = B(0, R)$

$$-\Delta v_k = V_k(x)e^{v_k}, \ V_k(x) \geq 0 \text{ in } B$$

$$V_k \to V \text{ unif. in } \overline{B}, \ \max_{\overline{B}} v_k \to +\infty$$

$$\max_{\overline{B} \setminus B_r} v_k \to -\infty, \ \forall r \in (0, R)$$

$$\lim_k \int_B V_k e^{v_k} = \alpha, \ \int_B e^{v_k} \leq C \qquad \alpha \in 8\pi\mathbf{N}$$

1. non-radial bifurcation on annulus (SS. Lin 89 Nagasaki-S. 90b)

2. Effective bound of blowup points for simply-connected domains (S.-Nagasaki 89 Grossi-F. Takahashi 10)

3. Classification of singular limits (Nagasaki-S. 90a S. 22)

4. Spherical mean value theorem (S. 90)

5. Localization (Brezis-Merle 91)

6. Entire solution (W. Chen-C. Li 91)

7. sup+inf inequality (Sharfir 92)

8. Field-particle duality (Wolanski 92)

9. Uniqueness (S. 92, Bartolucci-C.S. Lin 15)

10. Singular perturbation (Weston 78 S. 93 Baraket-Pacard 98 Esposito-Grossi-Pistoia 05 del Pino-Kowarzyk-Musso 05)

11. Blowup analysis (Li-Shafrir 94)

12. Chern-Simons theory (Tarantello 96)

13. Global bifurcation (Nagasaki-S. 89 Mizoguchi-S. 97 Senba-S. 00 Chang-Chen-Lin 03 Bartolucci-Jevnikar-Lee-Yang 18)

14. Mini-max solution (Struwe-Tarantello 98 Ding-Jost-Li-Wang 99)

15. Local uniform estimate (Y.Y. Li 99)

16. Variable coefficient (Ma-Wei 01)

17. Refined asymptotics (Chen-Lin 02)

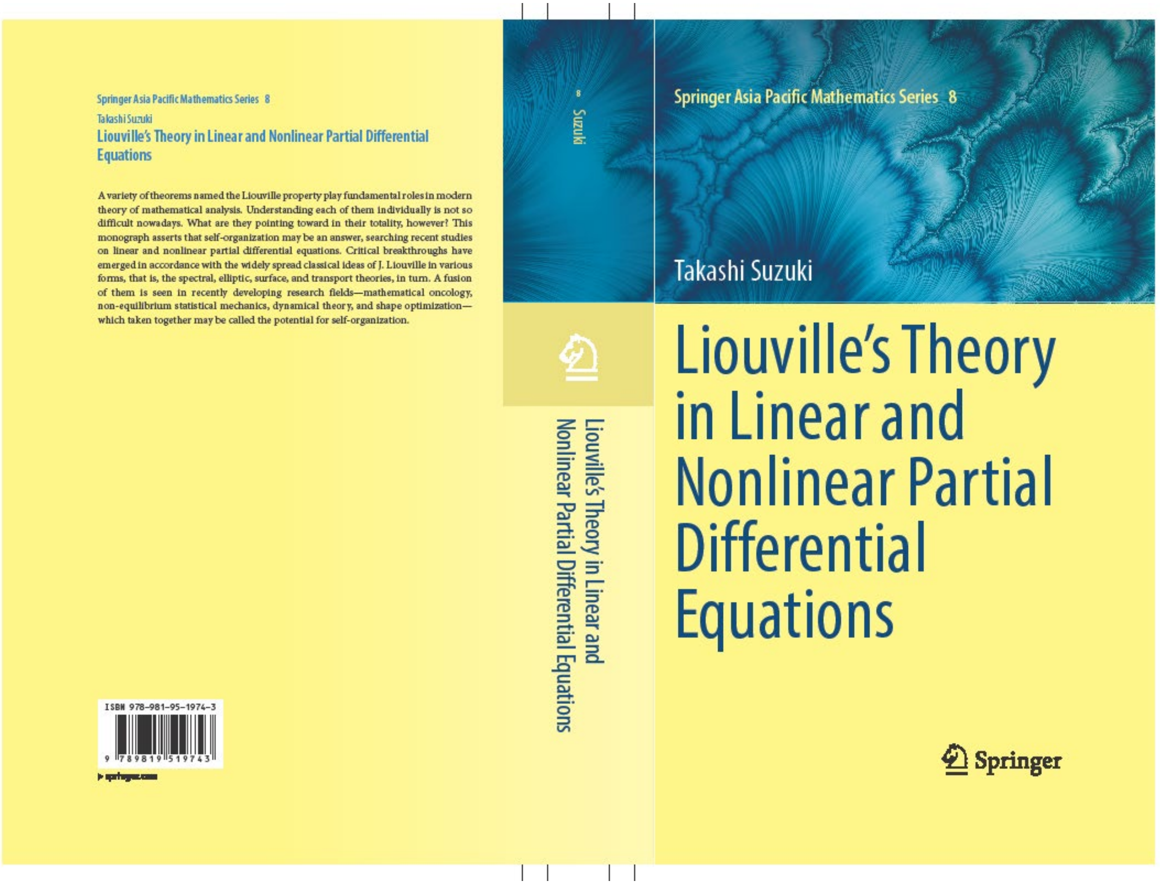
18. Topological degree (Li 99 Chen-Lin 03 Malchiodi 08)

19. Asymptotic nondegeneracy (Gladiali-Grossi 04 Grossi-Ohtsuka-S. 11 Sato-S. 18)

20. Isometric profile (Lin-Lucia 06)

21. Deformation lemma (Lucia 07)

22. Morse index (Gladiali-Grossi 09 Sato-S. 23)



Isoperimetric Inequality

$\Omega \subset \mathbf{R}^2$ simply-connected bounded domain $\quad \partial\Omega$ smooth

$$p = p(x) > 0, \quad -\Delta \log p \leq p \text{ in } \Omega \qquad \lambda = \int_\Omega p \qquad H_c^1(\Omega) = H_0^1(\Omega) \oplus \mathbf{R}$$

$$\lambda < 4\pi \ \overset{\text{Bandle 75}}{\Longrightarrow} \ \nu_1 = \inf \left\{ \int_\Omega |\nabla\phi|^2 \mid \phi \in H_0^1(\Omega), \ \int_\Omega p\phi^2 = 1 \right\} > 1$$

$$\lambda < 8\pi \ \overset{\text{S. 92}}{\Longrightarrow} \ \nu_2 = \inf \left\{ \int_\Omega |\nabla\phi|^2 \mid \phi \in H_c^1(\Omega), \ \int_\Omega p\phi^2 = 1, \ \int_\Omega p\phi = 0 \right\} > 1 \ \Longrightarrow \ \textbf{Theorem C}$$

Bol's inequality

$$S^2 = \{|x| = 1 \mid x = (x_1, x_2, x_3) \in \mathbf{R}^3\}$$

$$\ell(\partial\omega)^2 \geq m(\omega)(4\pi - m(\omega))$$ isoperimetric inequality

$$\omega \subset S^2$$

new rearrangement

Laplace-Beltrami operator

associated Legendre equation

c.f. Bartolucchi-Lin 14

Multi-Intensity Model

$$H_\ell(x_1,\dots,x_\ell)=\sum_{i=1}^{\ell}\frac{\alpha_i^2}{2}R(x_j)+\sum_{1\le i<j\le\ell}\alpha_i\alpha_j G(x_i,x_j)$$

1. Stochastic Case (Neri 04)

relative intensity

1 species $\alpha_i=\hat\alpha$ $\quad \gamma=\frac{\hat\alpha}{|\hat\alpha|}\in[-1,1]=I$

is a random variable subject to the probability density P

$$-\Delta v=\lambda\frac{\int_I \alpha e^{\alpha v}P(d\alpha)}{\int_I\int_\Omega e^{\alpha v}P(d\alpha)},\ v|_{\partial\Omega}=0$$

$$J_\lambda(v)=\frac12\|\nabla v\|_2^2-\lambda\log\int_I\left(\int_\Omega e^{\alpha v}\right)\ P(d\alpha)$$

Euler-Lagrange equation

2. Deterministic Case (Onsager's note, Sawada-S. 08)

n species $\tau_i\ell$ particles take the intensity $\gamma_i\hat\alpha=\alpha_i$ $\quad 1\le i\le n$

$$0<\tau_i<1,\ -1\le\gamma_i\le 1,\ \sum_i\tau_i=1$$

$P=\sum_i\tau_i\delta_{\gamma_i}$ discrete probability measure can be continuously distributed

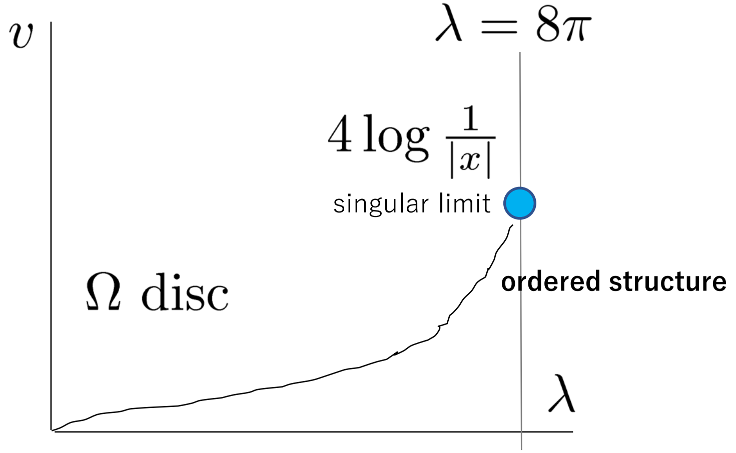
$$-\Delta v=\lambda\int_I\frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}}P(d\alpha),\ v|_{\partial\Omega}=0$$

$$J_\lambda(v)=\frac12\|\nabla v\|_2^2-\lambda\int_I\log\left(\int_\Omega e^{\alpha v}\right)P(d\alpha)$$

$v\in H_0^1(\Omega)$

single intensity $P=\delta_1$

1. blowup analysis
2. asymptotic non-degeneracy, Morse index calculation
3. deformation theory, topological degree calculation



Trudinger-Moser inequality $j_\lambda\equiv\inf_{H_0^1}J_\lambda>-\infty$

extremal value $\overline{\lambda}\equiv\sup\{\lambda\mid j_\lambda>-\infty\}$

$j_{\overline{\lambda}}>-\infty$ & not attained ⟹ *the first* order structure

Extremal Boundedness

$P(d\alpha)$ probability measure on $[-1,1]$ $\quad j_\lambda \equiv \inf_{H_0^1} J_\lambda \quad \overline{\lambda} \equiv \sup\{\lambda \mid j_\lambda > -\infty\}$

Ω closed Riemann surface $\quad E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$

stochastic case

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_I \left(\int_\Omega e^{\alpha v}\right) P(d\alpha)$$

Ricciardi-Zecca12
S.-Zhang 13

$$1 \in \mathrm{supp}\ P(d\alpha) \Longrightarrow \overline{\lambda} = 8\pi,\ j_{\overline{\lambda}} > -\infty$$

deterministic case

$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \int_I \left(\log \int_\Omega e^{\alpha v}\right) P(d\alpha)$$

Theorem E Ricciardi-S. 14

$$\overline{\lambda} = \inf\left\{ \frac{8\pi P(K_\pm)}{\left(\int_{K_\pm} \alpha P(d\alpha)\right)^2} \middle| K_\pm \subset I_\pm \cap \mathrm{supp}\ P \right\}$$

deterministic case (continued)

Theorem F $\quad P$ discrete probability measure $\Longrightarrow j_{\overline{\lambda}} > -\infty$

S.-Toyota, in preparation

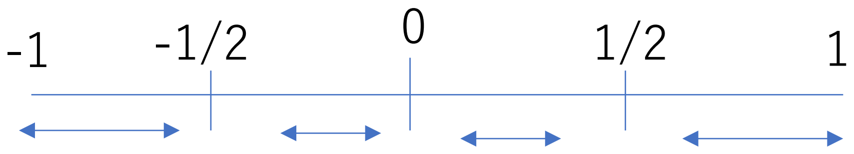
Theorem G continuous probability measure

$$P = \tau_+ P_+ + \tau_- P_-, \ \tau_\pm > 0, \ \tau_+ + \tau_- = 1$$

$$P_\pm = \tau_{1\pm} P_{1\pm} + \tau_{2\pm} P_{2\pm}, \ \tau_{i\pm} > 0, \ \tau_{1\pm} + \tau_{2\pm} = 0$$

$$0 < \alpha_{2\pm} < \frac{4\pi}{\overline{\lambda} \int_{I_\pm} |\alpha| P(d\alpha)}, \ \frac{1}{2} < \alpha_{1\pm} \leq 1$$

Y.Y. Li estimate
mass separation $\Longrightarrow j_{\overline{\lambda}} > -\infty$



The same result for bounded domains

Results $S^2 = \{x=(x_1,x_2,x_3)\in \mathbf{R}^3 \mid |x|=1\}, \quad D = \{x = (x = (x_1,x_2) \in \mathbf{R}^2 \mid |x|<1\}$

$$P = \sum_{i=1}^n \tau_i \delta_{\gamma_i}, \ \tau_i > 0, \ \sum_i \tau_i = 1, \ 0 < \gamma_1 < \cdots < \gamma_n = 1$$

discrete (for simplicity) $\quad \mathrm{supp}\ P \subset I_+ = [0,1]$

stochastic intensity (Neri)

$$-\Delta v = \lambda \left(\frac{\int_{I_+} \alpha e^{\alpha v} P(d\alpha)}{\int_{I_+}\int_\Omega e^{\alpha v} P(d\alpha)} - \frac{1}{|\Omega|} \right), \ \int_\Omega v = 0$$

deterministic intensity (Sawada-S.)

$$-\Delta v = \lambda \int_{I_+} \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|}\right) P(d\alpha), \ \int_\Omega v = 0$$

$\Omega = S^2$ $\quad n \geq 2$

$$\lambda \leq \overline{\lambda} = 8\pi \implies v = 0 \impliedby \lambda \leq \lambda_* = \frac{8\pi}{\int_{I_+} \alpha P(d\alpha)}$$

$P = \delta_1$

$\lambda < \lambda_* = 8\pi$

c.f. C.S. Lin 00a

$\Omega = D$

$$-\Delta v = \lambda \frac{\int_{I_+} \alpha e^{\alpha v} P(d\alpha)}{\int_{I_+}\int_\Omega e^{\alpha v} P(d\alpha)}, \ v|_{\partial\Omega} = 0$$

$$-\Delta v = \lambda \int_{I_+} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}} P(d\alpha), \ v|_{\partial\Omega} = 0$$

$$\lambda \geq \overline{\lambda} = 8\pi \implies \nexists v \impliedby \lambda \geq \hat{\lambda} = \frac{8\pi}{\left(\int_{I_+} \alpha P(d\alpha)\right)^2}$$

Remark $\quad \hat{\lambda} \geq \overline{\lambda}$

$K_+ = I_+$

$$P = \tau\delta_1 + (1-\tau)\delta_\gamma \quad \overline{\lambda} = \hat{\lambda} \iff \gamma \geq \frac{\sqrt{\tau}}{1+\sqrt{\tau}}$$

Ricciardi-Takahashi 19

$0 < \tau, \gamma < 1$

$\lambda \geq \overline{\lambda}$

$$n \geq 3, \ \gamma_1 > \frac{1}{2} \longrightarrow \overline{\lambda} = \hat{\lambda}$$

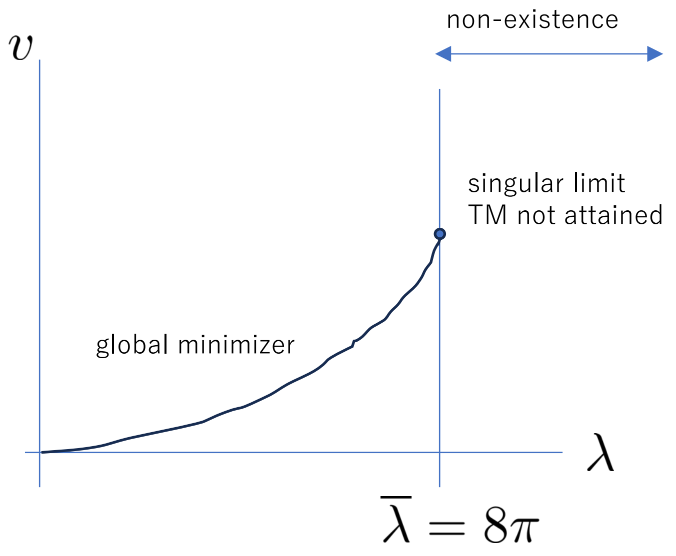
radial symmetry (GNN)+transformation

Bifurcation Diagram

$$\Omega = D \qquad P=\sum_{i=1}^{n}\tau_i\delta_{\gamma_i},\ \tau_i>0,\ \sum_i \tau_i=1,\ 0<\gamma_1<\cdots<\gamma_n=1,\ n\ge 1$$

Neri

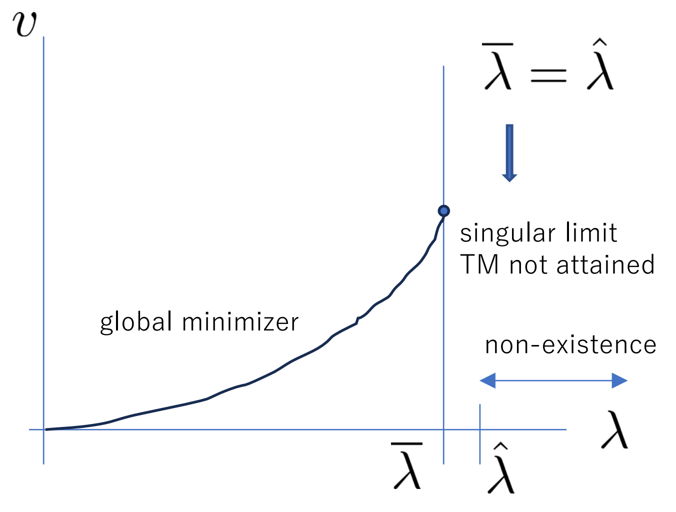
$$-\Delta v = \lambda \frac{\int_{I_+}\alpha e^{\alpha v}P(d\alpha)}{\int_{I_+}\int_\Omega e^{\alpha v}P(d\alpha)}, \quad v|_{\partial\Omega}=0$$



any solution is radially symmetric
Gidas-Ni-Nirenberg 79

Sawada-S.

$$-\Delta v = \lambda \int_{I_+}\frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}}P(d\alpha), \quad v|_{\partial\Omega}=0$$



$$\overline{\lambda}=\inf\left\{\frac{8\pi P(K_+)}{\left(\int_{K_+}\alpha P(d\alpha)\right)^2}\,\middle|\, K_+\subset I_+\cap \operatorname{supp} P\right\}$$

$$\le \hat{\lambda} = \frac{8\pi}{\left(\int_\Omega \alpha P(d\alpha)\right)^2}$$

$$\Omega = S^2 \qquad P = \sum_{i=1}^{n} \tau_i \delta_{\gamma_i}, \ \tau_i > 0, \ \sum_i \tau_i = 1, \ 0 < \gamma_1 < \cdots < \gamma_n = 1$$

Neri

$$-\Delta v = \lambda \left(\frac{\int_{I_+} \alpha e^{\alpha v} P(d\alpha)}{\int_{I_+} \int_\Omega e^{\alpha v} P(d\alpha)} - \frac{1}{|\Omega|} \right), \ \int_\Omega v = 0$$

Sawada-S.

$$-\Delta v = \lambda \int_{I_+} \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) P(d\alpha), \ \int_\Omega v = 0$$

linearized operator at $v = 0$ on $E = \{v \in H^1(\Omega) \mid \int_\Omega v = 0\}$

$$L_0 = -\Delta - \frac{\lambda}{|\Omega|} \int_{I_+} \alpha^2 P(d\alpha)$$

$$L_0 = -\Delta - \frac{\lambda}{|\Omega|} \int_\Omega \alpha P(d\alpha)$$

spectrum of the Laplace-Beltrami operator on S^2

$\mu_1(-\Delta) = 1, \ \phi_0 = \text{constant} \quad \mu_2(-\Delta) = 2, \ \phi_i = x_i, \ i = 1, 2, 3 \qquad |\Omega| = 4\pi$

bifurcation from odd multiplicity ⟹

$$\lambda_* = \frac{8\pi}{\int_{I_+} \alpha^2 P(d\alpha)} \qquad \lambda_* = \frac{8\pi}{\int_\Omega \alpha P(d\alpha)}$$

first bifurcation point from the branch of trivial solutions

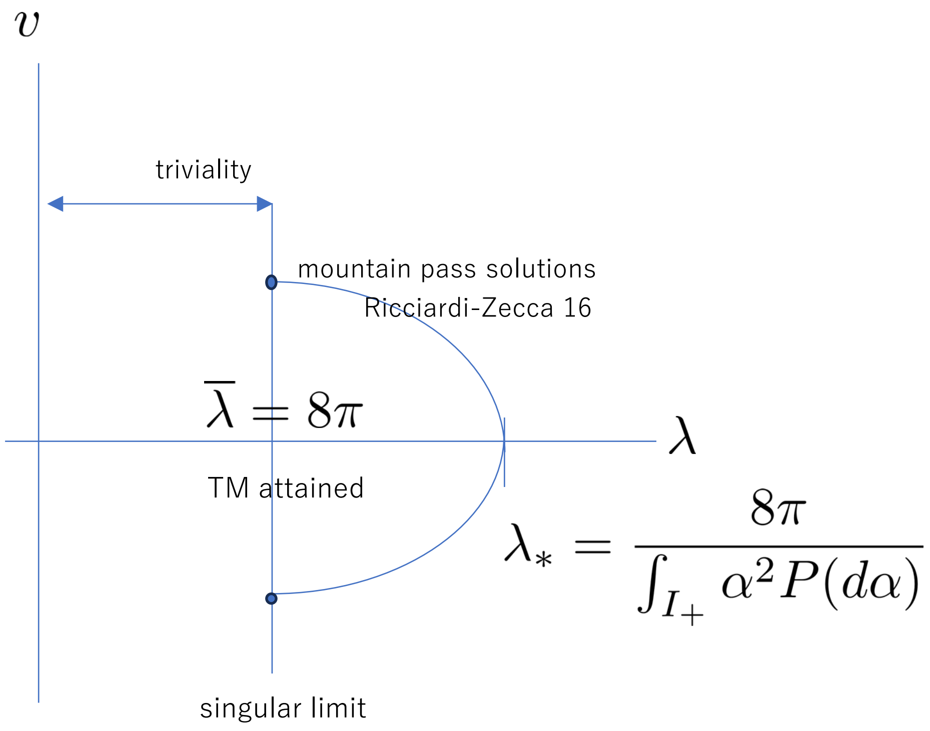
$$\Omega = S^2 \qquad P = \sum_{i=1}^{n} \tau_i \delta_{\gamma_i}, \ \tau_i > 0, \ \sum_i \tau_i = 1, \ 0 < \gamma_1 < \cdots < \gamma_n = 1, \ n \geq 2$$

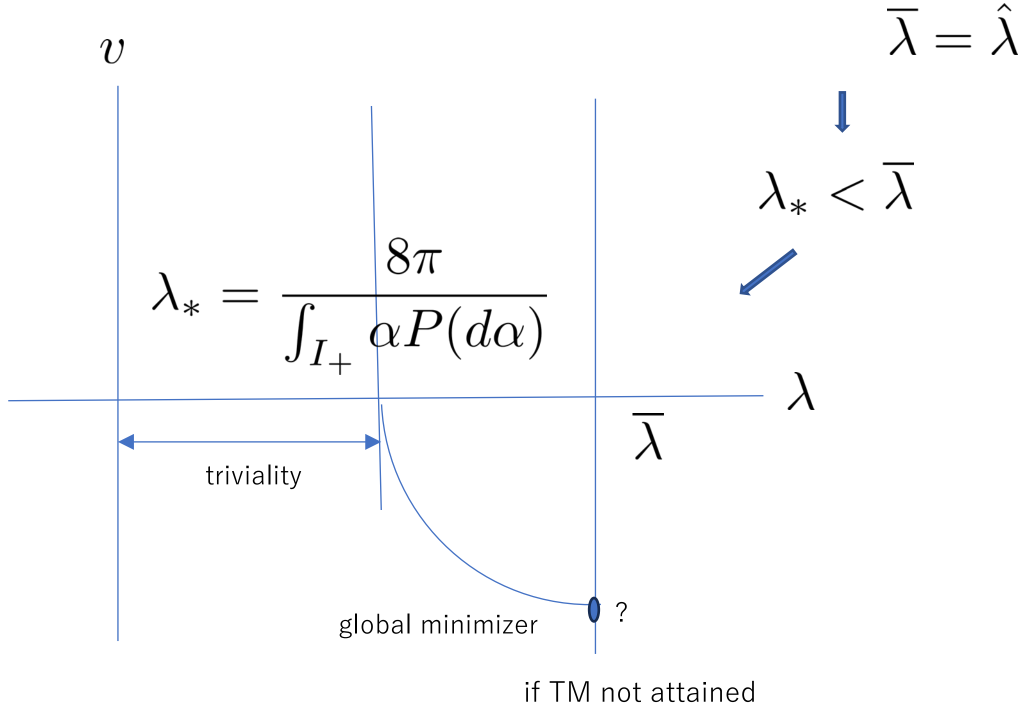
Neri

$$-\Delta v = \lambda \left(\frac{\int_{I_+} \alpha e^{\alpha v} P(d\alpha)}{\int_{I_+} \int_\Omega e^{\alpha v} P(d\alpha)} - \frac{1}{|\Omega|} \right), \ \int_\Omega v = 0$$

Sawada-S.

$$-\Delta v = \lambda \int_{I_+} \alpha \left(\frac{e^{\alpha v}}{\int_\Omega e^{\alpha v}} - \frac{1}{|\Omega|} \right) P(d\alpha), \ \int_\Omega v = 0$$





up to $O(3)$ symmetry

Isoperimetric Inequality of PSS Type

Payne-Sperb-Stakgold 77 $f=f(t)\geq 0$ real analytic $F(t)=\int_0^t f(t)dt$

Lemma 1 $-\Delta v=f(v)$ in $\Omega\subset\mathbf{R}^2$, $v|_{\partial\Omega}=0$ $\Longrightarrow$ $\left(\int_\Omega f(v)\right)^2\geq 8\pi\int_\Omega F(v)$

Proof $m(t)=|\omega_t|,\ E(t)=\int_{\omega_t}f(v),\ \omega_t=\{v>t\}$ $\quad E(t)=\int_{v>t}-\Delta v=-\int_{v=t}\frac{\partial v}{\partial\nu}=\int_{v=t}|\nabla v|$

co-area formula

$$E'(t)=-\int_{v=t}\frac{f(v)}{|\nabla v|}$$

isoperimetric inequality

$$-(EE')(t)=f(t)\int_{v=t}\frac{1}{|\nabla v|}\cdot\int_{v=t}|\nabla v|\geq f(t)\ell(\partial\omega_t)^2\geq 4\pi F'(t)m(t)$$

co-area formula

$$\Longrightarrow\ \frac{1}{2}\left(\int_\Omega f(v)\right)^2\geq -4\pi\int_0^\infty F(t)m'(t)dt=4\pi\int_\Omega F(v)\qquad\square$$

Remark

Pohozaev identity, star-shaped

$$4F(v)=\int_{\partial\Omega}\left(\frac{\partial v}{\partial\nu}\right)^2(x\cdot\nu)\geq\frac{1}{\int_{\partial\Omega}(x\cdot\nu)^{-1}}\left(\frac{\partial v}{\partial\nu}\right)^2=\frac{1}{\int_{\partial\Omega}(x\cdot\nu)^{-1}}\left(\int_\Omega f(v)\right)^2$$

$$\Longrightarrow 4\int_{\partial\Omega}(x\cdot\nu)^{-1}\cdot\int_\Omega F(v)\geq\left(\int_\Omega f(v)\right)^2\geq 8\pi\int_\Omega F(v)$$

Theorem 1 $P = \sum_{i=1}^{n} \tau_i \delta_{\gamma_i},\ \tau_i > 0,\ \sum_i \tau_i = 1,\ 0 < \gamma_1 < \cdots < \gamma_n = 1,\ n \geq 1$

$\Omega = D,\quad -\Delta v = \lambda \int_{I_+} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v}} P(d\alpha),\ v|_{\partial\Omega} = 0 \quad \Longrightarrow \quad \lambda < \hat{\lambda} = \frac{8\pi}{\left(\int_{I_+} \alpha P(d\alpha)\right)^2}$

Proof $\left(\int_\Omega f(v)\right)^2 = 8\pi \int_\Omega F(v) \qquad f(t) = \lambda \int_{I_+} \frac{\alpha e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha) \geq 0 \qquad F(t) = \lambda \int_{I_+} \frac{e^{\alpha t} - 1}{\int_\Omega e^{\alpha v}} P(d\alpha)$

$$\int_\Omega f(v) = \lambda \int_{I_+} \alpha P(d\alpha) \qquad \int_\Omega F(v) = \lambda \int_{I_+} \frac{\int_\Omega e^{\alpha v} - |\Omega|}{\int_\Omega e^{\alpha v}} P(d\alpha) = \lambda \int_{I_+} \left(1 - \frac{1}{\frac{1}{|\Omega|}\int_\Omega e^{\alpha v}}\right) P(d\alpha) < \lambda$$

$v = v(x) > 0$

$\Longrightarrow \quad \lambda \left(\int_{I_+} \alpha P(d\alpha)\right)^2 < 8\pi \qquad \lambda < \hat{\lambda} \qquad \square$

Lemma 2 $\Omega = S^2$ $f = f(t) \geq 0,\ t \in \mathbf{R}$ real-analytic $f, f' \in L^1(-\infty, 0)$

$$-\Delta v = f(v) - \frac{1}{|\Omega|}\int_\Omega f(v)$$

$$\int_\Omega v = 0$$

$$v = v(x) \not\equiv 0$$

$\Longrightarrow$

Bol's inequality

$$\frac{1}{2}\left(\int_\Omega f(v)\right)^2 \geq 4\pi \int_\Omega F(v) + \frac{1}{2}\int_{-\infty}^{\infty} \left\{ \frac{f'(t)}{|\Omega|}\int_\Omega f(v) - 2f(t) \right\} m^2(t)dt$$

$$m(t) = |\omega_t|,\ \omega_t = \{v > t\}$$

$$F(t) = \int_{-\infty}^{t} f(t)dt$$

Sawada-S. functional

Corollary $f(t) = \lambda \int_{I_+} \frac{\alpha e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha)$ $P = \sum_{i=1}^{n} \tau_i \delta_{\gamma_i},\ \tau_i > 0,\ \sum_i \tau_i = 1,\ 0 < \gamma_1 < \cdots < \gamma_n = 1$

$$v(x) \not\equiv 0 \overset{\lambda \leq \lambda_*}{\Longrightarrow} \int_{I_+} A(\alpha)P(d\alpha) \leq \int_{I_+} \alpha A(\alpha)P(d\alpha)$$

Theorem 2 $n \geq 2$ $\Longrightarrow$ contradiction $\Longrightarrow$ $v = 0$

Remark $\lambda \leq \lambda_* \Longrightarrow A(\alpha) \geq 0,\ \alpha \in I_+$

$$A(\alpha) = 8\pi - \lambda\alpha \int_{I_+} \alpha P(d\alpha) \geq 0$$

uniqueness open

$$A(\alpha) = 0 \Longleftrightarrow n = 1,\ \lambda = \lambda_* = 8\pi,\ \alpha = 1$$

Proof of Corollary

$$f(t) = \lambda \int_{I_+} \frac{\alpha e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha) \qquad \int_\Omega f(v) = \lambda \int_{I_+} \alpha P(d\alpha) \qquad f'(t) = \lambda \int_{I_+} \frac{\alpha^2 e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha)$$

$$F(t) = \lambda \int_{I_+} \frac{e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha) \qquad \int_\Omega F(v) = \lambda$$

$$\lambda \left(\int_{I_+} \alpha P(d\alpha) \right)^2 \geq 8\pi + \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \lambda \int_{I_+} \frac{\alpha^2 e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha) \cdot \int_{I_+} P(d\alpha) - \frac{1}{8\pi} \int_{I_+} \alpha \frac{\alpha e^{\alpha t}}{\int_\Omega e^{\alpha v}} P(d\alpha) \right\} m^2(t) dt$$

$$\lambda \leq \lambda_* \Longrightarrow A(\alpha) = 8\pi - \lambda \alpha \int_{I_+} \alpha P(d\alpha) \geq 0 \qquad \int_{I_+} A(\alpha) P(d\alpha) = 8\pi - \lambda \left(\int_{I_+} \alpha P(d\alpha) \right)^2$$

$$\int_{I_+} A(\alpha) P(d\alpha) \leq \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\{ \int_{I_+} \frac{\alpha e^{\alpha t}}{\int_\Omega e^{\alpha v}} A(\alpha) P(d\alpha) \right\} m^2(t) dt = \frac{1}{4\pi} \int_{I_+} \alpha A(\alpha) P(d\alpha) \cdot \int_{-\infty}^{\infty} \frac{e^{\alpha t} m^2(t)}{\int_{-\infty}^{\infty} e^{\alpha t} m(t) dt} dt$$

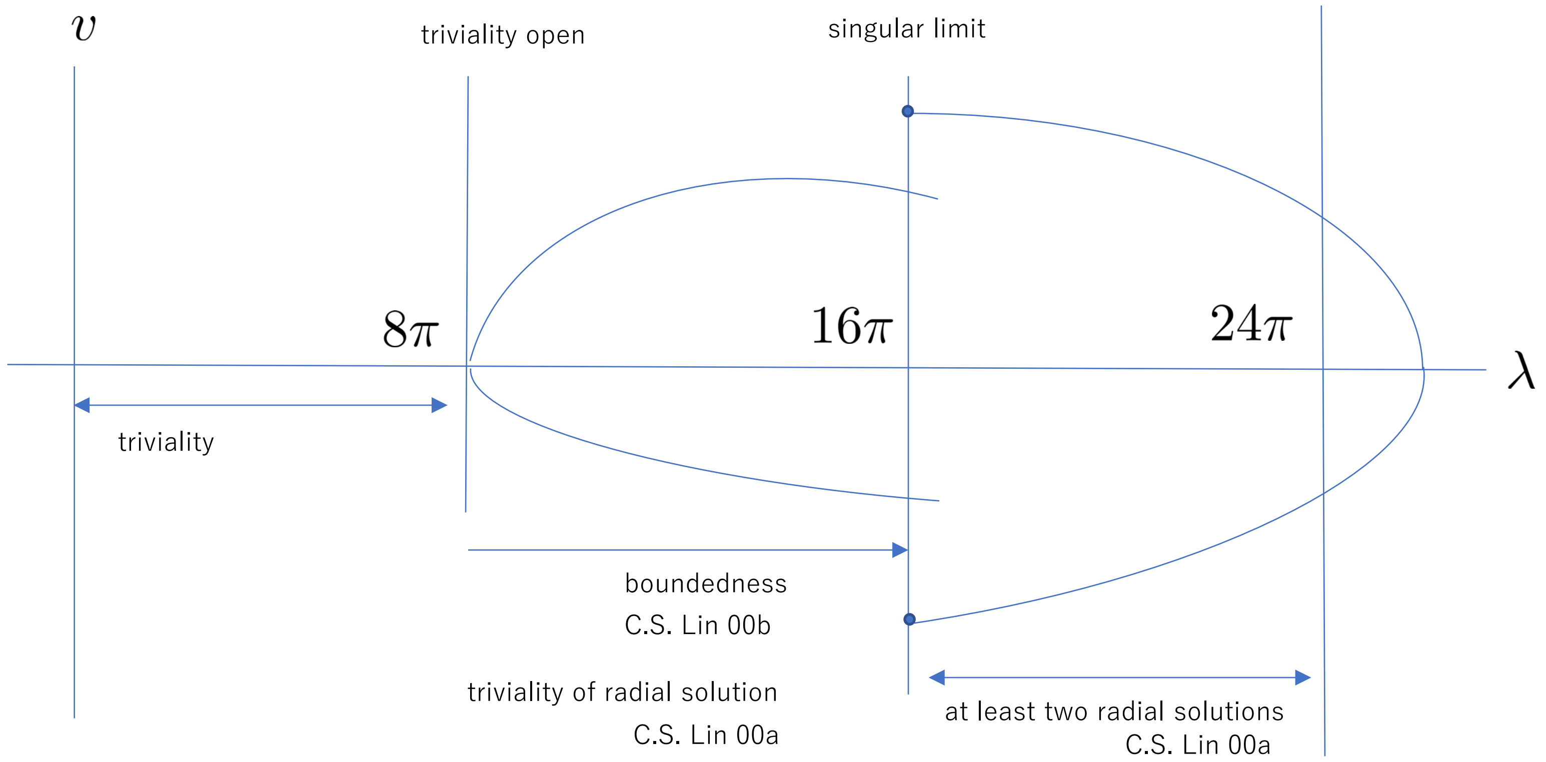
$$\leq \int_{I_+} \alpha A(\alpha) P(d\alpha) \qquad \square$$

Remark $\Omega = S^2,\ n = 1$

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right),\ \int_\Omega v = 0$$

topological degree calculation

C.C. Chen-C.S. Lin 03 Malchiodi 08



total degree (Leray-Schauder)

$$d_\lambda = \begin{pmatrix} k - \chi(\Omega) \\ k \end{pmatrix}$$

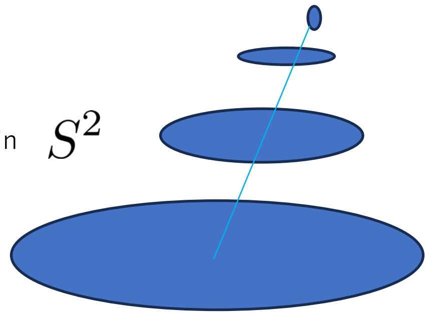
$$8\pi k < \lambda < 8\pi(k+1)$$

Euler characteristics

$$\chi(\Omega) = 2 - 2g(\Omega)$$

genus

8π ⟹ Each level set is a ball in S^2



up to $O(3)$ symmetry

From Quasi-stationary to Stationary

$$u_t + \beta\nabla\cdot(u\nabla^{\perp}v) = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \Omega\times(0,T)$$

Euler-Smoluchowski-Poisson equation

$$\frac{\partial u}{\partial\nu} - u\left(\frac{\partial v}{\partial\nu} + \beta\frac{\partial v}{\partial\tau}\right)\Big|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0 \qquad -\Delta v = u, \quad v|_{\partial\Omega} = 0$$

factorization (propagation of chaos)

$$P_N(x_1, x_2, \cdots, x_N, t) = \prod_{i=1}^{N} P_1(x_i, t)$$

high energy limit

$$\hat{\beta}N\alpha^2 = \beta, \ \alpha N = 1, \ \omega = P_1$$

$$\frac{\partial\omega}{\partial t} + \nabla^{\perp}\psi\cdot\nabla\omega = \nu\nabla\cdot(\nabla\omega + \beta\alpha\omega\nabla\psi)$$

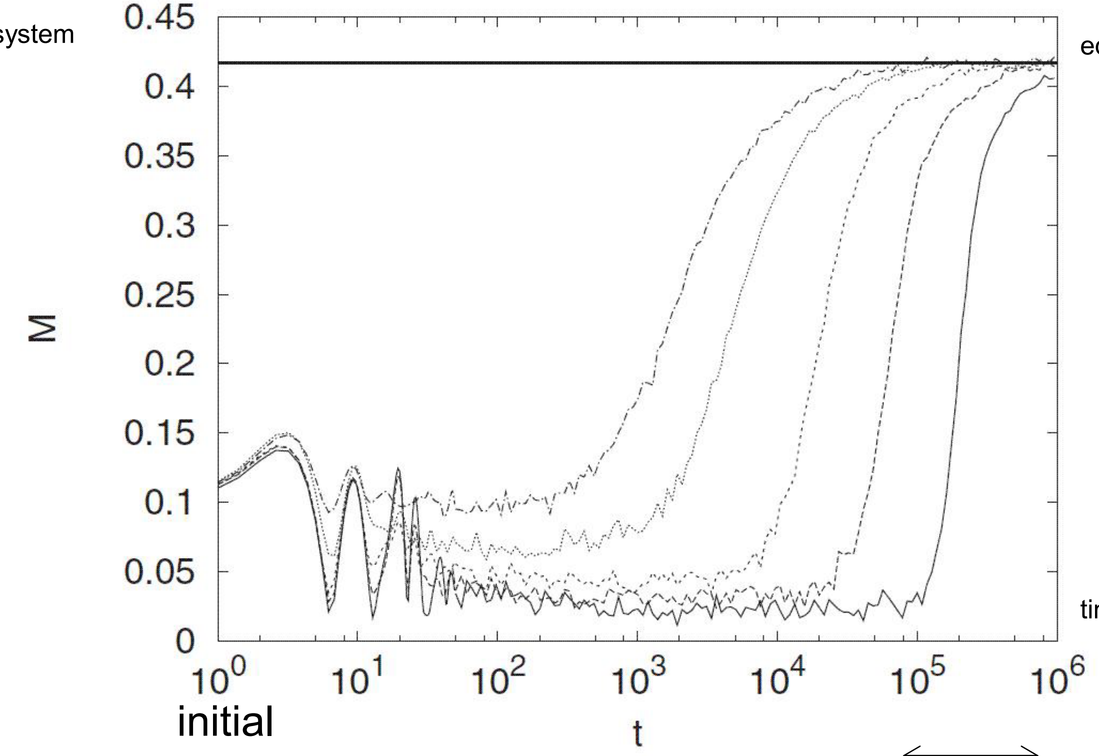
$$-\Delta\psi = \omega, \ \psi|_{\partial\Omega} = 0 \qquad \beta = -\lambda$$

negative inverse temperature

Hamilton system of many particles with inner interaction of long range

Staniscia-Chavanis-Ninno-Fanelli 09

state of the system



equilibrium (stationary)

time

initial

relaxation

quasi-equilibrium (stationary)

relaxation

recursive hierarchy?

Kyoto 2011. 8. 28-31

Chavanis 08 relaxation to the equilibrium in the point vortices, kinetic equation + maximum entropy production

c.f. Sire-Chavanis 02 motion of the mean field of many self-gravitating Brownian particles, BBGKY hierarchy + factorization

Mathematical Analysis

$\Omega \subset \mathbf{R}^2$ bounded domain, $\partial\Omega$ smooth

$$u_t + \beta\nabla\cdot(u\nabla^{\perp}v) = \nabla\cdot(\nabla u - u\nabla v) \text{ in } \Omega\times(0,T)$$

$$\frac{\partial u}{\partial\nu} - u\left(\frac{\partial v}{\partial\nu} + \beta\frac{\partial v}{\partial\tau}\right)\Big|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x) > 0$$

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0$$

Hamiltonian control of sub-collapse dynamics

Theorem H $T < +\infty$ ⟹ formation of collapse

$$u(x,t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx$$

$m(x_0) \in 8\pi\mathbf{N}$ collapse mass quantization possibly with sub-collapse collision

blowup set

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists x_k \to x_0,\ t_k \uparrow T,\ u(x_k,t_k) \to +\infty\}$$

$\mathcal{S} \subset \Omega$ exclusion of boundary blowup $\quad \sharp\mathcal{S} < +\infty$ finiteness of blowup points

$0 < f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega}\setminus\mathcal{S})$ regular part

backward self-similar transformation $\quad x_0 \in \Omega\cap\mathcal{S}$

$$y = (x-x_0)/(T-t)^{1/2}, \quad s = -\log(T-t)$$

$$z(y,s) = (T-t)u(x,t)$$

weak limit $\quad s_k \uparrow +\infty \quad$ subsequence

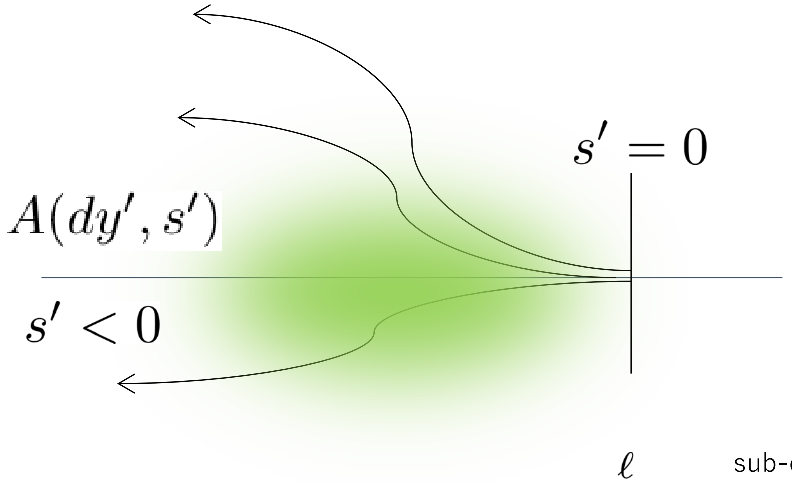
$$z(y,s+s_k)dy \rightharpoonup \exists\zeta(dy,s) \text{ in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2))$$

parabolic envelope

$$m(x_0) = \zeta(\mathbf{R}^2,s) \quad \langle |y|^2, \zeta(dy,s)\rangle \le C$$

scaling back

$$\zeta(dy,s) = e^{-s}A(dy',s'),\ y' = e^{-s/2}y,\ s' = -e^{-s}$$



⟹ formation of sub-collapses
residual vanishing

sub-collapse

$$A(dy',s') = \sum_{j=1}^{\ell} 8\pi\delta_{y_j'(s')}(dy')$$

Sub-collapse dynamics

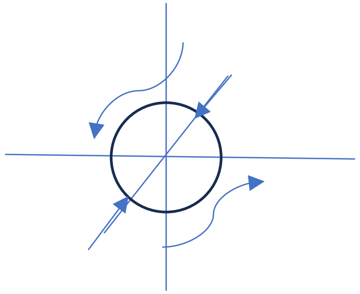
$$A(dy', s') = \sum_{j=1}^{\ell} 8\pi\delta_{y'_j(s')}(dy')$$

simple blowup point

$$\ell = 1 \ \Rightarrow \ A(dy', s') = 8\pi\delta_0(dy')$$

recursive hierarchy $\ell \geq 2$ $\qquad \Gamma(y') = \frac{1}{2\pi}\log\frac{1}{|y'|}$

$$\frac{dy'_j}{ds'} + 8\pi\beta\nabla_j^{\perp} H_\ell^0(y'_1, \cdots, y'_\ell) = 8\pi\nabla_j H_\ell^0(y'_1, \cdots, y'_\ell)$$



recursive hierarchy under scaling

$$H_\ell^0(y'_1, \cdots, y'_\ell) = \sum_{1\leq j<k\leq \ell} \Gamma(y'_j - y'_k)$$

Remark

collision of sub-collapses

$$\ell = 2 \ \Longrightarrow \ y'_{\pm}(s') = -2(-s')^{1/2}e^{\imath(\frac{\beta}{2}\log(-s')+c)}$$

c.f. blowup rate Herero-Velazquez 96, Mizoguchi 20, Collot-Ghoul-Masmoudi-Nguyen 22

2-intensity model

Chavanis 08

$$\frac{\partial u_1}{\partial t} + \beta\nabla\cdot u_1\nabla^{\perp}v = \Delta u_1 - \nabla\cdot u_1\nabla v$$

$$\frac{\partial u_2}{\partial t} - \beta\nabla\cdot u_2\nabla^{\perp}v = \Delta u_2 + \nabla\cdot u_2\nabla v$$

$$\frac{\partial u_1}{\partial \nu} - u_1\left(\frac{\partial v}{\partial \nu} + \beta\frac{\partial v}{\partial \tau}\right)\Bigg|_{\partial\Omega} = 0$$

$$\frac{\partial u_2}{\partial \nu} + u_2\left(\frac{\partial v}{\partial \nu} - \beta\frac{\partial v}{\partial \tau}\right)\Bigg|_{\partial\Omega} = 0$$

$$-\Delta v = u_1 - u_2, \quad v|_{\partial\Omega} = 0$$

$$(u, v)|_{t=0} = (u_0(x), v_0(x)) > 0$$

Stationary state

quantized blowup mechanism

Ohtsuka-S. 06, Jost-Wang-Ye-Zhou 08 quotenion

$$-\Delta v = \frac{\lambda_1 e^v}{\int_\Omega e^v} - \frac{\lambda_2 e^{-v}}{\int_\Omega e^{-v}}, \quad v|_{\partial\Omega} = 0$$

Summary

1. Thermodynamical background of the Boltzmann-Poisson equation is described in contrast with the mathematical modeling of biological events.

2. Order structure of many point vortices is analyzed through the Boltzmann-Poisson equation.

3. Two categories are formulated in accordance with the distribution of intensities; stochastic (Neri) and deterministic (Sawada-S.).

4. Uniqueness and non-existence of the solution are shown for the associated mean field equation on sphere and disc.

5. The proof relies on the isoperimetric inequality of Payne-Sperb-Stakgold type that on sphere is new.

6. Combined with the existence of the solution obtained by the variational method, emergence of ordered structure is observed at the first level of negative temperature formulated by the Trudinger-Moser inequality.

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