

Energy decay of solutions to the wave equation with space-dependent effective damping localized near infinity

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第3回奈良女大・埼大関数不等式研究

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Plan of talk

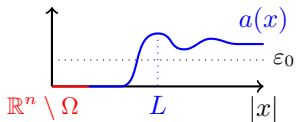
1. Introduction: damped wave equation & energy decay

$$u_{tt} - \Delta u + u_t = 0, \quad t > 0, x \in \mathbb{R}^n$$

2. Localized damping near infinity

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

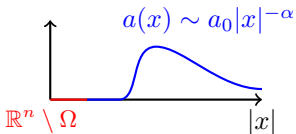
Results by Nakao (2001) & Ikehata (2003)



3. Effective damping localized near infinity

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

Main result (in preparation)



1. Introduction: damped wave equation & energy decay

$$u_{tt} - \Delta u + u_t = 0, \quad t > 0, x \in \mathbb{R}^n$$

Damped wave equation

Damped wave equation

Consider the initial value problem for the damped wave equation in the whole space \mathbb{R}^n :

$$(1) \quad \begin{cases} u_{tt} - \Delta u + u_t = 0, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), u_t(0, x) = u_1(x), & x \in \mathbb{R}^n. \end{cases}$$

- $u = u(t, x)$: real-valued unknown function
- This equation describes the propagation of waves **decaying by the effect of damping or friction**.
- This equation is also called telegrapher's equation or dissipative wave equation, etc.

Energy

$$(2) \quad E(t) := \frac{1}{2} \int_{\mathbb{R}^n} (|u_t(t, x)|^2 + |\nabla_x u(t, x)|^2) dx \quad (t \geq 0)$$

is called the (total) energy of u .

Proposition 1

If $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then the solution u to (1) satisfies

$$E(t) + \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^2 dx ds = E(0) \quad (t \geq 0).$$

In particular, $E(t)$ is non-increasing.

Proof of Proposition 1

We formally calculate

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt} \left(\frac{1}{2} \int_{\mathbb{R}^n} (|u_t|^2 + |\nabla u|^2) dx \right) \\ &= \int_{\mathbb{R}^n} (u_t u_{tt} + \nabla u \cdot \nabla u_t) dx \\ &= \int_{\mathbb{R}^n} (u_t u_{tt} - u_t \Delta u) dx \\ &= - \int_{\mathbb{R}^n} |u_t|^2 dx.\end{aligned}$$

Integrating the above on $[0, t]$, we get

$$E(t) + \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^2 dx ds = E(0).$$



Sharp decay estimate of $E(t)$

Question (Energy decay problem)

$\lim_{t \rightarrow \infty} E(t) = 0$? If so, what is the sharp decay rate in t ?

In the whole space case (1), we can easily obtain the following answer.

Proposition 2

If $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then the solution u to (1) satisfies

$$E(t) \leq C(1+t)^{-1} (\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2) \quad (t \geq 0).$$

Moreover, the decay order $(1+t)^{-1}$ is sharp.

Proof of Proposition 2

There are several ways for the proof (Fourier analysis, solution formula, ...). But here we use the energy method for the later purpose.

We put

$$E_0(t) := \int_{\mathbb{R}^n} \left(u(t, x)u_t(t, x) + \frac{1}{2}|u(t, x)|^2 \right) dx$$

and formally calculate

$$\begin{aligned} \frac{d}{dt} E_0(t) &= \int_{\mathbb{R}^n} (u_t^2 + uu_{tt} + uu_t) dx \\ &= \int_{\mathbb{R}^n} (u_t^2 + u\Delta u) dx \\ &= \int_{\mathbb{R}^n} (u_t^2 - |\nabla u|^2) dx. \end{aligned}$$

Then, we need to control the positive **bad term** u_t^2 .

Combining $E(t)$ and $E_0(t)$, we define the modified energy by

$$\mathcal{E}(t) := 2(1+t)E(t) + E_0(t).$$

Then, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= \frac{d}{dt} [2(1+t)E(t) + E_0(t)] \\ &= 2E(t) + 2(1+t)\frac{d}{dt}E(t) + \frac{d}{dt}E_0(t) \\ &= 2E(t) - 2(1+t) \int_{\mathbb{R}^n} u_t^2 dx + \int_{\mathbb{R}^n} (u_t^2 - |\nabla u|^2) dx \\ &= \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2 - 2(1+t)u_t^2 + u_t^2 - |\nabla u|^2) dx \\ &= -2t \int_{\mathbb{R}^n} u_t^2 dx \\ &\leq 0. \end{aligned}$$

Integrating the preceding inequality on $[0, t]$, we get

$$\mathcal{E}(t) \leq \mathcal{E}(0) \quad (t \geq 0).$$

On the other hand, by the Schwarz inequality, one can easily show

$$\begin{aligned} \mathcal{E}(t) &:= 2(1+t) \int_{\mathbb{R}^n} (|u_t|^2 + |\nabla u|^2) dx + \int_{\mathbb{R}^n} \left(uu_t + \frac{1}{2}|u|^2 \right) dx \\ &\sim (1+t)E(t) + \|u(t)\|_{L^2}^2. \end{aligned}$$

Thus, we conclude

$$E(t) \leq C(1+t)^{-1} (\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2) \quad (t \geq 0).$$

For the sharpness of the order $(1+t)^{-1}$, we need to investigate the asymptotic profile of $u(t, x)$.

Actually, we can prove that

$$u(t) \sim e^{t\Delta}(u_0 + u_1) \quad \text{for large } t$$

by using the Fourier analysis, solution formula, or energy method, etc. (we omit the details).

Moreover, the sharpness of the estimate

$$\|\partial_t^j \partial_x^\alpha e^{t\Delta} f\|_{L^p} \leq C t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - j - \frac{|\alpha|}{2}} \|f\|_{L^q} \quad (t > 0, f \in L^q(\mathbb{R}^n))$$

for $1 \leq q \leq p \leq \infty$, $j \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{Z}_{\geq 0}^n$ is well known (by, e.g., the scaling argument).

So, we have the sharpness of the order $(1+t)^{-1}$ in the estimate

$$E(t) \leq C(1+t)^{-1} (\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2) \quad (t \geq 0).$$



Remark: L^p - L^q type decay

Similarly to the L^p - L^q estimates for the heat equation

$$\|\partial_t^j \partial_x^\alpha e^{t\Delta} f\|_{L^p} \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-j-\frac{|\alpha|}{2}} \|f\|_{L^q} \quad (t > 0, f \in L^q(\mathbb{R}^n))$$

for $1 \leq q \leq p \leq \infty$, $j \in \mathbb{Z}_{\geq 0}$, $\alpha \in \mathbb{Z}_{\geq 0}^n$, we can get the following better decay estimate if we assume the additional decay for the initial data at spatial infinity.

Proposition 3 (Matsumura (1976))

$$E(t) \leq C(1+t)^{-n(\frac{1}{q}-\frac{1}{2})-1} (\|u_0\|_{H^1}^2 + \|u_1\|_{L^2}^2 + \|u_0\|_{L^q}^2 + \|u_1\|_{L^q}^2) \quad (t \geq 0)$$

for $1 \leq q \leq 2$.

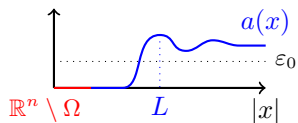
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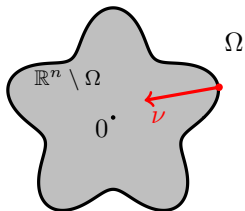
$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

Results by Nakao (2001) & Ikehata (2003)



Exterior domain Ω

- $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) : exterior domain ($\mathbb{R}^n \setminus \Omega$: compact)
- $\partial\Omega$: smooth
- $\mathbb{R}^n \setminus \Omega$: star-shaped with respect to $0 \notin \bar{\Omega}$
- ν : outward normal unit vector of $\partial\Omega$



Wave equation with space-dependent damping

Wave equation with space-dependent damping

Consider the initial-boundary value problem for the damped wave equation in Ω :

$$(3) \quad \begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega. \end{cases}$$

- $a(x) \in C^\infty(\bar{\Omega}) \cap L^\infty(\Omega)$, $a(x) \geq 0$
- $a(x)$ describes the strength of the damping at $x \in \Omega$.
- For $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the existence of unique solution can be proved by the Hille–Yosida theorem.

Energy

$$E(t) := \frac{1}{2} \int_{\Omega} (|u_t(t, x)|^2 + |\nabla_x u(t, x)|^2) dx \quad (t \geq 0).$$

In the same way as Proposition 1, we can show

$$E(t) + \int_0^t \int_{\Omega} a(x) |u_t(s, x)|^2 dx ds = E(0) \quad (t \geq 0).$$

In particular, $E(t)$ is non-increasing.

Question (Energy decay problem)

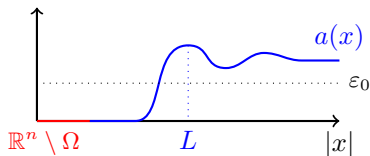
$\lim_{t \rightarrow \infty} E(t) = 0$? If so, how the decay rate in t depends on $a(x)$?

Results of Nakao (2001) & Ikehata (2003)

Assumption 1

There exist $L > 0$ and $\varepsilon_0 > 0$ such that

$$a(x) \geq \varepsilon_0 \quad \text{for } |x| \geq L.$$



Nakao (2001)

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then the solution u to (3) satisfies

$$E(t) \leq C(1+t)^{-1} \left(\|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \right) \quad (t \geq 0).$$

Ikehata (2003)

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and

$$I_0 := \int_{\Omega} |u_1(x) + a(x)u_0(x)|^2 d(x)^2 dx < +\infty,$$

where

$$d(x) := \begin{cases} |x| & (n \geq 3), \\ |x| \log(B|x|) & (n = 2) \end{cases}$$

with some $B \gg 1$, then the solution u to (3) satisfies

$$E(t) \leq C(1+t)^{-2} \left(\|u_0\|_{H^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 + I_0 \right) \quad (t \geq 0).$$

This result implies that additional decay assumption on initial data gives faster decay of energy.

Outline of Proof of Nakao (2001)

Lemma 4 (cf. Morawetz (1961), J. L. Lions (1988), Zuazua (1990))

Let $h(x) = (h_1(x), \dots, h_n(x)) \in \text{Lip}(\overline{\Omega})$. Then, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u_t h(x) \cdot \nabla_x u \, dx \\ &= -\frac{1}{2} \int_{\Omega} (\nabla_x \cdot h) (|u_t|^2 - |\nabla_x u|^2) \, dx + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \nu \cdot h \, dS \\ & \quad - \int_{\Omega} \sum_{i,j=1}^n \partial_i u \partial_j u \partial_i h_j \, dx - \int_{\Omega} a(x) u_t (h \cdot \nabla_x u) \, dx. \end{aligned}$$

We want to take $h(x)$ so that

$$\nabla_x \cdot h(x) > 0 \text{ and } \sum_{i,j=1}^n \partial_i u \partial_j u \partial_i h_j > 0 \text{ in } |x| < L \quad \text{and} \quad \nu \cdot h \leq 0 \text{ on } \partial\Omega.$$

To this end, we choose $h(x)$ as

$$h(x) := \phi(|x|)x$$

with

$$\phi(r) := \begin{cases} \varepsilon_0 & (r \leq L), \\ \frac{\varepsilon_0 L}{r} & (r > L). \end{cases}$$

Then, it is obvious that

$$\nabla_x \cdot h(x) = \varepsilon_0 n > 0 \quad \text{and} \quad \sum_{i,j=1}^n \partial_i u \partial_j u \partial_i h_j = \varepsilon_0 |\nabla_x u|^2 \quad \text{in} \quad |x| < L$$

Moreover, since $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to 0, we have

$$\nu \cdot h(x) = \varepsilon_0 \nu \cdot x \leq 0 \quad \text{on} \quad x \in \partial\Omega.$$

Based on Lemma 4, we define the following modified energy.

Let $\mu, \lambda > 0$ and

$$\mathcal{E}(t) := \int_{\Omega} u_t h(x) \cdot \nabla_x u \, dx + \mu \widetilde{E}_0(t) + (1 + \lambda t) E(t).$$

Here, we put

$$\widetilde{E}_0(t) := \int_{\Omega} \left(u(t, x) u_t(t, x) + \frac{a(x)}{2} |u(t, x)|^2 \right) dx.$$

Then, we can obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq -\frac{n}{2} \varepsilon_0 \int_{|x| < L} |u_t|^2 \, dx - (1 + \lambda t) \int_{\Omega} a(x) |u_t|^2 \, dx + (\mu + \lambda) \int_{\Omega} |u_t|^2 \, dx \\ &\quad - (\mu - \lambda + \varepsilon_0) \int_{\Omega} |\nabla_x u|^2 \, dx + \frac{n}{2} \varepsilon_0 \int_{|x| < L} |\nabla_x u|^2 \, dx + (\text{harmless terms}). \end{aligned}$$

Choosing μ, λ appropriately, we can show (RHS) ≤ 0 . □

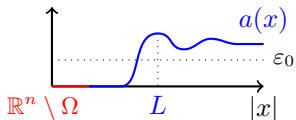
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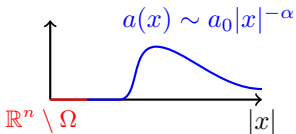
Results by Nakao (2001) & Ikehata (2003)



3. Effective damping localized near infinity

$$u_{tt} - \Delta u + a(x)u_t = 0, \quad t > 0, x \in \Omega$$

Main result (in preparation)



Space-dependent effective damping

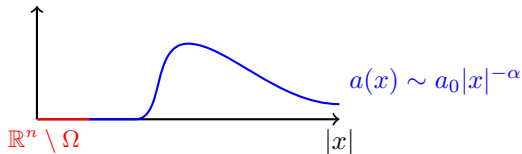
Let us change the assumption of Nakao & Ikehata to the following.

Assumption 2

$a \in C^\infty(\overline{\Omega})$, $a(x) \geq 0$, and there exist $\alpha \in [0, 1)$ and $a_0 > 0$ such that

$$\lim_{|x| \rightarrow \infty} |x|^\alpha a(x) = a_0.$$

The damping term having the above behavior is so-called **effective damping**.



Remark: If $\alpha > 1$, then the energy does not decay to 0 in general (Mochizuki (1976))

Main result

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be an exterior domain with smooth boundary, and assume that $\mathbb{R}^n \setminus \Omega$ is star-shaped with respect to 0 ($\notin \bar{\Omega}$).

Consider the initial-boundary value problem of

$$(4) \quad \begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega. \end{cases}$$

Theorem 5 (In preparation)

Under Assumption 2, if $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies

$$I_0 := \int_{\Omega} (|u_1(x)|^2 + |\nabla_x u_0(x)|^2 + |u_0(x)|^2 \langle x \rangle^{-\alpha}) \langle x \rangle^{\lambda(2-\alpha)} dx < +\infty$$

with some $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$, where $\langle x \rangle := \sqrt{1 + |x|^2}$, then the solution u to (4) satisfies

$$E(t) \leq C(1+t)^{-\lambda-1} I_0 \quad (t \geq 0).$$

Remarks (1)

Let us compare our result

$$E(t) \leq C(1+t)^{-\lambda-1} I_0, \quad \lambda \in \left[0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1}\right)$$

with those of Nakao and Ikehata.

- $\alpha = \lambda = 0 \Rightarrow E(t) \leq C(1+t)^{-1}$.

This corresponds to the result of Nakao (2001).

- $\alpha = 0 \Rightarrow$ the interval for λ is $\left[0, \frac{n}{2} \cdot \frac{n}{2n-1}\right)$.

- $n = 2 \Rightarrow \frac{n}{2} \cdot \frac{n}{2n-1} = \frac{2}{3} < 1$
- $n = 3 \Rightarrow \frac{n}{2} \cdot \frac{n}{2n-1} = \frac{9}{10} < 1$
- $n \geq 4 \Rightarrow \frac{n}{2} \cdot \frac{n}{2n-1} > 1$

So, we do not cover Ikehata's (2003) conclusion $E(t) \leq C(1+t)^{-2}$ for $n = 2, 3$.

(The assumption on the initial data is also different.)

Remarks (2)

- For general $\alpha \in [0, 1)$:
If we strengthen Assumption 2 so that $a(x)$ is positive everywhere and $a(x) \sim a_0|x|^{-\alpha}$ as $|x| \rightarrow \infty$, we have the following:

(Sobajima–W. (2021))

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies

$$I_0 := \int_{\Omega} (|u_1(x)|^2 + |\nabla_x u_0(x)|^2 + |u_0(x)|^2 \langle x \rangle^{-\alpha}) \langle x \rangle^{\lambda(2-\alpha)} dx < +\infty$$

with some $\lambda \in [0, \frac{n-\alpha}{2-\alpha})$, then the solution u to (4) satisfies

$$E(t) \leq C(1+t)^{-\lambda-1} I_0 \quad (t \geq 0).$$

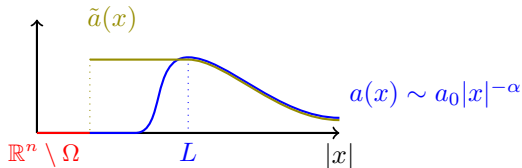
So, we believe that our restriction “ $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$ ” is technical and should be relaxed to “ $\lambda \in [0, \frac{n-\alpha}{2-\alpha})$ ”.

Outline of Proof

Let $L > 0$ be sufficiently large so that $a(x) \geq \frac{a_0}{2}|x|^{-\alpha}$ for $|x| > L$.

We define $\tilde{a}(x)$ so that

$$\tilde{a}(x) = a(x) \text{ in } |x| > L \quad \text{and} \quad \tilde{a}(x) > 0 \text{ everywhere.}$$



For $\tilde{a}(x)$, Sobajima–W. (2021) constructed a weight function $\Phi(t, x)$ such that

$$\Phi(t, x) \sim (t_0 + t + \langle x \rangle^{2-\alpha})^{-\beta}$$

and

$$\tilde{a}(x)\partial_t\Phi(t, x) - \Delta\Phi(t, x) \geq c\tilde{a}(x) (t_0 + t + \langle x \rangle^{2-\alpha})^{-\beta-1},$$

where

$$\beta \in \left[0, \frac{n-\alpha}{2-\alpha}\right), \quad t_0 \geq 1$$

are parameters and c is a constant.

The above inequality means that Φ is a supersolution of the heat equation $\tilde{a}(x)\partial_tv - \Delta v = 0$.

By this weight function Φ , we define the weighted energies by

$$E_1(t) := \frac{1}{2} \int_{\Omega} (|u_t|^2 + |\nabla_x u|^2) (t_0 + t + \langle x \rangle^{2-\alpha})^{\lambda + \frac{\alpha}{2-\alpha}} dx,$$

$$E_0(t) := \int_{\Omega} \left(uu_t + \frac{a(x)}{2} |u|^2 \right) \Phi(t, x)^{-1+2\delta} dx,$$

where $\delta \in (0, 1/2)$ is a parameter.

These $E_1(t)$ and $E_0(t)$ satisfy the following good energy estimates.

$$E_1(t) := \frac{1}{2} \int_{\Omega} (|u_t|^2 + |\nabla_x u|^2) (t_0 + t + \langle x \rangle^{2-\alpha})^{\lambda + \frac{\alpha}{2-\alpha}} dx$$

For simplicity, we put

$$\Psi(t, x) := t_0 + t + \langle x \rangle^{2-\alpha}.$$

Lemma 6

$$\begin{aligned} \frac{d}{dt} E_1(t) &= - \int_{\Omega} a(x) |u_t|^2 \Psi^{\lambda + \frac{\alpha}{2-\alpha}} dx \\ &\quad - \left(\lambda + \frac{\alpha}{2-\alpha} \right) \int_{\Omega} u_t (\nabla_x u \cdot \nabla_x \Psi) \Psi^{\lambda + \frac{\alpha}{2-\alpha} - 1} dx \\ &\quad + \frac{1}{2} \left(\lambda + \frac{\alpha}{2-\alpha} \right) \int_{\Omega} (|u_t|^2 + |\nabla_x u|^2) \Psi^{\lambda + \frac{\alpha}{2-\alpha} - 1} dx. \end{aligned}$$

$$E_0(t) := \int_{\Omega} \left(uu_t + \frac{a(x)}{2} |u|^2 \right) \Phi(t, x)^{-1+2\delta} dx$$

Lemma 7

$$\begin{aligned} \frac{d}{dt} E_0(t) &\leq -\frac{\delta}{1-\delta} \int_{\Omega} |\nabla_x u|^2 \Phi^{-1+2\delta} dx \\ &\quad - \frac{1-2\delta}{2} \int_{\Omega} |u|^2 \Phi^{-1+2\delta} \cdot c\tilde{a}(x) \Psi^{-\beta-1} dx \\ &\quad + \int_{\Omega} |u_t|^2 \Phi^{-1+2\delta} dx \\ &\quad - (1-2\delta) \int_{\Omega} uu_t \Phi^{-2+2\delta} \partial_t \Phi dx \\ &\quad - \frac{1-2\delta}{2} \int_{\Omega} |u|^2 \Phi^{-1+2\delta} (a(x) - \tilde{a}(x)) \partial_t \Phi dx. \end{aligned}$$

Lemmas 6 & 7 are essentially given in Sobajima–W. (2021).

The crucial point is to apply Lemma 4 with

$$h(t, x) := \begin{cases} \mu\Phi(t, x)^{-1+2\delta}x & (|x| < L), \\ \mu\Phi(t, x)^{-1+2\delta}\frac{L}{|x|}x & (|x| \geq L), \end{cases}$$

where $\mu > 0$ is a suitably small constant.

Then, we have:

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} u_t h(t, x) \cdot \nabla_x u \, dx \\ &= -\frac{1}{2} \int_{\Omega} (\nabla_x \cdot h) (|u_t|^2 - |\nabla_x u|^2) \, dx + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 \nu \cdot h \, dS \\ & \quad - \int_{\Omega} \sum_{i,j=1}^n \partial_i u \partial_j u \partial_i h_j \, dx - \int_{\Omega} a(x) u_t (h \cdot \nabla_x u) \, dx \\ & \quad - \int_{\Omega} u \partial_t h(t, x) \cdot \nabla_x u \, dx. \end{aligned}$$

Finally, we define the modified energy by

$$\mathcal{E}(t) := \int_{\Omega} u_t h(t, x) \cdot \nabla_x u \, dx + \nu E_0(t) + E_1(t),$$

where $\nu > 0$ is a suitably small constant.

Then, we obtain

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &\leq \int_{\Omega} \left(-\frac{1}{2} \nabla_x \cdot h(t, x) + \nu \Phi^{-1+2\delta} - a(x) \Psi^{\lambda + \frac{\alpha}{2-\alpha}} \right) |u_t|^2 \, dx \\ &\quad + \int_{\Omega} \left(\frac{1}{2} \nabla_x h(t, x) - \nu \frac{\delta}{1-\delta} \Phi^{-1+2\delta} \right) |\nabla_x u|^2 \, dx \\ &\quad + \dots \end{aligned}$$

Taking the parameters $\delta, \mu, \nu, \beta, t_0$ appropriately (in this step, we use the assumption $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$), we can get (RHS) ≤ 0 . □

Summary

$$(4) \quad \begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & t > 0, x \in \Omega, \\ u = 0, & t > 0, x \in \partial\Omega, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \Omega. \end{cases}$$

$a \in C^\infty(\bar{\Omega})$, $a(x) \geq 0$, and there exist $\alpha \in [0, 1)$ and $a_0 > 0$ such that

$$\lim_{|x| \rightarrow \infty} |x|^\alpha a(x) = a_0.$$

Theorem 5

If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies

$$I_0 := \int_{\Omega} (|u_1(x)|^2 + |\nabla_x u_0(x)|^2 + |u_0(x)|^2 \langle x \rangle^{-\alpha} \langle x \rangle^{\lambda(2-\alpha)} dx < +\infty$$

with some $\lambda \in [0, \frac{n-\alpha}{2-\alpha} \cdot \frac{n}{2n-1})$, then the solution u to (4) satisfies

$$E(t) \leq C(1+t)^{-\lambda-1} I_0 \quad (t \geq 0).$$

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