The right side of (56) tends to ∞ as $R \to \infty$. Hence there exists R_0 such that $|P(z)| > \mu$ if $|z| > R_0$. Since $|P|$ is continuous on the closed disc with center at 0 and radius R_0, Theorem 4.16 shows that $|P(z_0)| = \mu$ for some z_0.

We claim that $\mu = 0$.

If not, put $Q(z) = P(z + z_0)/P(z_0)$. Then Q is a nonconstant polynomial, $Q(0) = 1$, and $|Q(z)| \geq 1$ for all z. There is a smallest integer k, $1 \leq k \leq n$, such that

(57) $$Q(z) = 1 + b_k z^k + \cdots + b_n z^n, \qquad b_k \neq 0.$$

By Theorem 8.7(d) there is a real θ such that

(58) $$e^{ik\theta} b_k = -|b_k|.$$

If $r > 0$ and $r^k |b_k| < 1$, (58) implies

$$|1 + b_k r^k e^{ik\theta}| = 1 - r^k |b_k|,$$

so that

$$|Q(re^{i\theta})| \leq 1 - r^k\{|b_k| - r|b_{k+1}| - \cdots - r^{n-k}|b_n|\}.$$

For sufficiently small r, the expression in braces is positive; hence $|Q(re^{i\theta})| < 1$, a contradiction.

Thus $\mu = 0$, that is, $P(z_0) = 0$.

Exercise 27 contains a more general result.

FOURIER SERIES

8.9 Definition A *trigonometric polynomial* is a finite sum of the form

(59) $$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \qquad (x \text{ real}),$$

where $a_0, \ldots, a_N, b_1, \ldots, b_N$ are complex numbers. On account of the identities (46), (59) can also be written in the form

(60) $$f(x) = \sum_{-N}^{N} c_n e^{inx} \qquad (x \text{ real}),$$

which is more convenient for most purposes. It is clear that every trigonometric polynomial is periodic, with period 2π.

If n is a nonzero integer, e^{inx} is the derivative of e^{inx}/in, which also has period 2π. Hence

(61) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\, dx = \begin{cases} 1 & (\text{if } n = 0), \\ 0 & (\text{if } n = \pm 1, \pm 2, \ldots). \end{cases}$$

Let us multiply (60) by e^{-imx}, where m is an integer; if we integrate the product, (61) shows that

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx \tag{62}$$

for $|m| \le N$. If $|m| > N$, the integral in (62) is 0.

The following observation can be read off from (60) and (62): The trigonometric polynomial f, given by (60), is *real* if and only if $c_{-n} = \overline{c_n}$ for $n = 0, \ldots, N$.

In agreement with (60), we define a *trigonometric series* to be a series of the form

$$\sum_{-\infty}^{\infty} c_n e^{inx} \qquad (x \text{ real}); \tag{63}$$

the Nth partial sum of (63) is defined to be the right side of (60).

If f is an integrable function on $[-\pi, \pi]$, the numbers c_m defined by (62) for all integers m are called the *Fourier coefficients* of f, and the series (63) formed with these coefficients is called the *Fourier series* of f.

The natural question which now arises is whether the Fourier series of f converges to f, or, more generally, whether f is determined by its Fourier series. That is to say, if we know the Fourier coefficients of a function, can we find the function, and if so, how?

The study of such series, and, in particular, the problem of representing a given function by a trigonometric series, originated in physical problems such as the theory of oscillations and the theory of heat conduction (Fourier's "Théorie analytique de la chaleur" was published in 1822). The many difficult and delicate problems which arose during this study caused a thorough revision and reformulation of the whole theory of functions of a real variable. Among many prominent names, those of Riemann, Cantor, and Lebesgue are intimately connected with this field, which nowadays, with all its generalizations and ramifications, may well be said to occupy a central position in the whole of analysis.

We shall be content to derive some basic theorems which are easily accessible by the methods developed in the preceding chapters. For more thorough investigations, the Lebesgue integral is a natural and indispensable tool.

We shall first study more general systems of functions which share a property analogous to (61).

8.10 Definition Let $\{\phi_n\}$ $(n = 1, 2, 3, \ldots)$ be a sequence of complex functions on $[a, b]$, such that

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}\,dx = 0 \qquad (n \ne m). \tag{64}$$

Then $\{\phi_n\}$ is said to be an *orthogonal system of functions* on $[a, b]$. If, in addition,

(65) $$\int_a^b |\phi_n(x)|^2 \, dx = 1$$

for all n, $\{\phi_n\}$ is said to be *orthonormal.*

For example, the functions $(2\pi)^{-\frac{1}{2}} e^{inx}$ form an orthonormal system on $[-\pi, \pi]$. So do the real functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots .$$

If $\{\phi_n\}$ is orthonormal on $[a, b]$ and if

(66) $$c_n = \int_a^b f(t)\overline{\phi_n(t)} \, dt \qquad (n = 1, 2, 3, \ldots),$$

we call c_n the nth Fourier coefficient of f relative to $\{\phi_n\}$. We write

(67) $$f(x) \sim \sum_1^\infty c_n \phi_n(x)$$

and call this series the Fourier series of f (relative to $\{\phi_n\}$).

Note that the symbol $\sim$ used in (67) implies nothing about the convergence of the series; it merely says that the coefficients are given by (66).

The following theorems show that the partial sums of the Fourier series of f have a certain minimum property. We shall assume here and in the rest of this chapter that $f \in \mathscr{R}$, although this hypothesis can be weakened.

8.11 Theorem *Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Let*

(68) $$s_n(x) = \sum_{m=1}^n c_m \phi_m(x)$$

be the nth partial sum of the Fourier series of f, and suppose

(69) $$t_n(x) = \sum_{m=1}^n \gamma_m \phi_m(x).$$

Then

(70) $$\int_a^b |f - s_n|^2 \, dx \le \int_a^b |f - t_n|^2 \, dx,$$

and equality holds if and only if

(71) $$\gamma_m = c_m \qquad (m = 1, \ldots, n).$$

That is to say, among all functions t_n, s_n gives the best possible mean square approximation to f.

Proof Let $\int$ denote the integral over $[a, b]$, Σ the sum from 1 to n. Then

$$\int f\bar{t}_n = \int f \sum \bar{\gamma}_m \bar{\phi}_m = \sum c_m \bar{\gamma}_m$$

by the definition of $\{c_m\}$,

$$\int |t_n|^2 = \int t_n \bar{t}_n = \int \sum \gamma_m \phi_m \sum \bar{\gamma}_k \bar{\phi}_k = \sum |\gamma_m|^2$$

since $\{\phi_m\}$ is orthonormal, and so

$$\begin{aligned}\int |f - t_n|^2 &= \int |f|^2 - \int f\bar{t}_n - \int \bar{f}t_n + \int |t_n|^2 \\ &= \int |f|^2 - \sum c_m \bar{\gamma}_m - \sum \bar{c}_m \gamma_m + \sum \gamma_m \bar{\gamma}_m \\ &= \int |f|^2 - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2,\end{aligned}$$

which is evidently minimized if and only if $\gamma_m = c_m$.

Putting $\gamma_m = c_m$ in this calculation, we obtain

(72) $$\int_a^b |s_n(x)|^2\,dx = \sum_1^n |c_m|^2 \leq \int_a^b |f(x)|^2\,dx,$$

since $\int |f - t_n|^2 \geq 0$.

8.12 Theorem *If $\{\phi_n\}$ is orthonormal on $[a, b]$, and if*

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

then

(73) $$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2\,dx.$$

In particular,

(74) $$\lim_{n\to\infty} c_n = 0.$$

Proof Letting $n \to \infty$ in (72), we obtain (73), the so-called "Bessel inequality."

8.13 Trigonometric series From now on we shall deal only with the trigonometric system. We shall consider functions f that have period 2π and that are Riemann-integrable on $[-\pi, \pi]$ (and hence on every bounded interval). The Fourier series of f is then the series (63) whose coefficients c_n are given by the integrals (62), and

(75) $$s_N(x) = s_N(f; x) = \sum_{-N}^{N} c_n e^{inx}$$

is the Nth partial sum of the Fourier series of f. The inequality (72) now takes the form

(76) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}|s_N(x)|^2\,dx = \sum_{-N}^{N}|c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx.$$

In order to obtain an expression for s_N that is more manageable than (75) we introduce the *Dirichlet kernel*

(77) $$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin(N+\frac{1}{2})x}{\sin(x/2)}.$$

The first of these equalities is the definition of $D_N(x)$. The second follows if both sides of the identity

$$(e^{ix}-1)D_N(x) = e^{i(N+1)x} - e^{-iNx}$$

are multiplied by $e^{-ix/2}$.

By (62) and (75), we have

$$s_N(f;x) = \sum_{-N}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt\,e^{inx}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{-N}^{N} e^{in(x-t)}\,dt,$$

so that

(78) $$s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_N(x-t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt.$$

The periodicity of all functions involved shows that it is immaterial over which interval we integrate, as long as its length is 2π. This shows that the two integrals in (78) are equal.

We shall prove just one theorem about the pointwise convergence of Fourier series.

8.14 Theorem *If, for some x, there are constants $\delta > 0$ and $M < \infty$ such that*

(79) $$|f(x+t) - f(x)| \le M|t|$$

for all $t \in (-\delta, \delta)$, then

(80) $$\lim_{N\to\infty} s_N(f;x) = f(x).$$

Proof Define

(81) $$g(t) = \frac{f(x-t) - f(x)}{\sin(t/2)}$$

for $0 < |t| \le \pi$, and put $g(0) = 0$. By the definition (77),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\, dx = 1.$$

Hence (78) shows that

$$s_N(f; x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) \sin\left(N + \frac{1}{2}\right)t\, dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[g(t)\cos\frac{t}{2}\right]\sin Nt\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[g(t)\sin\frac{t}{2}\right]\cos Nt\, dt.$$

By (79) and (81), $g(t)\cos(t/2)$ and $g(t)\sin(t/2)$ are bounded. The last two integrals thus tend to 0 as $N \to \infty$, by (74). This proves (80).

Corollary *If $f(x) = 0$ for all x in some segment J, then* $\lim s_N(f; x) = 0$ *for every $x \in J$.*

Here is another formulation of this corollary:

If $f(t) = g(t)$ for all t in some neighborhood of x, then

$$s_N(f; x) - s_N(g; x) = s_N(f - g; x) \to 0 \textit{ as } N \to \infty.$$

This is usually called the *localization theorem.* It shows that the behavior of the sequence $\{s_N(f; x)\}$, as far as convergence is concerned, depends only on the values of f in some (arbitrarily small) neighborhood of x. Two Fourier series may thus have the same behavior in one interval, but may behave in entirely different ways in some other interval. We have here a very striking contrast between Fourier series and power series (Theorem 8.5).

We conclude with two other approximation theorems.

8.15 Theorem *If f is continuous (with period 2π) and if $\varepsilon > 0$, then there is a trigonometric polynomial P such that*

$$|P(x) - f(x)| < \varepsilon$$

for all real x.

Proof If we identify x and $x + 2\pi$, we may regard the 2π-periodic functions on R^1 as functions on the unit circle T, by means of the mapping $x \to e^{ix}$. The trigonometric polynomials, i.e., the functions of the form (60), form a self-adjoint algebra $\mathscr{A}$, which separates points on T, and which vanishes at no point of T. Since T is compact, Theorem 7.33 tells us that $\mathscr{A}$ is dense in $\mathscr{C}(T)$. This is exactly what the theorem asserts.

A more precise form of this theorem appears in Exercise 15.

8.16 Parseval's theorem *Suppose f and g are Riemann-integrable functions with period 2π, and*

(82) $$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \qquad g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}.$$

Then

(83) $$\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 \, dx = 0,$$

(84) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx = \sum_{-\infty}^{\infty} c_n \bar{\gamma}_n,$$

(85) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

Proof Let us use the notation

(86) $$\|h\|_2 = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(x)|^2 \, dx\right\}^{1/2}.$$

Let $\varepsilon > 0$ be given. Since $f \in \mathscr{R}$ and $f(\pi) = f(-\pi)$, the construction described in Exercise 12 of Chap. 6 yields a continuous 2π-periodic function h with

(87) $$\|f - h\|_2 < \varepsilon.$$

By Theorem 8.15, there is a trigonometric polynomial P such that $|h(x) - P(x)| < \varepsilon$ for all x. Hence $\|h - P\|_2 < \varepsilon$. If P has degree N_0, Theorem 8.11 shows that

(88) $$\|h - s_N(h)\|_2 \le \|h - P\|_2 < \varepsilon$$

for all $N \ge N_0$. By (72), with $h - f$ in place of f,

(89) $$\|s_N(h) - s_N(f)\|_2 = \|s_N(h - f)\|_2 \le \|h - f\|_2 < \varepsilon.$$

Now the triangle inequality (Exercise 11, Chap. 6), combined with (87), (88), and (89), shows that

(90) $$\|f - s_N(f)\|_2 < 3\varepsilon \qquad (N \ge N_0).$$

This proves (83). Next,

(91) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} s_N(f)\bar{g}\, dx = \sum_{-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{g(x)}\, dx = \sum_{-N}^{N} c_n \bar{\gamma}_n,$$

and the Schwarz inequality shows that

(92) $$\left|\int f\bar{g} - \int s_N(f)\bar{g}\right| \le \int |f - s_N(f)||g| \le \left\{\int |f - s_N|^2 \int |g|^2\right\}^{1/2},$$

which tends to 0, as $N \to \infty$, by (83). Comparison of (91) and (92) gives (84). Finally, (85) is the special case $g = f$ of (84).

A more general version of Theorem 8.16 appears in Chap. 11.

THE GAMMA FUNCTION

This function is closely related to factorials and crops up in many unexpected places in analysis. Its origin, history, and development are very well described in an interesting article by P. J. Davis (*Amer. Math. Monthly*, vol. 66, 1959, pp. 849–869). Artin's book (cited in the Bibliography) is another good elementary introduction.

Our presentation will be very condensed, with only a few comments after each theorem. This section may thus be regarded as a large exercise, and as an opportunity to apply some of the material that has been presented so far.

8.17 Definition For $0 < x < \infty$,

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt. \tag{93}$$

The integral converges for these x. (When $x < 1$, both 0 and ∞ have to be looked at.)

8.18 Theorem

(*a*) *The functional equation*

$$\Gamma(x+1) = x\Gamma(x)$$

holds if $0 < x < \infty$.

(*b*) $\Gamma(n+1) = n!$ *for* $n = 1, 2, 3, \ldots$.

(*c*) $\log \Gamma$ *is convex on* $(0, \infty)$.

Proof An integration by parts proves (*a*). Since $\Gamma(1) = 1$, (*a*) implies (*b*), by induction. If $1 < p < \infty$ and $(1/p) + (1/q) = 1$, apply Hölder's inequality (Exercise 10, Chap. 6) to (93), and obtain

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \le \Gamma(x)^{1/p}\Gamma(y)^{1/q}.$$

This is equivalent to (*c*).

It is a rather surprising fact, discovered by Bohr and Mollerup, that these three properties characterize Γ completely.

INTEGRATION OF COMPLEX FUNCTIONS

Suppose f is a complex-valued function defined on a measure space X, and $f = u + iv$, where u and v are real. We say that f is measurable if and only if both u and v are measurable.

It is easy to verify that sums and products of complex measurable functions are again measurable. Since

$$|f| = (u^2 + v^2)^{1/2},$$

Theorem 11.18 shows that $|f|$ is measurable for every complex measurable f.

Suppose μ is a measure on X, E is a measurable subset of X, and f is a complex function on X. We say that $f \in \mathscr{L}(\mu)$ on E provided that f is measurable and

$$\int_E |f|\, d\mu < +\infty, \tag{97}$$

and we define

$$\int_E f\, d\mu = \int_E u\, d\mu + i \int_E v\, d\mu$$

if (97) holds. Since $|u| \le |f|$, $|v| \le |f|$, and $|f| \le |u| + |v|$, it is clear that (97) holds if and only if $u \in \mathscr{L}(\mu)$ and $v \in \mathscr{L}(\mu)$ on E.

Theorems 11.23(*a*), (*d*), (*e*), (*f*), 11.24(*b*), 11.26, 11.27, 11.29, and 11.32 can now be extended to Lebesgue integrals of complex functions. The proofs are quite straightforward. That of Theorem 11.26 is the only one that offers anything of interest:

If $f \in \mathscr{L}(\mu)$ on E, there is a complex number c, $|c| = 1$, such that

$$c \int_E f\, d\mu \ge 0.$$

Put $g = cf = u + iv$, u and v real. Then

$$\left| \int_E f\, d\mu \right| = c \int_E f\, d\mu = \int_E g\, d\mu = \int_E u\, d\mu \le \int_E |f|\, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int g\, d\mu$ is real.

FUNCTIONS OF CLASS $\mathscr{L}^2$

As an application of the Lebesgue theory, we shall now extend the Parseval theorem (which we proved only for Riemann-integrable functions in Chap. 8) and prove the Riesz-Fischer theorem for orthonormal sets of functions.

11.34 Definition Let X be a measurable space. We say that a complex function $f \in \mathscr{L}^2(\mu)$ on X if f is measurable and if

$$\int_X |f|^2 \, d\mu < +\infty.$$

If μ is Lebesgue measure, we say $f \in \mathscr{L}^2$. For $f \in \mathscr{L}^2(\mu)$ (we shall omit the phrase "on X" from now on) we define

$$\|f\| = \left\{ \int_X |f|^2 \, d\mu \right\}^{1/2}$$

and call $\|f\|$ the $\mathscr{L}^2(\mu)$ norm of f.

11.35 Theorem *Suppose $f \in \mathscr{L}^2(\mu)$ and $g \in \mathscr{L}^2(\mu)$. Then $fg \in \mathscr{L}(\mu)$, and*

(98)
$$\int_X |fg| \, d\mu \le \|f\| \, \|g\|.$$

This is the Schwarz inequality, which we have already encountered for series and for Riemann integrals. It follows from the inequality

$$0 \le \int_X (|f| + \lambda |g|)^2 \, d\mu = \|f\|^2 + 2\lambda \int_X |fg| \, d\mu + \lambda^2 \|g\|^2,$$

which holds for every real λ.

11.36 Theorem *If $f \in \mathscr{L}^2(\mu)$ and $g \in \mathscr{L}^2(\mu)$, then $f + g \in \mathscr{L}^2(\mu)$, and*

$$\|f + g\| \le \|f\| + \|g\|.$$

Proof The Schwarz inequality shows that

$$\begin{aligned} \|f + g\|^2 &= \int |f|^2 + \int f\bar{g} + \int \bar{f}g + \int |g|^2 \\ &\le \|f\|^2 + 2\|f\| \, \|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned}$$

11.37 Remark If we define the distance between two functions f and g in $\mathscr{L}^2(\mu)$ to be $\|f - g\|$, we see that the conditions of Definition 2.15 are satisfied, except for the fact that $\|f - g\| = 0$ does not imply that $f(x) = g(x)$ for all x, but only for almost all x. Thus, if we identify functions which differ only on a set of measure zero, $\mathscr{L}^2(\mu)$ is a metric space.

We now consider $\mathscr{L}^2$ on an interval of the real line, with respect to Lebesgue measure.

11.38 Theorem *The continuous functions form a dense subset of $\mathscr{L}^2$ on $[a, b]$.*

More explicitly, this means that for any $f \in \mathscr{L}^2$ on $[a, b]$, and any $\varepsilon > 0$, there is a function g, continuous on $[a, b]$, such that

$$\|f - g\| = \left\{\int_a^b |f - g|^2\, dx\right\}^{1/2} < \varepsilon.$$

Proof We shall say that f is approximated in $\mathscr{L}^2$ by a sequence $\{g_n\}$ if $\|f - g_n\| \to 0$ as $n \to \infty$.

Let A be a closed subset of $[a, b]$, and K_A its characteristic function. Put

$$t(x) = \inf |x - y| \qquad (y \in A)$$

and

$$g_n(x) = \frac{1}{1 + nt(x)} \qquad (n = 1, 2, 3, \ldots).$$

Then g_n is continuous on $[a, b]$, $g_n(x) = 1$ on A, and $g_n(x) \to 0$ on B, where $B = [a, b] - A$. Hence

$$\|g_n - K_A\| = \left\{\int_B g_n^2\, dx\right\}^{1/2} \to 0$$

by Theorem 11.32. Thus characteristic functions of closed sets can be approximated in $\mathscr{L}^2$ by continuous functions.

By (39) the same is true for the characteristic function of any measurable set, and hence also for simple measurable functions.

If $f \geq 0$ and $f \in \mathscr{L}^2$, let $\{s_n\}$ be a monotonically increasing sequence of simple nonnegative measurable functions such that $s_n(x) \to f(x)$. Since $|f - s_n|^2 \leq f^2$, Theorem 11.32 shows that $\|f - s_n\| \to 0$.

The general case follows.

11.39 Definition We say that a sequence of complex functions $\{\phi_n\}$ is an *orthonormal* set of functions on a measurable space X if

$$\int_X \phi_n \bar{\phi}_m\, d\mu = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

In particular, we must have $\phi_n \in \mathscr{L}^2(\mu)$. If $f \in \mathscr{L}^2(\mu)$ and if

$$c_n = \int_X f \bar{\phi}_n\, d\mu \qquad (n = 1, 2, 3, \ldots),$$

we write

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

as in Definition 8.10.

The definition of a trigonometric Fourier series is extended in the same way to $\mathscr{L}^2$ (or even to $\mathscr{L}$) on $[-\pi, \pi]$. Theorems 8.11 and 8.12 (the Bessel inequality) hold for any $f \in \mathscr{L}^2(\mu)$. The proofs are the same, word for word.

We can now prove the Parseval theorem.

11.40 Theorem *Suppose*

(99)
$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

where $f \in \mathscr{L}^2$ *on* $[-\pi, \pi]$. *Let* s_n *be the nth partial sum of* (99). *Then*

(100)
$$\lim_{n\to\infty} \|f - s_n\| = 0,$$

(101)
$$\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 \, dx.$$

Proof Let $\varepsilon > 0$ be given. By Theorem 11.38, there is a continuous function g such that

$$\|f - g\| < \frac{\varepsilon}{2}.$$

Moreover, it is easy to see that we can arrange it so that $g(\pi) = g(-\pi)$. Then g can be extended to a periodic continuous function. By Theorem 8.16, there is a trigonometric polynomial T, of degree N, say, such that

$$\|g - T\| < \frac{\varepsilon}{2}.$$

Hence, by Theorem 8.11 (extended to $\mathscr{L}^2$), $n \geq N$ implies

$$\|s_n - f\| \leq \|T - f\| < \varepsilon,$$

and (100) follows. Equation (101) is deduced from (100) as in the proof of Theorem 8.16.

Corollary *If* $f \in \mathscr{L}^2$ *on* $[-\pi, \pi]$, *and if*

$$\int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = 0 \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

then $\|f\| = 0$.

Thus if two functions in $\mathscr{L}^2$ have the same Fourier series, they differ at most on a set of measure zero.

11.41 Definition Let f and $f_n \in \mathscr{L}^2(\mu)$ $(n = 1, 2, 3, \ldots)$. We say that $\{f_n\}$ converges to f in $\mathscr{L}^2(\mu)$ if $\|f_n - f\| \to 0$. We say that $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$ if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$, $m \geq N$ implies $\|f_n - f_m\| \leq \varepsilon$.

11.42 Theorem *If $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$, then there exists a function $f \in \mathscr{L}^2(\mu)$ such that $\{f_n\}$ converges to f in $\mathscr{L}^2(\mu)$.*

This says, in other words, that $\mathscr{L}^2(\mu)$ is a *complete* metric space.

Proof Since $\{f_n\}$ is a Cauchy sequence, we can find a sequence $\{n_k\}$, $k = 1, 2, 3, \ldots$, such that

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k} \qquad (k = 1, 2, 3, \ldots).$$

Choose a function $g \in \mathscr{L}^2(\mu)$. By the Schwarz inequality,

$$\int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \frac{\|g\|}{2^k}.$$

Hence

$$\sum_{k=1}^{\infty} \int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \|g\|. \tag{102}$$

By Theorem 11.30, we may interchange the summation and integration in (102). It follows that

$$|g(x)| \sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty \tag{103}$$

almost everywhere on X. Therefore

$$\sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < +\infty \tag{104}$$

almost everywhere on X. For if the series in (104) were divergent on a set E of positive measure, we could take $g(x)$ to be nonzero on a subset of E of positive measure, thus obtaining a contradiction to (103).

Since the kth partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)),$$

which converges almost everywhere on X, is

$$f_{n_{k+1}}(x) - f_{n_1}(x),$$

we see that the equation

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

defines $f(x)$ for almost all $x \in X$, and it does not matter how we define $f(x)$ at the remaining points of X.

We shall now show that this function f has the desired properties. Let $\varepsilon > 0$ be given, and choose N as indicated in Definition 11.41. If $n_k > N$, Fatou's theorem shows that

$$\|f - f_{n_k}\| \le \liminf_{i \to \infty} \|f_{n_i} - f_{n_k}\| \le \varepsilon.$$

Thus $f - f_{n_k} \in \mathscr{L}^2(\mu)$, and since $f = (f - f_{n_k}) + f_{n_k}$, we see that $f \in \mathscr{L}^2(\mu)$. Also, since ε is arbitrary,

$$\lim_{k \to \infty} \|f - f_{n_k}\| = 0.$$

Finally, the inequality

$$\|f - f_n\| \le \|f - f_{n_k}\| + \|f_{n_k} - f_n\| \tag{105}$$

shows that $\{f_n\}$ converges to f in $\mathscr{L}^2(\mu)$; for if we take n and n_k large enough, each of the two terms on the right of (105) can be made arbitrarily small.

11.43 The Riesz-Fischer theorem *Let $\{\phi_n\}$ be orthonormal on X. Suppose $\Sigma |c_n|^2$ converges, and put $s_n = c_1\phi_1 + \cdots + c_n\phi_n$. Then there exists a function $f \in \mathscr{L}^2(\mu)$ such that $\{s_n\}$ converges to f in $\mathscr{L}^2(\mu)$, and such that*

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Proof For $n > m$,

$$\|s_n - s_m\|^2 = |c_{m+1}|^2 + \cdots + |c_n|^2,$$

so that $\{s_n\}$ is a Cauchy sequence in $\mathscr{L}^2(\mu)$. By Theorem 11.42, there is a function $f \in \mathscr{L}^2(\mu)$ such that

$$\lim_{n \to \infty} \|f - s_n\| = 0.$$

Now, for $n > k$,

$$\int_X f\bar{\phi}_k \, d\mu - c_k = \int_X f\bar{\phi}_k \, d\mu - \int_X s_n \bar{\phi}_k \, d\mu,$$

so that

$$\left| \int_X f\bar{\phi}_k \, d\mu - c_k \right| \le \|f - s_n\| \cdot \|\phi_k\| + \|f - s_n\|.$$

Letting $n \to \infty$, we see that

$$c_k = \int_X f\bar{\phi}_k \, d\mu \qquad (k = 1, 2, 3, \ldots),$$

and the proof is complete.

11.44 Definition An orthonormal set $\{\phi_n\}$ is said to be *complete* if, for $f \in \mathscr{L}^2(\mu)$, the equations

$$\int_X f\bar{\phi}_n \, d\mu = 0 \qquad (n = 1, 2, 3, \ldots)$$

imply that $\|f\| = 0$.

In the Corollary to Theorem 11.40 we deduced the completeness of the trigonometric system from the Parseval equation (101). Conversely, the Parseval equation holds for every complete orthonormal set:

11.45 Theorem *Let $\{\phi_n\}$ be a complete orthonormal set. If $f \in \mathscr{L}^2(\mu)$ and if*

(106) $$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

then

(107) $$\int_X |f|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$

Proof By the Bessel inequality, $\Sigma |c_n|^2$ converges. Putting

$$s_n = c_1\phi_1 + \cdots + c_n\phi_n,$$

the Riesz-Fischer theorem shows that there is a function $g \in \mathscr{L}^2(\mu)$ such that

(108) $$g \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

and such that $\|g - s_n\| \to 0$. Hence $\|s_n\| \to \|g\|$. Since

$$\|s_n\|^2 = |c_1|^2 + \cdots + |c_n|^2,$$

we have

(109) $$\int_X |g|^2 \, d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$

Now (106), (108), and the completeness of $\{\phi_n\}$ show that $\|f - g\| = 0$, so that (109) implies (107).

Combining Theorems 11.43 and 11.45, we arrive at the very interesting conclusion that every complete orthonormal set induces a 1-1 correspondence between the functions $f \in \mathscr{L}^2(\mu)$ (identifying those which are equal almost everywhere) on the one hand and the sequences $\{c_n\}$ for which $\Sigma |c_n|^2$ converges, on the other. The representation

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

together with the Parseval equation, shows that $\mathscr{L}^2(\mu)$ may be regarded as an infinite-dimensional euclidean space (the so-called "Hilbert space"), in which the point f has coordinates c_n, and the functions ϕ_n are the coordinate vectors.

EXERCISES

1. If $f \geq 0$ and $\int_E f\,d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. *Hint:* Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.
2. If $\int_A f\,d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.
3. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.
4. If $f \in \mathscr{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathscr{L}(\mu)$ on E.
5. Put

$$g(x) = \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases}$$
$$f_{2k}(x) = g(x) \qquad (0 \leq x \leq 1),$$
$$f_{2k+1}(x) = g(1 - x) \qquad (0 \leq x \leq 1).$$

Show that

$$\liminf_{n \to \infty} f_n(x) = 0 \qquad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x)\,dx = \tfrac{1}{2}.$$

[Compare with (77).]