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Functional Analysis

11/24, 12/1 (Friday)

Fourier series

(67) $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}, \quad 0 \leqq x \leqq 1$

c_n: n-th Fourier coefficient

(62') $c_m = \int_0^1 f(x) e^{-2\pi i m x} dx$.

(64, 65) O.N.S

$\{\phi_n\}_{n=1}^{\infty}$ sequence of ft's s.t.

$$\int_0^1 \phi_n(x)\overline{\phi_m(x)}\,dx = \delta_{n,m} = \begin{cases} 1 & (n=m) \\ 0 & (n\neq m) \end{cases}$$

E.g. $\{e^{2\pi i n x}\}_{n=-\infty}^{\infty}$ is an O.N.S.

8.11 Thm (for general O.N.S.)

(68) $s_n(x) := \sum_{m=1}^{n} c_m \phi_m(x)$,

where $c_m := \int_0^1 f(x)\overline{\phi_m(x)}\,dx$,

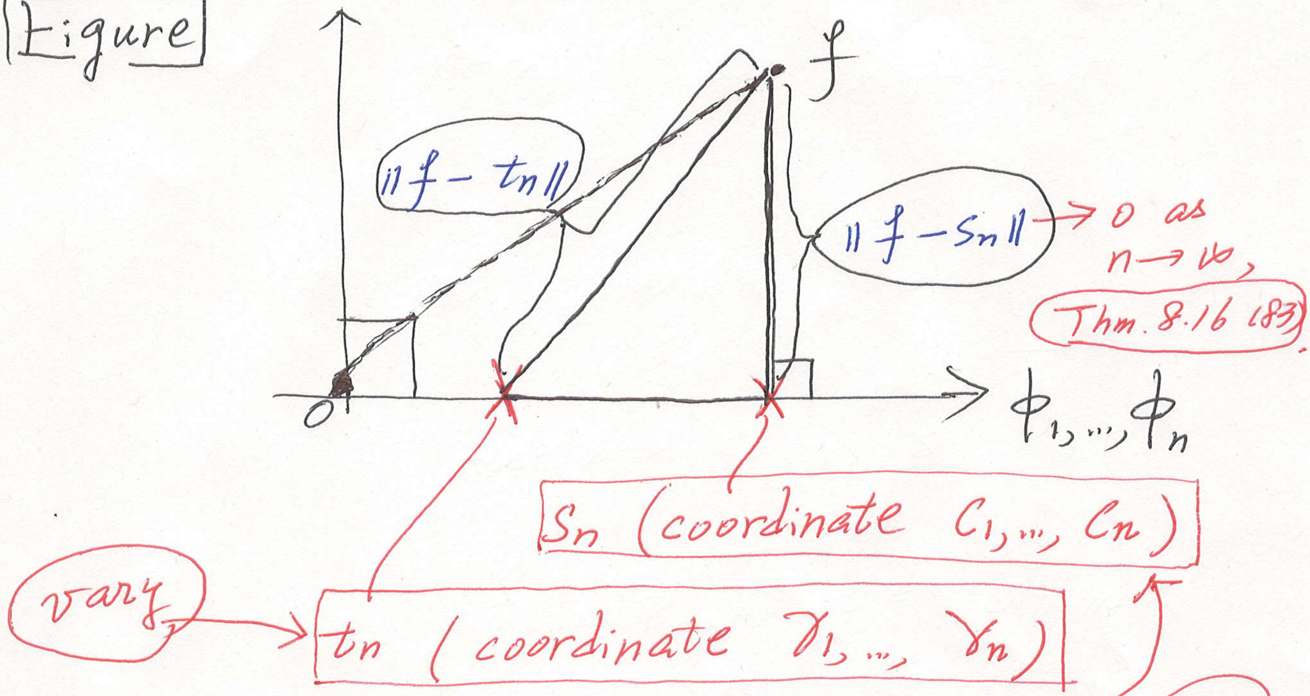
and suppose

(69) $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$.

$\Rightarrow$ (70) $\|f - s_n\| \leqq \|f - t_n\|$,

(71) the equality $\Leftrightarrow$ $\forall \gamma_m = c_m \ (1 \leqq m \leqq n)$

$\boxed{\text{Figure}}$



Check that

$$\int_0^1 f\,\overline{t_n}\,dx = \sum c_m \overline{\gamma_m},$$

$$\int_0^1 |t_n|^2\,dx = \sum |\gamma_m|^2, \quad \text{and}$$

$$(*) \quad \|f - t_n\|^2 := \int_0^1 |f - t_n|^2\,dx = \int_0^1 |f|^2\,dx - \sum c_m \overline{\gamma_m} - \sum \overline{c_m}\gamma_m + \sum |\gamma_m|^2$$

$$= \int_0^1 |f|^2\,dx - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2$$

$$\underset{(=)}{\geqq} \int_0^1 |f|^2\,dx - \sum |c_m|^2.$$

$\forall \gamma_m = c_m$ $(1 \leqq m \leqq n)$

Putting $t_n = S_n$ in (*),

$$\| f - S_n \|^2 = \int_0^1 |f|^2 dx - \sum |c_m|^2 .$$

Therefore

$$\| f - t_n \|^2 \geqq \| f - S_n \|^2 .$$

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$\boxed{\text{Fourier}}$

(76)
$$\int_0^1 | S_N(x) |^2 dx = \sum_{-N}^{N} | c_n |^2$$
$$\leqq \int_0^1 |f|^2 dx .$$

(77)
$$D_N(x) := \sum_{-N}^{N} e^{2\pi i n x}$$
$$= \frac{\sin 2\pi (N+\frac{1}{2})x}{\sin \pi x} .$$

$$(e^{2\pi i x} - 1) D_N(x)$$
$$= e^{2\pi i (N+1)x} - e^{-2\pi i N x}$$

$$D_N(x) = \frac{e^{\pi i (2N+1)x} - e^{-\pi i (2N+1)x}}{e^{\pi i x} - e^{-\pi i x}} .$$

(78)
$$S_N(x) = \sum_{-N}^{N} \left(\int_0^1 f(t) e^{-2\pi i n t} dt \right) e^{2\pi i n x}$$
$$= \int_0^1 f(t) D_N(x-t)\, dt .$$

8.14 Thm $\exists x \in [0,1]$, $\exists \delta > 0$, $\exists M < \infty$ s.t.

(79) $|f(x+t) - f(x)| \leqq M|t|$, $|t| < \delta$

$\Rightarrow$ (80) $S_N(f, x) \to f(x)$, $N \to \infty$.

$\boxed{\because}$ Put

(81) $$\underbrace{g(t)}_{\text{bounded}} := \begin{cases} \dfrac{f(x-t) - f(x)}{\sin(\pi t)}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

Then

$$S_N(f, x) - f(x) = \int_0^1 g(t) \sin[(2N+1)\pi t]\, dt$$
$$\to 0, \quad N \to \infty. \quad //$$

8.16 Thm (Parseval)

For functions f, g $\leftrightarrow$ Fourier coefficients $\{c_n\}$, $\{\gamma_n\}$,

(83) $$\int_0^1 |f(x) - S_N(f, x)|^2 dx = 0, \quad N \to \infty,$$

(84) $$\int_0^1 f(x) \overline{g(x)}\, dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n},$$

(85) $$\int_0^1 |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

$\boxed{\because}$ Notation : $\|h\|_2 := \left(\int_0^1 |h(x)|^2 dx\right)^{1/2}$.

(87) $\exists h$: conti., $\|f - h\|_2 < \varepsilon$.

$\exists P$: trigonometric polynomial s.t.

$\|h - P\|_2 < \varepsilon$.

If $\deg P = N_0$, then (by $\boxed{\text{Thm. 8.11}}$)

(88) $\|h - S_N(h)\|_2 \leqq \|h - P\|_2 \ (< \varepsilon)$

for $\forall N \geqq N_0$.

(89) $\|S_N(h) - S_N(f)\|_2 \leqq \|h - f\|_2 \ (< \varepsilon)$.

Therefore,

(90) $\forall N \geqq N_0$, $\|f - S_N(f)\|_2 < 3\varepsilon$,

completing the proof of (83).

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$\boxed{\text{p.325}\sim}$ X meas. sp., μ measure

— Definition of $L^2(X, \mu)$

$\|f\|_2 := \left(\int_X |f|^2 d\mu\right)^{1/2}$.

— 11.40 Thm (Parseval for $L^2(X, \mu)$)

<u>11.42 Thm</u> ($L^2(X,\mu)$ is a complete metric space.)

$\boxed{\because}$ For $\forall$ Cauchy seq $\{f_n\}$, $\exists \{n_k\}$

s.t. $\quad \| f_{n_k} - f_{n_{k+1}} \| < 2^{-k} \quad (k=1,2,\dots)$.

Then we have (104) :

$$\sum_{k=1}^{\infty} | f_{n_{k+1}}(x) - f_{n_k}(x) | < +\infty$$

for a.e. $x \in X$.

The equation $f(x) := \lim_{k\to\infty} f_{n_k}(x)$ defines $f(x)$ for a.e. $x \in X$.

We can finally show that $f \in L^2(X,\mu)$ and $\| f_n - f \| \to 0 \ (n\to\infty)$. ///

<u>Remark</u> $\quad L^2(X,\mu)$ may be regarded as an ∞-dimensional Euclidean space (= Hilbert), in which the point f has coordinates c_n, and the functions ϕ_n (where $\{\phi_n\}_{n=1}^{\infty}$ is a complete orthonormal system) are the coordinate vectors.