Let X be a vector space / $\mathbb{C}$.

Q.1 Prove that $C[a,b]$, the space of $\mathbb{C}$-valued continuous functions defined on $[a,b]$, is a vector space.

Let $\|\cdot\| : X \to \mathbb{R}_{\geqq 0}$ satisfy

(N1) $\|u\| = 0 \iff u = 0 \in X$;

(N2) $\|\alpha u\| = |\alpha| \, \|u\|$, $\alpha \in \mathbb{C}$;

(N3) $\|u + v\| \leqq \|u\| + \|v\|$.

triangle ineq.

Then $\|u\|$ is called the norm of $u \in X$.

Q.2 For $u \in C[a,b]$, put $\|u\| := \sup_{x \in [a,b]} |u(x)|$.

Prove that $\|u\|$ is a norm of $u \in C[a,b]$.

Let X be a normed vector space. If

"$\|u_m - u_n\| \to 0 \quad (m, n \to \infty)$"

is equivalent to

"$\exists u \in X$ s.t. $\|u_n - u\| \to 0 \ (n \to \infty)$"

for every sequence $\{u_n\} \subset X$, then X is said to be complete. A complete normed space is called a Banach space.

$\boxed{Q.3}$ Prove that $C[a,b]$ is a Banach space.

Let $(\cdot,\cdot) : X \times X \to \mathbb{C}$ satisfy

(H1) $(u,v) = \overline{(v,u)}$;

(H2) $(\alpha u, v) = \alpha (u,v)$;

(H3) $(u+v, w) = (u,w) + (v,w)$;

(H4) $(u,u) \geqq 0$;

(H5) $(u,u) = 0 \Leftrightarrow u = 0$.

Then (u,v) is called the inner product of $u, v \in X$. If we put $\|u\| := (u,u)^{1/2}$, then we can verify that $\|u\|$ satisfies (N1)-(N3). A complete inner product space is called a Hilbert space.

$\boxed{Q.4}$ For $u, v \in C[a,b]$, put $(u,v) := \int_a^b u(x)\overline{v(x)}\,dx$. Prove that (u,v) is an inner product of $u, v \in C[a,b]$. Is this inner product space a Hilbert space?

Let $C^1[a,b] := \{ u \in C[a,b] \mid u$ is differentiable and $u' \in C[a,b]. \}$ If we put

$$\|u\|_{C^1} := \sup_{x\in[a,b]} |u(x)| + \sup_{x\in[a,b]} |u'(x)|,$$

then we can prove that $C^1[a,b]$ is a Banach space.

$\boxed{Q.5}$ Prove that $C^1[a,b]$ is a Banach space.

Further function spaces to be considered :

$\Omega \subset \mathbb{R}^n$ m'ble set, $|\Omega| > 0$.

(1) Lebesgue space $L^p(\Omega)$, $1 \leqq p < \infty$.

For $u \in L^p(\Omega)$, $\|u\|_p := \left(\int_\Omega |u(x)|^p dx \right)^{1/p} < \infty$.

For $u, v \in L^2(\Omega)$, $(u,v)_{L^2(\Omega)} := \int_\Omega u(x)\overline{v(x)}\, dx$.

(2) Sobolev space $W^{m,p}(\Omega)$, $1 \leqq p < \infty$, $m = 0, 1, 2, \dots$

For $u \in W^{m,p}(\Omega)$,

$$\|u\|_{m,p} := \left(\sum_{|\alpha| \leqq m} \|D^\alpha u\|_p^p \right)^{1/p}.$$

For $u, v \in W^{m,2}(\Omega)$,

$$(u,v) := \sum_{|\alpha| \leqq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.$$