Improvements and generalizations of two Hardy type inequalities

Megumi Sano (Nara Women's University)

奈良女大・埼大 関数不等式研究
@ 奈良女子大学 理学部 G 棟 2 階 G201
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（板書講演の資料）

§0 My research and interest

- Functional inequalities (Embeddings)
  ⋆ Optimal constant, (Non-)Existence of optimizer
  ⋆ (Non-)Compactness of embedding
  ⋆ The Sobolev inequality, The Hardy inequality etc.
  ⋆ Their relationship, for example,
  Hardy ineq. can be derived from N-dim. Sobolev ineq. as "$N \nearrow \infty$". (Ref. S., 2020) ※ The derivation from Hardy to Sobolev has been well-known for ages.

- Elliptic and parabolic PDEs
  ⋆ (Non-)Existence, Regularity (Singularity), Stability and Asymptotic behavior of solution etc.

- Some "critical" situation
  ⋆ Sobolev embedding $W^{1,p} \subset L^{\frac{Np}{N-p}}$ $({}^{\exists}C \, \|u\|_{\frac{Np}{N-p}} \le \|\nabla u\|_p, \forall u)$
  holds for $p < N$, but $W^{1,N} \not\subset L^\infty$. $p = N$ is critical case!

Plan of my talk

- §1 Two Hardy type inequalities
  - The classical Hardy inequality $(H)_I$: Interior sing.
  - The geometric Hardy inequality $(H)_B$: Boundary sing.
- §2 Main result
  - Thm 1 (Improvements of $(H)_I$ and $(H)_B$)
  - Known results
- §3 Proof of Thm 1
  - Differences in use of the divergence thm.
  - Thm 2 (Generalizations of $(H)_I$ and $(H)_B$)
  - Relation between virtual optimizers
- §4 Higher order cases $+\alpha$ (Open problems)

§1 Two Hardy type inequalities

Let B be the unit ball in $\mathbb{R}^N$, $N \geq 2$ and $1 < p < \infty$.

Interior singularity ($\alpha < N$)

$$(H)_I \quad \left(\frac{N-\alpha}{p}\right)^p \int_B \frac{|u|^p}{|x|^\alpha}\,dx \leq \int_B \frac{|\nabla u|^p}{|x|^{\alpha-p}}\,dx \quad \left({}^\forall u \in C_0^1(B)\right)$$

Proof: $\int_B \frac{|u|^p}{|x|^\alpha}\,dx = \int_B \frac{1}{N-\alpha}\mathrm{div}\left(\frac{x}{|x|^\alpha}\right)|u|^p\,dx = \frac{-p}{N-\alpha}\int_B \frac{|u|^{p-2}u}{|x|^{\alpha-1}}\left(\nabla u \cdot \frac{x}{|x|}\right)dx$

Facts

- The constant $\left(\frac{N-\alpha}{p}\right)^p$ is optimal and is not attained for $u \not\equiv 0$.
- $|x|^{-\frac{N-\alpha}{p}}$ is so-called "*the virtual optimizer*." (also, *sol.* of EL eq.) If we assume that V attains the equality of $(H)_I$. Then we have $V(x) = c|x|^{-\frac{N-\alpha}{p}} (c \in \mathbb{R})$. However, $\int_B \frac{|\nabla V|^p}{|x|^{\alpha-p}}\,dx = \infty$ if $c \neq 0$.

§1 Two Hardy type inequalities

Boundary singularity ($\beta > 1$)

$$(H)_B \quad \left(\frac{\beta - 1}{p}\right)^p \int_B \frac{|u|^p}{(1-|x|)^\beta}\,dx \le \int_B \frac{|\nabla u|^p}{(1-|x|)^{\beta-p}}\,dx \quad \left({}^\forall u \in C_0^1(B)\right)$$

Facts

- The constant $\left(\frac{\beta-1}{p}\right)^p$ is optimal and is not attained for $u \not\equiv 0$.
- $(1-|x|)^{\frac{\beta-1}{p}}$ is the virtual optimizer. (※ not sol. of EL eq.)

Aim

To combine $(H)_I$ and $(H)_B$ including best constants

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Aim

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§2 Main result

Thm 1 (Improvements of $(H)_I$ and $(H)_B$, S., '22)

Let $\alpha < N, \beta > 1$ and $\gamma \in (0, \frac{N-\alpha}{\beta-1}]$. Then $^\forall u \in C_0^1(B)$,

$$\left(\frac{\beta-1}{p}\gamma\right)^p \int_B \frac{|u|^p}{|x|^\alpha (1-|x|^\gamma)^\beta}\, dx \le \int_B \frac{|\nabla u|^p}{|x|^{\alpha-p}(1-|x|^\gamma)^{\beta-p}}\, dx.$$

The constant $(\frac{\beta-1}{p}\gamma)^p$ is optimal and is not attained for $u \not\equiv 0$.

Cor 1

(i) Let $\beta = p, \gamma = \frac{N-\alpha}{p-1}$. Then Thm 1 implies $(H)_I$. ($\because \frac{1}{(1-|x|)^\beta} \ge 1$)

(ii) Let $\alpha = p, \gamma = 1 (\le \frac{N-p}{\beta-1})$. Then Thm 1 implies $(H)_B$. ($\because \frac{1}{|x|^\alpha} \ge 1$)

(continued in the next slide ↓)

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(continued in the next slide ↓)

Cor 1

(iii) Let $\alpha = \beta = \gamma = p = 2\,(\rightsquigarrow N \geq 4)$. Then Thm 1 implies

$$\int_B \frac{|u|^2}{(1-|x|^2)^2}\,dx \leq \int_B |\nabla u|^2\,dx. \left(\overset{\text{Möbius trans.}}{\Longleftrightarrow} (H)_B \text{ in } \mathbb{R}^N_+ \text{ with } \beta = p = 2\right)$$

Ref.(Remainder term) Hardy-Sobolev-Maz'ya inequality

(iv) Since $1 - |x|^\gamma = \gamma \log \frac{1}{|x|} + o(\gamma)\,(\gamma \to 0)$, Thm 1 implies

$$\left(\frac{\beta - 1}{p}\right)^p \int_B \frac{|u|^p}{|x|^\alpha \left(\log \frac{1}{|x|}\right)^\beta}\,dx \leq \int_B \frac{|\nabla u|^p}{|x|^{\alpha-p}\left(\log \frac{1}{|x|}\right)^{\beta-p}}\,dx.$$

Remark 1 (i) The case $\alpha = \beta = p$ in Thm 1 is known.

$$\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla f|^p}{f^p}|u|^p\,dx \le \int_\Omega |\nabla u|^p\,dx \quad \left({}^\forall u \in C_0^1(\Omega),\ f \ge 0\right)$$

Ref. D'Ambrosio, '05, Li-Wang, '06($p = 2, f = G_{\Omega,a}$: Green's funct.), Adimurthi-Sekar, '06($f = G_{\Omega,a}$), D'Ambrosio-Dipierro, '14($-\Delta_p f \ge 0$)

$$\left(\frac{p-1}{p}\right)^p \frac{|\nabla G_{\Omega,0}|^p}{|G_{\Omega,0}|^p} = \begin{cases} \left(\frac{N-p}{p}\right)^p |x|^{-p} & (p < N,\ \Omega = \mathbb{R}^N), \\ \left(\frac{N-p}{p}\right)^p |x|^{-p}(1 - |x|^{\frac{N-p}{p-1}})^{-p} & (p < N,\ \Omega = B), \\ \left(\frac{N-1}{N}\right)^N |x|^{-N}(\log \frac{1}{|x|})^{-N} & (p = N,\ \Omega = B) \end{cases}$$

Let $p < N, \Omega = B, f(x) = |x|^{-\gamma} - 1\ (\gamma > 0)$.
Then f satisfies $-\Delta_p f \ge 0$ for $\gamma \le \frac{N-p}{p-1}$.

Remark 1 (ii) Some generalization is known as follows.

$$(\bigstar) \quad \left(\frac{|p-1-\tilde{\alpha}|}{p}\right)^p \int_\Omega f^{\tilde{\alpha}} \frac{|\nabla f|^p}{f^p} |u|^p \, dx \le \int_\Omega f^{\tilde{\alpha}} |\nabla u|^p \, dx \quad ({}^\forall u)$$

Ref. D., '05, D.-D., '14 $\left(-(p-1-\tilde{\alpha})\Delta_p f \ge 0, f \ge 0\right)$

(※) However, **(★)** $\neq$ **Thm 1** in general. This difference comes from how to use the divergence thm.

(iii) The case $\gamma = \frac{N-\alpha}{\beta-1}$ is shown by Fabricant-Kutev-Rangelov('13).
(※) Attainability and remainder terms are not studied in this paper.

The case $\beta = p, \gamma = \frac{N-\alpha}{p-1}$ is shown by Ioku('19) via harmonic transplantation.

§3 Proof of Thm 1

[Ineq.]

$$
\begin{aligned}
&\int_B \frac{|u|^p}{|x|^\alpha} \frac{(\beta-1)\gamma}{(1-|x|^\gamma)^\beta}\, dx \\
&= \int_B \frac{|u|^p}{|x|^\alpha} \left[\frac{1}{|x|^{(\beta-1)\gamma}} \mathrm{div} \left(\frac{x}{(|x|^{-\gamma}-1)^{\beta-1}} \right) - \frac{N}{(1-|x|^\gamma)^{\beta-1}} \right] dx \\
&= \int_B \frac{-p\,|u|^{p-2}u}{|x|^{\alpha-1}(1-|x|^\gamma)^{\beta-1}} \left(\nabla u \cdot \frac{x}{|x|} \right) + \frac{\{N-\alpha-(\beta-1)\gamma\}\,|u|^p}{|x|^\alpha(1-|x|^\gamma)^{\beta-1}}\, dx \\
&\overset{(1)}{\le} p \int_B \frac{|u|^{p-1}\,|\nabla u|}{|x|^{\alpha-1}(1-|x|^\gamma)^{\beta-1}}\, dx \\
&\overset{(2)}{\le} \left(\int_B \frac{|u|^p}{|x|^\alpha(1-|x|^\gamma)^\beta}\, dx \right)^{1-\frac{1}{p}} \left(\int_B \frac{|\nabla u|^p}{|x|^{\alpha-p}(1-|x|^\gamma)^{\beta-p}}\, dx \right)^{\frac{1}{p}}
\end{aligned}
$$

[Equality condition] Assume that u attains the equality.

(1) $\Longrightarrow$ u: nonnegative, radially symmetric, decreasing, $\gamma = \frac{N-\alpha}{\beta-1}$.

(2) $\Longrightarrow$ u satisfies $-u'(r) = \frac{\beta-1}{p}\gamma\frac{u(r)}{r(1-r^\gamma)}, r \in (0,1)$.

$\Longrightarrow$ $u(r) = c\left(|x|^{-\gamma} - 1\right)^{\frac{\beta-1}{p}}$ $(c \in \mathbb{R})$ (virtual optimizer of Thm 1)

$$\left(\frac{p}{(\beta-1)\gamma}\right)^p \int_B \frac{|\nabla u|^p}{|x|^{\alpha-p}(1-|x|^\gamma)^{\beta-p}}\,dx = \int_B \frac{|u|^p}{|x|^\alpha(1-|x|^\gamma)^\beta}\,dx$$

$$= c^p\,|\mathbb{S}^{N-1}| \int_0^1 \frac{dr}{r(1-r^\gamma)}$$

$$\geq C(\varepsilon)\int_0^\varepsilon \frac{dr}{r} + D(\varepsilon) + E(\varepsilon)\int_{1-\varepsilon}^1 \frac{dr}{1-r} = \infty \quad \text{if } c \neq 0.$$

[Optimality of $(\frac{\beta-1}{p}\gamma)^p$] Test $f_A(|x|) = (1-|x|^\gamma)^A\,\phi_\delta(|x|)\,(A > \frac{\beta-1}{p})$, where $\phi_\delta \equiv 1$ on $B \setminus B_{1-\delta}$, $\phi_\delta \equiv 0$ on $B_{1-2\delta}$ $(0 < \delta \ll 1)$. □

If we use another identity:

$$\operatorname{div}\left(\frac{x}{|x|^{\alpha}(1-|x|^{\gamma})^{\beta-1}}\right)|u|^p = \underbrace{\frac{(N-\alpha)|u|^p}{|x|^{\alpha}(1-|x|^{\gamma})^{\beta-1}}}_{\text{(Case I)}} + \underbrace{\frac{(\beta-1)\gamma\,|u|^p}{|x|^{\alpha-\gamma}(1-|x|^{\gamma})^{\beta}}}_{\text{(Case II)}}$$

and the divergence thm., then we get the followings.

Thm 2 (Generalizations of $(H)_I$ and $(H)_B$, S., '22)

(Case I : $\alpha < N, \beta \geq 1, \gamma > 0$) ※ (The case $\beta = 1$) = $(H)_I$

$$\left(\frac{N-\alpha}{p}\right)^p \int_B \frac{|u|^p}{|x|^{\alpha}(1-|x|^{\gamma})^{\beta-1}}\,dx \leq \int_B \frac{|\nabla u|^p}{|x|^{\alpha-p}(1-|x|^{\gamma})^{\beta-1}}\,dx.$$

(Case II : $\alpha \leq N, \beta > 1, \gamma > 0$) ※ (The case $\alpha = \gamma = 1$) = $(H)_B$

$$\left(\frac{\beta-1}{p}\gamma\right)^p \int_B \frac{|u|^p}{|x|^{\alpha-\gamma}(1-|x|^{\gamma})^{\beta}}\,dx \leq \int_B \frac{|\nabla u|^p}{|x|^{\alpha-\gamma+(\gamma-1)p}(1-|x|^{\gamma})^{\beta-p}}\,dx.$$

If we use another identity:

$$\operatorname{div}\left(\frac{x}{|x|^{\alpha}(1-|x|^{\gamma})^{\beta-1}}\right)|u|^{p} = \underbrace{\frac{(N-\alpha)|u|^{p}}{|x|^{\alpha}(1-|x|^{\gamma})^{\beta-1}}}_{\text{(Case I)}} + \underbrace{\frac{(\beta-1)\gamma\,|u|^{p}}{|x|^{\alpha-\gamma}(1-|x|^{\gamma})^{\beta}}}_{\text{(Case II)}}$$

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Thm 2 (Generalizations of $(H)_I$ and $(H)_B$, S., '22)

(Case I : $\alpha < N,\ \beta \geq 1,\ \gamma > 0$) ※ (The case $\beta = 1$) = $(H)_I$

$$\left(\frac{N-\alpha}{p}\right)^{p}\int_{B}\frac{|u|^{p}}{|x|^{\alpha}(1-|x|^{\gamma})^{\beta-1}}\,dx \leq \int_{B}\frac{|\nabla u|^{p}}{|x|^{\alpha-p}(1-|x|^{\gamma})^{\beta-1}}\,dx.$$

(Case II : $\alpha \leq N,\ \beta > 1,\ \gamma > 0$) ※ (The case $\alpha = \gamma = 1$) = $(H)_B$

$$\left(\frac{\beta-1}{p}\gamma\right)^{p}\int_{B}\frac{|u|^{p}}{|x|^{\alpha-\gamma}(1-|x|^{\gamma})^{\beta}}\,dx \leq \int_{B}\frac{|\nabla u|^{p}}{|x|^{\alpha-\gamma+(\gamma-1)p}(1-|x|^{\gamma})^{\beta-p}}\,dx.$$

Remark 2 (Relation between virtual optimizers)

(Thm 2 Case I, $(H)_I$) $|x|^{-\frac{N-\alpha}{p}}$ (Thm 2 Case II, $(H)_B$) $(1-|x|^{\gamma})^{\frac{\beta-1}{p}}$

(Thm 1) $(|x|^{-\gamma}-1)^{\frac{\beta-1}{p}} = |x|^{-\frac{N-\alpha}{p}}(1-|x|^{\gamma})^{\frac{\beta-1}{p}}$ $\left(\text{when } \gamma = \dfrac{N-\alpha}{\beta-1}\right)$

Let $k, m \in \mathbb{N}, k \geq 2, p > 1$, and

$$\nabla^k u = \begin{cases} \Delta^m u & \text{if } k = 2m, \\ \nabla \Delta^m u & \text{if } k = 2m+1, \end{cases} \quad A_{k,p} = \begin{cases} B_{k,p,m} & \text{if } k = 2m, \\ \frac{N-kp+2mp}{p} B_{k,p,m} & \text{if } k = 2m+1, \end{cases}$$

$$B_{k,p,m} = \prod_{j=0}^{m-1} \frac{\{N-kp+2jp\}\{N(p-1)+kp-2(j+1)p\}}{p^2}.$$

Remark 2 (Relation between virtual optimizers)

(Thm 2 Case I, $(H)_I$) $|x|^{-\frac{N-\alpha}{p}}$ (Thm 2 Case II, $(H)_B$) $(1-|x|^{\gamma})^{\frac{\beta-1}{p}}$

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$$B_{k,p,m} = \prod_{j=0}^{m-1} \frac{\{N-kp+2jp\}\{N(p-1)+kp-2(j+1)p\}}{p^2}.$$

§4 Higher order cases (Open problems)

Conjectures ($k \geq 2$: Improvements of $(R)_I$ and $(R)_B$)

(I) $1 < p < \frac{N}{k}$, $\gamma \in (0, \frac{N-kp}{p-1}]$. Then $^\forall u \in C_0^k(B)$,

$$\left(\frac{p-1}{N-kp}\gamma A_{k,p}\right)^p \int_B \frac{|u|^p}{|x|^{kp}(1-|x|^\gamma)^p}\,dx \leq \int_B |\nabla^k u|^p\,dx.$$

(II) $1 < p < \frac{N}{k}$, $\gamma \in (0, \frac{N-kp}{kp-1}]$. Then $^\forall u \in C_0^k(B)$,

$$\left(\prod_{j=1}^{k} \frac{jp-1}{p}\gamma\right)^p \int_B \frac{|u|^p}{|x|^{kp}(1-|x|^\gamma)^{kp}}\,dx \leq \int_B |\nabla^k u|^p\,dx.$$

Constants of (I) and (II) are optimal and are not attained for $u \not\equiv 0$.

Remark 3 (I) The case u: radial or $p = 2$ is true. (Thm 3)
(II) The case where $k = p = 2$ is true. (Thm 4)

As corollaries of Conjectures, we can get the followings.

$$(R)_I \quad A^p_{k,p} \int_B \frac{|u|^p}{|x|^{kp}}\, dx \le \int_B |\nabla^k u|^p\, dx \quad \left(1 < p < \frac{N}{k}\right)$$

(**Ref.** Rellich, '56, Davies, '98, Mitidieri, '00, Gazzola et.al, '03 etc.)

$$(R)_B \quad \left(\prod_{j=1}^{k} \frac{jp-1}{p}\right)^p \int_B \frac{|u|^p}{(1-|x|)^{kp}}\, dx \le \int_B |\nabla^k u|^p\, dx \quad \left(1 < p < \frac{N}{k}\right)$$

(**Ref.** Owen, '99 $(p = 2)$, ※ Barbatis('07) mentioned that
even if $k = 2$, the case $p \neq 2$ is difficult to show the above ineq.)

$$(R)_{I,\text{limit}} \quad \tilde{A}^p_{k,N} \int_B \frac{|u|^p}{|x|^N (\log \frac{1}{|x|})^p}\, dx \le \int_B |\nabla^k u|^p\, dx \quad \left(p = \frac{N}{k}\right)$$

(**Ref.** Horiuchi et.al,'04, Nguyen,arXiv,'17, Ruzhansky-Suragan,'19)

$$(R)_{B,\text{limit}} \quad \left(\prod_{j=1}^{k} \frac{jp-1}{p}\right)^p \int_B \frac{|u|^p}{|x|^N (\log \frac{1}{|x|})^N}\, dx \le \int_B |\nabla^k u|^p\, dx \quad \left(p = \frac{N}{k}\right)$$

※ $(R)_{B,\text{limit}}$ is probably not known.

Remark 4 (Virtual minimizers of $(R)_{I,\mathrm{limit}}$ and $(R)_{B,\mathrm{limit}}$)

$$\text{(Virtual minimizer of } (R)_{I,\mathrm{limit}}\text{)} \quad \left(\log\frac{1}{|x|}\right)^{\frac{N-k}{N}} \notin W_0^{k,\frac{N}{k}}(B)$$

$$\| \text{ (if } k = 1)$$

$$\text{(Virtual minimizer of } (R)_{B,\mathrm{limit}}\text{)} \quad \left(\log\frac{1}{|x|}\right)^{k\frac{N-1}{N}} \notin W_0^{k,\frac{N}{k}}(B)$$

$$※ \quad \left(\log\frac{1}{|x|}\right)^{\alpha} \in W^{k,\frac{N}{k}}(B_\delta) \quad \left(\alpha < \frac{N-k}{N}\right)$$

$$※ \quad \left(\log\frac{1}{|x|}\right)^{\alpha} \in W^{k,\frac{N}{k}}(B \setminus B_{1-\delta}) \quad \left(\alpha > k\frac{N-1}{N}\right)$$

Extra: Fractional case (Open problem)

The fractional Hardy inequality (Frank-Seiringer, 2008)

Let $s \in (0,1)$, $1 < p < \frac{N}{s}$. Then $^\forall u \in \dot{W}^{s,p}(\mathbb{R}^N)$,

$$C_{N,s,p} \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{ps}}\, dx \le \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dx.$$

The constant $C_{N,s,p}$ is optimal and is not attained for $u \not\equiv 0$.

In the case $p = \frac{N}{s}$, the non-sharp critical Hardy inequality:

$$C \int_B \frac{|u|^p}{|x|^N \left(\log \frac{e}{|x|}\right)^p}\, dx \le \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{2N}}\, dx$$

is known by Edmund-Tribel, 1999, but its optimal constant and non-attainability are open.

Thank you very much for your kind attention!