A mathematical analysis of the mean field equations with variable intensities

奈良女大・埼大 関数不等式研究 @ 奈良女子大学

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1 Introduction

[MFE: Mean Field Equation]

$$\begin{cases} -\Delta v = \lambda \dfrac{e^v}{\int_\Omega e^v \, dx} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \tag{MF}$$

[Neri(2004)]

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_{[-1,1]} \dfrac{\alpha e^{\alpha v}}{\iint_{[-1,1]\times\Omega} e^{\alpha' v} \, \mathcal{P}(d\alpha')dx} \, \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{N}$$

[Onsager(Eyink-Srenivasan(2006)), Sawada-Suzuki(2008)]

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_{[-1,1]} \dfrac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v} dx} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{OSS}$$

[Onsager(Eyink-Srenivasan(2006)), Sawada-Suzuki(2008)]

$$\begin{cases} -\Delta v = \lambda \displaystyle\int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v} dx} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{OSS}$$

$\Omega \subset \mathbf{R}^2$: bounded domain with smooth boundary $\partial\Omega$

$v(x) = -\beta\psi(x)$, $\psi = \psi(x)$: stream function of the Euler flow

$\lambda = -\beta$, $\beta \in \mathbf{R}$: inverse temperature

$\mathcal{P} \in \mathcal{M}([-1,1])$: Borel probability measure defined on $[-1,1]$

$\mathcal{M}([-1,1])$: space of measures on $[-1,1]$

Liouville equation

$$\begin{cases} -\Delta u = e^u \text{ in } \Omega \\ u = \text{const. on } \partial\Omega, \quad \int_\Omega e^u dx < +\infty. \end{cases} \tag{L}$$

(MF) $\Leftrightarrow$ (L)

(MF) $\Rightarrow$ (L):

$$u = v + \log\lambda - \log\int_\Omega e^v dx$$

(L) $\Rightarrow$ (MF):

$$v = u - u|_{\partial\Omega}\,, \qquad \lambda = \int_\Omega e^u dx$$

[Nagasaki-Suzuki (1990)] $\{(\lambda_k, u_k)\}$ with $\lambda_k \downarrow 0$: sol. seq. of

$$-\Delta u = \lambda e^u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

Passing to a subseq. (still denoted by the same notation), we have $\int_\Omega \lambda_k e^{u_k} dx \to 8\pi l$ for some $l \in \mathbf{N} \cup \{0, +\infty\}$ and either of (a)-(c) below:

(a) $l = 0 \Rightarrow u_k \to 0$ unif. on $\bar{\Omega}$

(b) $l = +\infty \Rightarrow u_k \to +\infty$ loc. unif. in Ω

(c) $l \in \mathbf{N} \Rightarrow \exists \mathcal{S} = \{x_j^0\}_{j=1}^l \subset \Omega$ ($x_i^0 \neq x_j^0$ if $i \neq j$) s.t.

$$u_k \to 8\pi \sum_{j=1}^{l} G(\cdot, x_j^0) \quad \text{loc. unif. in } \Omega,$$

$$\nabla_{x_j} \left(\frac{1}{2} \sum_{j=1}^{l} R(x_j) + \sum_{1 \le i < j \le l} G(x_i, x_j) \right) \Bigg|_{(x_1,\ldots,x_l)=(x_1^0,\ldots,x_l^0)} = 0 \quad (j = 1, \ldots, l)$$

Green function: $-\Delta G(\cdot, y) = \delta_y$ in Ω, $G(\cdot, y) = 0$ on $\partial\Omega$ (for $y \in \Omega$)

Robin function: $R(x) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|} \Big|_{y=x}$.

Simple cases [Caglioti-Lions-Marchioro-Pulvirenti (1992)]
Consider (MF).

(1) $\Omega = B_1$: unit disk, where $B_R = \{x \in \mathbf{R}^2 \mid |x| < R\}$ $(R > 0)$
[Gidas-Ni-Nirenberg (1979)] $\Rightarrow$ sol. is radial: $v = v(x) = v(r)$, $r = |x|$
We solve the ODE and then obtain

$$v(r) = 2 \log \frac{1}{1 - \frac{\lambda}{8\pi}(1 - r^2)}$$

$\lambda \geq 8\pi \Rightarrow \not\exists$ sol. (by the Pohozaev identity).

$$\lambda \uparrow 8\pi \;\Rightarrow\; \lambda \frac{e^v}{\int_\Omega e^v \, dx} =: \rho_\lambda \overset{*}{\rightharpoonup} 8\pi\delta_0 \quad \text{in } \mathcal{M}(\bar{B}_1) = [C(\bar{B}_1)]',$$

$$\psi \to \frac{1}{2\pi} \log \frac{1}{|x|} \quad \text{loc. unif. in } B_1 \setminus \{0\}.$$

Here, "$\rho_\beta \overset{*}{\rightharpoonup} 8\pi\delta_0$ in $\mathcal{M}(\bar{B}_1)$" $\Leftrightarrow$ "$\int_{B_1} \rho_\beta(x)\varphi(x)dx \to 8\pi\varphi(0)$, $\forall\varphi \in C(\bar{B}_1)$."
This property is that of (c) for $\ell = 1$ in the theorem of [Nagasaki-Suzuki (1990)].

(2) $\Omega = B_1 \setminus B_\ell$ $(0 < \ell < 1)$: annulus $\Rightarrow$ $\exists$ radial sol. for $\forall \lambda \in \mathbf{R}$.
More precisely,

$$v(r) = 2 \log \frac{(1+A) r^{\sqrt{2E}/2}}{r(1 + A r^{\sqrt{2E}})},$$

where $A = A(\ell, \lambda)$ and $E = E(\ell, \lambda)$ are the functions satisfying

$$\lim_{\lambda \to +\infty} A(\ell, \lambda) = +\infty, \quad \lim_{\lambda \to +\infty} E(\ell, \lambda) = +\infty$$

for every $0 < \ell < 1$.
This property is that of (b) in the theorem of [Nagasaki-Suzuki (1990)].

Entire solutions for (L) [Chen-Li (1991)]

$$-\Delta u = e^u \text{ in } \mathbf{R}^2, \quad \int_{\mathbf{R}^2} e^u < +\infty,$$

$\exists x_0 \in \mathbf{R}^2$, $\exists \mu > 0$ s.t.

$$u(x) = \log \frac{\mu^2}{\left(1 + \frac{\mu^2}{8}|x - x_0|^2\right)^2} =: U_{\mu, x_0}, \quad \int_\Omega e^u dx = 8\pi$$

Local properties [Brezis-Merle (1991)] & [Li-Shafrir (1994)]
Consider (L) without the boudary condition.
$\{u_k\}$: sol. seq. with

$$\int_\Omega e^{u_k} \leq \exists C \text{ :indep. of } k$$

Passing to a subseq. (still denoted by $\{u_k\}$), we have the following alternatives:
(i) $\{u_k\}$: loc. unif. bdd. in Ω.
(ii) $u_k \to -\infty$ loc. unif. in Ω.
(iii) Define the blow-up set by

$$\Sigma := \{x_0 \in \Omega \mid \Omega \ni \exists x_k \to x_0 \quad \text{s.t.} \quad u_k(x_k) \to +\infty\}.$$

Then, $\Sigma \neq \emptyset$, $\sharp\Sigma < +\infty$, $u_k \to -\infty$ loc. unif. in $\Omega \setminus \Sigma$, and

$$e^{u_k} \stackrel{*}{\rightharpoonup} \sum_{x_0 \in \Sigma} m(x_0)\delta_{x_0} \quad \text{in } \mathcal{M}(\Omega) \quad \text{with } m(x_0) \in 8\pi\mathbf{N},$$

$$\text{i.e.,} \quad \int_\Omega e^{u_k}\varphi(x)dx \to \sum_{x_0 \in \Sigma} m(x_0)\varphi(x_0), \quad \forall \varphi \in C_0(\Omega).$$

Profile of blow-up solution sequences

[Y.Y. Li (1999)] Assume that (iii) above occurs. Furthermore,

$$x_k \to x_0 = 0 \in \mathcal{S}\text{: local maximizer of } u_k \text{ around } 0,$$
$$\exists r_0 > 0 \text{ s.t. } B_{2r_0} \cap \mathcal{S} = \{0\}, \quad \max_{\partial B_{2r_0}} u_k - \max_{\partial B_{2r_0}} u_k \leq \exists C' \text{ :indep. of } k$$

$\Rightarrow \exists C > 0$:indep. of k s.t.

$$|u_k(x + x_k) - U_{\mu_k, x_k}(x + x_k)| \leq C, \ \forall x \in B_{r_0}, \ \forall k,$$

where

$$\mu_k = e^{u_k(x_k)}, \quad U_{\mu_k, x_k} = U_{\mu_k, x_k}(x)\text{: entire sol. above.}$$

[Y.Y. Li (1999)] "$\cdots$, we tend to believe that a good enough pointwise estimate of blowup solutions $\{u_\lambda\}$ as $\lambda \to 8\pi m$ is the most crucial step in evaluating the jump-value of d_λ at $8\pi m$. Once we know the jump-values for m less than some m_0, we obtain a formula of d_λ in $(-\infty, 8\pi m_0) \setminus \bigcup_{m=1}^{m_0 - 1}\{8\pi m\}$. The main purpose of this paper is to start making good pointwise estimates for blowup solutions $\{u_\lambda\}$ as $\lambda \to 8\pi m$."

[Chen-Lin (2002, 2003)] $\cdots$ This conjecture is positive if Ω is not a simply connected domain: $d_\lambda = {}_mC_{m+1-g} \neq 0$.

Contents

- A simple case
- Concentration and compactness
- Known results
- Quantization for (OSS) of type $(+1, +\gamma)$

2 A simple case

In this section, we assume that $\Omega = D$ (unit disk) and

$$\mathcal{P}(d\alpha) = \tau\delta_1(d\alpha) + (1-\tau)\delta_\gamma(d\alpha),$$

where $\tau, \gamma \in (0,1)$.
Then, (N) and (OSS) take the form

$$\begin{cases} -\Delta v = \lambda \dfrac{\tau e^v + (1-\tau)\gamma e^{\gamma v}}{\int_D \{\tau e^v + (1-\tau) e^{\gamma v}\}\, dx} & \text{in } D \\ \quad v = 0 & \text{on } \partial D \end{cases} \tag{N_D}$$

and

$$\begin{cases} -\Delta v = \lambda \left(\tau \dfrac{e^v}{\int_D e^v\, dx} + (1-\tau)\gamma \dfrac{e^{\gamma v}}{\int_D e^{\gamma v}\, dx} \right) & \text{in } D \\ \quad v = 0 & \text{on } \partial D, \end{cases} \tag{OSS_D}$$

respectively.

We organize [Neri (2004)], [Ricciardi-Zecca (2012)], [Ricciardi-Suzuki (2014)] and [Ricciardi-T. (2019)] to get the following theorem.

Theorem
(1) For any $\tau, \gamma \in (0,1)$, $\exists$ sol. of $(\mathrm{OSS}_D) \Leftrightarrow \lambda < \bar{\lambda}_{\tau,\gamma}$.
Note that $\bar{\lambda}_{\tau,\gamma}$ is the optimal constant of the Trudinger-Moser inequality:

$$\bar{\lambda}_{\tau,\gamma} = 8\pi \min\left\{ \frac{1}{\{\tau + (1-\tau)\gamma\}^2}, \frac{1}{\tau} \right\} = \begin{cases} \frac{8\pi}{\tau}, & \text{if } 0 < \gamma \le \frac{\sqrt{\tau}}{1+\sqrt{\tau}} \\ \frac{8\pi}{\{\tau+(1-\tau)\gamma\}^2}, & \text{if } \frac{\sqrt{\tau}}{1+\sqrt{\tau}} < \gamma < 1, \end{cases}$$

$$\inf_{v \in H_0^1(\Omega)} J_\lambda^{(\mathrm{d})}(v) \begin{cases} > -\infty & \text{for } \lambda < \bar{\lambda}_{\tau,\gamma} \\ = -\infty & \text{for } \lambda > \bar{\lambda}_{\tau,\gamma}, \end{cases}$$

$$J_\lambda^{(\mathrm{d})}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \lambda\tau \ln \int_\Omega e^v\,dx - \lambda(1-\tau)\ln\int_\Omega e^{\gamma v}\,dx.$$

(2) $\forall \gamma \in (0,1)$, $\exists \underline{\tau}_\gamma \in (0,1)$ s.t. for all $\tau \in (0, \underline{\tau}_\gamma)$, $\exists$ sol. of (N_D) even if $0 < \lambda - 8\pi \ll 1$.
Note that 8π is the optimal constant of the Trudinger-Moser inequality:

$$\inf_{v \in H_0^1(\Omega)} J_\lambda^{(\mathrm{s})}(v) \begin{cases} > -\infty & \text{for } \lambda < 8\pi \\ = -\infty & \text{for } \lambda > 8\pi, \end{cases}$$

$$J_\lambda^{(\mathrm{s})}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \lambda \ln\left(\tau\int_\Omega e^v\,dx + (1-\tau)\int_\Omega e^{\gamma v}\,dx\right).$$

3 Concentration and compactness

We organize [Ohtsuka-Suzuki (2006)], [Ohtsuka-Ricciardi-Suzuki (2010)], [Ricciardi-Zecca (2012)], [Suzuki-Zhang (2013)], [Ricciardi-T. (2016)] and related works to get the following theorem.

Theorem

(λ_k, v_k): sol. seq. of (N) or (OSS) with $\lambda_k \to \exists \lambda_0 \geq 0$

Then, passing to a subsequence, we have the following alternatives:

[I] Compactness: $\limsup_{k\to\infty} \|v_k\|_\infty < +\infty$ ($\mathcal{S} = \emptyset$)

$$\exists v \in H_0^1(\Omega) \text{ s.t. } v_k \to v \text{ in } H_0^1(\Omega) \text{ \& } v\text{: sol. of (N) or (OSS)}$$

[II] Concentration: $\limsup_{k\to\infty} \|v_k\|_\infty = \infty$ ($\mathcal{S} \neq \emptyset$)

$$\sharp\mathcal{S} < \infty \quad \text{and} \quad \mathcal{S} \cap \partial\Omega = \emptyset$$

Here, $\mathcal{S}$ is the blow-up set:

$$\mathcal{S} := \mathcal{S}_+ \cup \mathcal{S}_-, \quad \mathcal{S}_\pm := \{x_0 \in \overline{\Omega} \mid \exists x_k \in \Omega \text{ s.t. } x_k \to x_0 \text{ \& } v_k(x_k) \to \pm\infty\}.$$

a) $0 \leq \exists s_\pm \in L^1(\Omega) \cap L^\infty_{loc}(\Omega \setminus \mathcal{S})$ s.t.

$$\nu_{k,\pm} := \lambda_k \int_{I_\pm} |\alpha| V(\alpha, v_k) e^{\alpha v_k} \mathcal{P}(d\alpha) \overset{*}{\rightharpoonup} \nu_\pm = s_\pm + \sum_{x_0 \in \mathcal{S}_\pm} m_\pm(x_0) \delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega})$$

with $m_\pm(x_0) \geq 4\pi$ for every $x_0 \in \mathcal{S}_\pm$, where

$$I_+ = [0,1], \quad I_- = [-1,0), \quad V(\alpha, v) = \begin{cases} \left(\iint_{[-1,1]\times\Omega} e^{\alpha v} \mathcal{P}(d\alpha) dx \right)^{-1} & \text{(for (N))} \\ \left(\int_\Omega e^{\alpha v}\, dx \right)^{-1} & \text{(for (OSS))} \end{cases}$$

b) $\exists \zeta_{x_0} \in \mathcal{M}(I)$ and $0 \leq \exists r \in L^1(I \times \Omega)$ s.t.

$$\mu_k := \lambda_k V(\alpha, v_k) e^{\alpha v_k} \mathcal{P}(d\alpha) dx$$
$$\overset{*}{\rightharpoonup} \mu = \mu(d\alpha dx) = r(\alpha, x) \mathcal{P}(d\alpha) dx + \sum_{x_0 \in \mathcal{S}} \zeta_{x_0}(d\alpha) \delta_{x_0}(dx) \text{ in } \mathcal{M}(I \times \bar{\Omega}),$$

where $I = I_+ \cup I_- = [-1, 1]$.

For both of (N) and (OSS), they are rewritten by the common form

$$-\Delta v_k = \nu_{k,+} - \nu_{k,-} \quad \text{in } \Omega.$$

c) For every $x_0 \in \mathcal{S}$,

$$8\pi \int_I \zeta_{x_0}(d\alpha) = \left(\int_I \alpha \zeta_{x_0}(d\alpha) \right)^2 \quad \text{((weak) mass relation)},$$

$$m_\pm(x_0) = \int_{I_\pm} |\alpha| \zeta_{x_0}(d\alpha), \quad s_\pm(x) = \int_{I_\pm} |\alpha| r(\alpha, x) \mathcal{P}(d\alpha),$$

$$m_\pm(x_0) \cdots \text{ as in } \textbf{a)}, \qquad \zeta_{x_0} \text{ and } r = r(\alpha, x) \cdots \text{ as in } \textbf{b)}.$$

Also, for every $x_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$,

$$m_\mp(x_0) = \int_{I_\mp} |\alpha| \zeta_{x_0}(d\alpha) = 0.$$

In particular,

$$m_\pm(x_0)^2 = 8\pi \int_I \zeta_{x_0}(d\alpha) \text{ (weak mass relation)}.$$

In addition, the following properties hold for (N).

Mass relation For every $x_0 \in \mathcal{S}$, we have

$$(m_+(x_0) - m_-(x_0))^2 = 8\pi(\beta_+(x_0)m_+(x_0) + \beta_-(x_0)m_-(x_0)),$$

where

$$\beta_\pm(x_0) = \begin{cases} |\alpha_\pm^*|^{-1} & \text{if } x_0 \in \mathcal{S}_\pm \\ 0 & \text{if } x_0 \notin \mathcal{S}_\pm \end{cases}, \quad \alpha_-^* = \min \operatorname{supp} \mathcal{P}, \quad \alpha_+^* = \max \operatorname{supp} \mathcal{P}.$$

Residual vanishing

$$s_\pm = 0 \quad \text{and} \quad r = 0,$$

$$\text{or} \quad \nu_k = \nu_{k,+} - \nu_{k,-} \stackrel{*}{\rightharpoonup} \sum_{x_0 \in \mathcal{S}} (m_+(x_0) - m_-(x_0))\delta_{x_0} \text{ in } \mathcal{M}(\bar{\Omega}).$$

Location of the blow-up points For every $x_0 \in \mathcal{S}$, we have

$$\nabla \left[H(x, x_0) + \sum_{x_0' \in \mathcal{S} \setminus \{x_0\}} \frac{(m_+(x_0') - m_-(x_0'))}{(m_+(x_0) - m_-(x_0))} G(x, x_0') \right] \Bigg|_{x = x_0} = 0,$$

where $H = H(x, y)$ is the regular part of the Green function:

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$

4 Known results

type $(+1,-1)$ or $\sinh$

$$-\Delta v = h_1 e^v - h_2 e^{-v} \qquad \text{in } \Omega$$

$$-\Delta v = \lambda \frac{\tau e^v - (1-\tau)e^{-v}}{\int_\Omega \{\tau e^v + (1-\tau)e^{-v}\}\, dx} \qquad \text{in } \Omega$$

$$-\Delta v = \lambda \left(\tau \frac{e^v}{\int_\Omega e^v\, dx} - (1-\tau)\frac{e^{-v}}{\int_\Omega e^{-v}\, dx} \right) \qquad \text{in } \Omega$$

[Ohtsuka-Suzuki (2006)] $\rightarrow$ quantization (conjecture)
[Jost-Wang-Ye-Zhou (2008)] $\rightarrow$ quantization (proof)
[Jevnikar-J.Wei-Yang (2018)] $\rightarrow$ quantization (a generalization)
[Jevnikar-J.Wei-Yang (2018)] $\rightarrow$ degree: (OSS)
[Bartolucci-Pistoia (2007)] $\rightarrow$ examples (concentration): $(\sinh)$
[Grossi-Pistoia (2013)] $\rightarrow$ examples (concentration): $(\sinh)$
[Esposito-Wei (2009)] $\rightarrow$ examples (concentration): (OSS)
[Ao-Jevnikar-Yang (2022)] $\rightarrow$ examples (concentration): (OSS)
and so on...

type $(+1,-\gamma)$ $(\gamma \in (0,1))$

$$-\Delta v = h_1 e^v - h_2 e^{-\gamma v} \qquad \text{in } \Omega$$

$$-\Delta v = \lambda \frac{\tau e^v - (1-\tau)\gamma e^{-\gamma v}}{\int_\Omega \{\tau e^v + (1-\tau) e^{-\gamma v}\}\, dx} \qquad \text{in } \Omega$$

$$-\Delta v = \lambda \left(\tau \frac{e^v}{\int_\Omega e^v\, dx} - (1-\tau)\gamma \frac{e^{-\gamma v}}{\int_\Omega e^{-\gamma v}\, dx} \right) \qquad \text{in } \Omega$$

[Ricciardi-Zecca (2016)] $\to$ existence: (OSS)
[Ricciardi-T.-Zecca-Zhang (2016)] $\to$ existence: (OSS)
[Jevnikar (2017)] $\to$ existence: (OSS)
[Jevnikar-Yang (2017)] $\to$ existence: (Liouville)
[Pistoia-Ricciardi (2016)] $\to$ examples (concentration): (Liouville)
[Pistoia-Ricciardi (2017)] $\to$ examples (concentration): (Liouville)
[Figueroa (2024)] $\to$ examples (concentration): (OSS)
[Esposito-Figueroa-Pistoia (2020)] $\to$ examples (pierced domains): (OSS)
[Figueroa (2021)] $\to$ examples (pierced domains): (Liouville)
[Figueroa (2023)] $\to$ examples (pierced domains): (Liouville & OSS)
and so on...

type $(+1,+\gamma)$ $(\gamma \in (0,1))$

$$-\Delta v = \lambda \frac{\tau e^v + (1-\tau)\gamma e^{\gamma v}}{\int_\Omega \{\tau e^v + (1-\tau) e^{\gamma v}\}\, dx} \qquad \text{in } \Omega$$

$$-\Delta v = \lambda \left(\tau \frac{e^v}{\int_\Omega e^v\, dx} + (1-\tau)\gamma \frac{e^{\gamma v}}{\int_\Omega e^{\gamma v}\, dx} \right) \qquad \text{in } \Omega$$

[Suzuki-Zhang (2013)] $\rightarrow$ various: (OSS)
[Ricciardi-T.-Zecca-Zhang (2016)] $\rightarrow$ existence: (OSS)
[Gui-Jevnikar-Moradifam (2018)] $\rightarrow$ uniqueness: (N)
[Jevnikar-Yang (2019)] $\rightarrow$ existence: (OSS)

Neri

$$-\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\iint_{[-1,1]\times\Omega} e^{\alpha' v}\,\mathcal{P}(d\alpha')dx}\,\mathcal{P}(d\alpha) \qquad \text{in } \Omega$$

[Ricciardi-Zecca (2016)] $\to$ existence (positive & general & non-degenerate)
[De Marchis-Ricciardi (2017)] $\to$ existence (positive & general)
[Toyota (2022)] $\to$ pointwise estimates (positive & general)

Onsager, Sawada-Suzuki

$$-\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\int_\Omega e^{\alpha v} dx}\mathcal{P}(d\alpha) \qquad \text{in } \Omega$$

[Suzuki-Toyota (2018)] $\to$ pointwise estimates (positive & continuous)
[Suzuki-Toyota (2019)]
$\to$ bounds below of the functional at the critical value (positive & continuous)

5 Quantization for (OSS) of type $(+1, +\gamma)$

$$\begin{cases} -\Delta v_k = \lambda_k \left(\tau \dfrac{e^{v_k}}{\int_\Omega e^{v_k}\, dx} + \gamma(1-\tau) \dfrac{e^{\gamma v_k}}{\int_\Omega e^{\gamma v_k}} \right) & \text{in } \Omega \\ v_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{OSS}$$

where $\tau \in (0,1)$, $\gamma \in (0,1)$ and $\lambda_k \to \exists \lambda_0 > 0$.
In this section, we assume that *concentration* occurs:

$$\lambda_k \tau \frac{e^{v_k}}{\int_\Omega e^{v_k}\, dx} \overset{*}{\rightharpoonup} \sum_{x_0 \in \mathcal{S}} m(x_0) \delta_{x_0} + r \qquad \text{in } \mathcal{M}(\overline{\Omega})$$

$$\lambda_k \gamma (1-\tau) \frac{e^{\gamma v_k}}{\int_\Omega e^{\gamma v_k}} \overset{*}{\rightharpoonup} \sum_{x_0 \in \mathcal{S}} m'(x_0) \delta_{x_0} + r' \qquad \text{in } \mathcal{M}(\overline{\Omega}),$$

recall $\mathcal{S}$ denotes the blow-up set:

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \Omega \ni \exists x_k \to x_0 \text{ s.t. } v_k(x_k) \to +\infty\}.$$

We know

$$\mathcal{S} = \{x_1, x_2, \ldots, x_N\} \subset \Omega \quad \text{for some } N \in \mathbf{N}.$$

__**Theorem 1**__ Concentration & residual vanishing $(r = r' = 0) \Rightarrow$

$$\lambda_0 = \overline{\lambda} N \text{ with } \overline{\lambda} = \frac{8\pi}{\{\tau + \gamma(1-\tau)\}^2}.$$

__**Theorem 2**__ Concentration & *not* residual vanishing $\Rightarrow$

$$\lambda_0 = \overline{\lambda} N \text{ with } \overline{\lambda} = \frac{8\pi}{\tau}.$$

__Remark__ $\overline{\lambda} = \bar{\lambda}_{\tau,\gamma}$ is the optimal constant of the Trudinger-Moser inequality.

Proof of Theorem 1 We introduce

$$w_k = \log\left(\lambda_k \tau \frac{e^{v_k}}{\int_\Omega e^{v_k}\,dx}\right) = v_k + \log(\lambda_k \tau) - \log\int_\Omega e^{v_k}$$

$$w_k' = \log\left(\lambda_k \gamma(1-\tau) \frac{e^{\gamma v_k}}{\int_\Omega e^{\gamma v_k}}\right) = \gamma v_k + \log(\lambda_k \gamma(1-\tau)) - \log\int_\Omega e^{\gamma v_k}$$

Then, (OSS) reduces to

$$\begin{cases} -\Delta w_k = e^{w_k} + e^{w_k'} & \text{in } \Omega \\ -\Delta w_k' = \gamma(e^{w_k} + e^{w_k'}) & \text{in } \Omega. \end{cases} \qquad (\mathrm{OSS}_w)$$

We can take $r_0 > 0$ such that

$$\overline{B}_{4r_0}(x_i) \subset \Omega, \quad \forall i \in \{1, 2, \ldots, N\}$$

$$B_{4r_0}(x_i) \cap B_{4r_0}(x_j) = \emptyset, \quad \forall i, j \in \{1, 2, \ldots, N\};\ i \neq j$$

Let $x_k^{(i)} \in \Omega$ be the maximizer of w_k on $\overline{B_{2r_0}(x_i)}$. Note that $x_k^{(i)} \in \Omega$ is also the maximizer of w_k' and v_k on $\overline{B_{2r_0}(x_i)}$. The argument in [Suzuki-Toyota (2018)] shows

$$\lim_{k\to+\infty} w_k(x_k^{(i)}) = \lim_{k\to+\infty} w_k'(x_k^{(i)}) = +\infty, \quad \forall i \in \{1, 2, \ldots, N\}.$$

Therefore, we can develop a blow-up analysis for (OSS_w).

Key estimates ([C.S. Lin (2007)], [Suzuki-Toyota (2018) (2019)], [Toyota (2022)])

$$w_k(x) - w_k(x_k^{(i)}) = -\frac{1}{2\pi}(m(x_i) + m'(x_i) + o(1)) \log\left(1 + \frac{|x - x_k^{(i)}|}{\sigma_k^{(i)}}\right) + O(1)$$

as $k \to \infty$ uniformly in $x \in B_{r_0}(x_k^{(i)})$ for every $i\{1, 2, \ldots, N\}$, where

$$\sigma_k^{(i)} = e^{-w_k(x_k^{(i)})/2} \ (\to 0 \ \text{ as } k \to \infty)$$

Since v_k is locally uniformly bounded in $\overline{\Omega} \setminus \{x_1, x_2, \ldots, x_N\}$ it holds

$$\begin{aligned} w_k(y^{(i)}) - w_k(x_k^{(i)}) &= \left\{ v_k(y^{(i)}) + \log(\lambda_k \tau) - \log \int_\Omega e^{v_k} \right\} - 2 \log \frac{1}{\sigma_k^{(i)}} \\ &= -\log \int_\Omega e^{v_k} - 2 \log \frac{1}{\sigma_k^{(i)}} + O(1) \end{aligned}$$

for any $x = y^{(i)} \in \partial B_{r_0/2}(x_k^{(i)})$ ($|y^{(i)} - x_k^{(i)}| = r_0/2$) and $i \in \{1, 2, \ldots, N\}$. On the other hand, the key estimates above give

$$\begin{aligned} w_k(y^{(i)}) - w_k(x_k^{(i)}) &= -\frac{1}{2\pi}(m(x_i) + m'(x_i) + o(1)) \log \left(1 + \frac{r_0/2}{\sigma_k^{(i)}} \right) + O(1) \\ &= -\frac{1}{2\pi}(m(x_i) + m'(x_i) + o(1)) \log \frac{1}{\sigma_k^{(i)}} + O(1) \end{aligned}$$

for any $x = y^{(i)} \in \partial B_{r_0/2}(x_k^{(i)})$ ($|y^{(i)} - x_k^{(i)}| = r_0/2$) and $i \in \{1, 2, \ldots, N\}$.

$$\therefore\ -\log \int_\Omega e^{v_k} = -\frac{1}{2\pi}(m(x_i) + m'(x_i) - 4\pi + o(1)) \log \frac{1}{\sigma_k^{(i)}} + O(1) \tag{1}$$

for every $i \in \{1, 2, \ldots, N\}$. Since the l.h.s. of (1) is independent i, we obtain

$$m(x_1) + m'(x_1) = \cdots\cdots = m(x_N) + m'(x_N)$$

We now have

$$m(x_1) + m'(x_1) = \cdots\cdots = m(x_N) + m'(x_N) \geq 8\pi$$

and the mass relation

$$(m(x_i) + m'(x_i))^2 = 8\pi(m(x_i) + m'(x_i)/\gamma), \quad \forall i \in \{1, 2, \ldots, N\}.$$

These properties imply

$$m(x_1) = \cdots\cdots = m(x_N) \text{ and } m'(x_1) = \cdots\cdots = m'(x_N).$$

On the other hand

$$\sum_{i=1}^{N} m(x_i) = \lambda_0 \tau \text{ and } \sum_{i=1}^{N} m'(x_i) = \lambda_0 \gamma (1 - \tau)$$

by the residual vanishing and $\lambda_k \to \lambda_0$.

$$\therefore\ m(x_1) = \cdots\cdots = m(x_N) = \frac{\lambda_0 \tau}{N} \text{ and } m'(x_1) = \cdots\cdots = m'(x_N) = \frac{\lambda_0 \gamma (1 - \tau)}{N}.$$

Finally, we combine these properties and the mass relation to get

$$\left\{ \frac{\lambda_0 \tau}{N} + \frac{\lambda_0 \gamma (1 - \tau)}{N} \right\}^2 = 8\pi \left\{ \frac{\lambda_0 \tau}{N} + \frac{\lambda_0 \gamma (1 - \tau)}{N} \cdot 1/\gamma \right\} = \frac{8\pi \lambda_0}{N},$$

or

$$\lambda_0 = \overline{\lambda} N \text{ with } \overline{\lambda} = \frac{8\pi}{\{\tau + \gamma(1 - \tau)\}^2}.$$

□

Proof of Theorem 2 It suffices to prove

$$m'(x_i) = 0, \quad \forall i \in \{1, 2, \ldots, N\}$$

thanks to the mass relation $(m(x_i) + m'(x_i))^2 = 8\pi(m(x_i) + m'(x_i)/\gamma)$.
If the residual vanishing does not occur then

$$\lim_{k\to\infty} \int_\Omega e^{\gamma v_k}\, dx < +\infty \tag{$*$}$$

by the Hölder inequality after passing to a subsequence.

case 1: $m(x_i) + m'(x_i) < 4\pi/\gamma$

The Brezis-Merle inequality shows that $e^{\gamma v_k}$ is uniformly bounded in $L^{p_i}(B_{r_i})$ for some $p_i > 1$ and $0 < r_i \ll 1$, and so $m'(x_i) = 0$ by the Hölder inequality.

case 2: $m(x_i) + m'(x_i) \geq 4\pi/\gamma$

Actually, this case does not hold. The argument in [Brezis-Merle (1991)] shows that $v_k \to \exists v$ a.e. in $B_{\varepsilon_i}(x_i)$ (or locally uniformly in $B_{\varepsilon_i}(x_i) \setminus \{x_i\}$) for some $0 < \varepsilon_i \ll 1$ s.t.

$$+\infty = \int_{B_{\varepsilon_i}(x_i)} e^{\gamma v}\, dx \leq \liminf_{k\to\infty} \int_{B_{\varepsilon_i}(x_i)} e^{\gamma v_k}\, dx \leq \lim_{k\to\infty} \int_\Omega e^{\gamma v_k}\, dx < +\infty$$

by $(*)$, a contradiction. $\square$

Thank you for your attention !