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A Discrete Transform and Decompositions of Distribution Spaces

MICHAEL FRAZIER*

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

AND

BJÖRN JAWERTH†

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208

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We study a representation formula of the form $f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q$ for a distribution f on $\mathbb{R}^n$. This formula is obtained by discretizing and localizing a standard Littlewood–Paley decomposition. The map taking f to the sequence $\{\langle f, \varphi_Q \rangle\}_Q$, with Q running over the dyadic cubes in $\mathbb{R}^n$, is called the φ-transform. The functions φ_Q and ψ_Q have a particularly simple form. Moreover, most of the familiar distribution spaces (L^p-spaces, $1 < p < +\infty$, H^p spaces, $0 < p \leqslant 1$, Sobolev and potential spaces, BMO, Besov and Triebel–Lizorkin spaces) are characterized by the magnitude of the φ-transform. This enables us to carry out a discrete Littlewood–Paley theory on the sequence spaces corresponding to these distribution spaces. The sequence space norms depend only on magnitudes; cancellation is accounted for in the φ_Q's and ψ_Q's. Consequently, analysis on the sequence space level is often easy. With this we can simplify, extend, and unify a variety of results in harmonic analysis. We obtain conditions for the boundedness of linear operators on these distribution spaces by considering corresponding conditions for matrices on the associated sequence spaces. Applications include a general version of the Hörmander (Fourier) multiplier theorem and results for kernel operators of Calderón–Zygmund type. We discuss certain other, more general, decomposition methods, including the "smooth atomic decomposition," and the "generalized φ-transform." The smooth atomic decomposition yields a simple method for dealing with restriction and extension phenomena for hyperplanes in $\mathbb{R}^n$. We also consider pointwise multipliers. For the characteristic function of a domain, we obtain boundedness results for a general class of domains which properly includes Lipschitz domains. Several interpolation methods are easily analyzed via the sequence spaces. For real interpolation, we obtain, among other things, an extension to the case $p = 0$. This in turn gives a new approach to the traditional atomic decomposition of Hardy spaces. © 1990 Academic Press, Inc.

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1. Introduction and Summary of Results

A fundamental technique in harmonic analysis is to represent a function or distribution as a linear combination of functions of an elementary form. Familiar examples include the Fourier series representation on the circle, and the "atomic decomposition" of the Hardy spaces $H^p(\mathbb{R}^n)$, $0<p\leqslant 1$ [Co; La]. A difficulty with Fourier series, however, is that few function spaces of interest, other than L^2, have simple characterizations in terms of the coefficients in the Fourier expansion. On the other hand, in the Hardy space atomic decomposition, the representing functions ("atoms") vary with the distribution being represented. Here we will study an elementary representation formula, introduced in [Fr-J1], which avoids both of these limitations. In particular, distributions on $\mathbb{R}^n$ will be represented in terms of a fixed, countable family of functions with convenient properties, and we will see that most of the function spaces of interest in harmonic analysis are characterized in terms of Littlewood–Paley expressions formed from the coefficients in the expansion.

In [Fr-J1] we have discussed our representation in the context of the Besov spaces $\dot{\mathbf{B}}_p^{\alpha q}(\mathbb{R}^n)$ and $\mathbf{B}_p^{\alpha q}(\mathbb{R}^n)$, $\alpha\in\mathbb{R}$, $0<p, q\leqslant +\infty$. Here we extend these results to include the Lebesgue spaces $L^p(\mathbb{R}^n)$, $1<p<+\infty$, the Hardy spaces $H^p(\mathbb{R}^n)$, $0<p\leqslant 1$, the Bessel potential spaces $L_\alpha^p(\mathbb{R}^n)$, $\alpha\in\mathbb{R}$, $1<p<+\infty$, and the space of functions of bounded mean oscillation

$\mathrm{BMO}(\mathbb{R}^n)$. In fact, we can deal with these cases in a unified manner by considering the more general Triebel–Lizorkin spaces $\dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$ (homogeneous) and $\mathbf{F}_p^{\alpha q}(\mathbb{R}^n)$ (inhomogeneous), $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$ (see Sections 2 and 12 for the definitions). Selecting the indices correctly gives the special cases above; it is well known (e.g., see [Tr2]) that $L^p \approx \dot{\mathbf{F}}_p^{02} \approx \mathbf{F}_p^{02}$ $(1 < p < +\infty)$, $H^p \approx \dot{\mathbf{F}}_p^{02}$ $(0 < p \leqslant 1)$, and $L_\alpha^p \approx \mathbf{F}_p^{\alpha 2}$ $(\alpha \in \mathbb{R},\ 1 < p < +\infty)$. We also have $\mathrm{BMO} \approx \dot{\mathbf{F}}_\infty^{02}$ (see Section 5). Although, for many, the main interest in our results will be in these special cases, we will treat the full Triebel–Lizorkin scale here. This allows greater generality, while the unified notation avoids tedious repetition. More importantly, our approach proceeds most naturally and transparently via the notation and techniques developed for the Besov and Triebel–Lizorkin spaces by Peetre, Triebel, and others. General references for this are [P3; Tr2].

We would like to emphasize that our results hold exactly for those function spaces that have some sort of Littlewood–Paley characterization. The reason for this is that our representation formula is set up so that the coefficients in the expansion exactly capture the information in the Littlewood–Paley norm defining the Triebel–Lizorkin spaces. Thus our approach can be described as a reformulation of Littlewood–Paley theory. Our main point is that the use of the representation formula makes this formulation particularly direct and simple. Classical Littlewood–Paley theory on the circle was developed by Littlewood, Paley, and the Zygmund school (see, e.g., [Z, Chaps. 14–15]), while in the more modern context in $\mathbb{R}^n$ it is largely due to Stein and his colleagues (see, e.g., [St, Chap. 4]).

In the introduction to [Fr-J1] we trace the background of our work through two general lines. One is the use of the Calderón reproducing formula to generate decompositions of functions into smooth bumps, as in [Cal2; Ch-F1; U1; Wi]. An alternate direction is that of Coifman and Rochberg [Co-R], Ricci and Taibleson [Ric-T], and others. We refer back to [Fr-J1] for discussion of these. However, there are many further references which could have been given at the time of [Fr-J1] or which should be mentioned specifically in connection with our current work. For example, various forerunners of our ideas can be found in [Co-W2; Tai-W]. Also, the theory of tent spaces [Co-M-S] shares many key features with our development; it can be regarded as a reformulation of Littlewood–Paley theory alternate to ours here. We also mention [Strö-T], where an earlier decomposition of L^p, $1 < p < +\infty$, along different lines than ours, is presented.

In any case, we want to make clear that virtually all of our techniques already exist in some antecedent form. Nevertheless their particular combination here leads to new conclusions and to sharpened versions of known results. Moreover, our presentation reveals an elementary discrete structure underlying a diverse range of topics in harmonic analysis.

The work most closely related to our current work, however, is the theory of "wavelets," as developed by Strömberg, Meyer, Lemarié, Coifman, and others (see, e.g., [Strö2; Le-M; Co-M]). Both our theory and theirs result from projects that have been ongoing for several years. Although the projects are independent, there has been a certain amount of mutual interchange and influence, which we would like to acknowledge. The wavelet theory can be regarded as a refinement of our earlier, more elementary, "almost orthogonal" decomposition in [Fr-J1]. (For a discussion of various almost orthogonal decompositions, including ours, see [DGM].) Wavelets are a collection of functions similar to the representing functions in our decomposition, but which are mutually orthogonal. In fact, wavelets form an unconditional basis for the usual function spaces in harmonic analysis listed above. Thus, unlike our theory, the wavelet theory is immediately connected to the vast literature on the construction of explicit unconditional bases for various function spaces. However, for the applications that we have considered (not related to bases), our more elementary decomposition has been sufficient. Thus, for reasons of simplicity (and perhaps stubbornness) we have presented our results without reference to the beautiful theory of wavelets. However, the reader will readily note that our conclusions generally apply as well to the wavelet decomposition.

In [Fr-J2] we have published a preliminary account of the results given here. There one can also find some further expository background and a few applications not presented here.

Our basic representation formula takes the form $f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q$, where the sum runs over all dyadic cubes Q in $\mathbb{R}^n$, and φ_Q and ψ_Q are translates and dilates of functions φ and ψ, respectively, to Q. (See Section 2. Here, and throughout the Introduction, the reader should refer to the main text for the precise statement.) The functions φ and ψ are assumed to satisfy (2.1–2.4) below; in particular, they are smooth, rapidly decreasing, and have compactly supported Fourier transform. We then have

$$\operatorname{supp} \hat{\varphi}_Q, \hat{\psi}_Q \subset \{\xi : 2^{\nu-1} \leq |\xi| \leq 2^{\nu+1}\} \qquad \text{if} \quad l(Q) = 2^{-\nu}.$$

Therefore $\langle \varphi_Q, \psi_P \rangle = 0$ unless $\frac{1}{2} \leq l(Q)/l(P) \leq 2$; even then $|\langle \varphi_Q, \psi_P \rangle|$ will be small if Q and P are far away from each other, since φ_Q and ψ_P decay rapidly away from Q and P, respectively. Thus our decomposition $f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q$ is "almost orthogonal."

We define a map S_φ, the φ-tranform, which takes the distribution f to the sequence of coefficients $\{\langle f, \varphi_Q \rangle\}_{Q \text{ dyadic}}$. For any sequence $s = \{s_Q\}_{Q \text{ dyadic}}$ of complex numbers, we define the map T_ψ, the inverse φ-transform, which takes s to $T_\psi s = \sum_Q s_Q \psi_Q$. Then our representation formula states that $f = T_\psi(S_\varphi f)$. To make these formal statements meaningful, we introduce quantitative assumptions on f and s which guarantee convergence in the appropriate sense. Let $\dot{\mathbf{f}}_p^{\alpha q}$ ($\alpha \in \mathbb{R}$, $0<p<+\infty$, $0<q\leqslant+\infty$) be the collection of all sequences $s=\{s_Q\}_{Q \text{ dyadic}}$ so that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}=\left\|\left(\sum_Q (|Q|^{-\alpha/n}\,|s_Q|\,\tilde{\chi}_Q)^q\right)^{1/q}\right\|_{L^p}<+\infty,$$

where the sum runs over all dyadic cubes in $\mathbb{R}^n$ and $\tilde{\chi}_Q=|Q|^{-1/2}\chi_Q$ is the L^2-normalized characteristic function of Q. Note that the quantity inside the L^p-norm is a generalized (to $q\neq 2$), discrete Littlewood–Paley expression. Our basic result is that $f=\sum_Q (S_\varphi f)_Q\,\psi_Q$, and that $f\in\dot{\mathbf{F}}_p^{\alpha q}$ if and only if $S_\varphi f\in\dot{\mathbf{f}}_p^{\alpha q}$ (with $\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}\approx\|S_\varphi f\|_{\dot{\mathbf{f}}_p^{\alpha q}}$). In fact, $\dot{\mathbf{F}}_p^{\alpha q}$ is a retract of $\dot{\mathbf{f}}_p^{\alpha q}$ under S_φ and T_ψ, or, in other words, we have the following theorem.

THEOREM 2.2. *The operators* $S_\varphi:\dot{\mathbf{F}}_p^{\alpha q}\to\dot{\mathbf{f}}_p^{\alpha q}$ *and* $T_\psi:\dot{\mathbf{f}}_p^{\alpha q}\to\dot{\mathbf{F}}_p^{\alpha q}$ *are bounded, and* $T_\psi\circ S_\varphi$ *is the identity on* $\dot{\mathbf{F}}_p^{\alpha q}$.

The proof of Theorem 2.2 is given in Section 2 and Appendix A (we frequently put technical, elementary computations in appendices); it is a variant of the proof given earlier in [Fr-J2]. We will use this result repeatedly to obtain applications regarding the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces in the following way. First we formulate and prove a corresponding assertion for $\dot{\mathbf{f}}_p^{\alpha q}$; this is generally easier because the $\dot{\mathbf{f}}_p^{\alpha q}$ norm is discrete and depends only on the magnitude of the sequence elements. Then the result for $\dot{\mathbf{F}}_p^{\alpha q}$ can be derived via Theorem 2.2. The general principle is that once Littlewood–Paley theory, in the guise of the φ-transform, has been applied to reduce the problem to the sequence space level, one only has to deal with "size" estimates of a combinatorial nature.

In Section 2 we also note Proposition 2.7, which is very simple but quite useful in applications. It states that we may replace χ_Q in the definition $\dot{\mathbf{f}}_p^{\alpha q}$ with χ_{E_Q} if, for each Q, $E_Q\subset Q$ and $|E_Q|/|Q|>\varepsilon>0$.

In Section 3 we study operators on $\dot{\mathbf{F}}_p^{\alpha q}$ by considering corresponding operators on $\dot{\mathbf{f}}_p^{\alpha q}$. Associated to a linear operator B on $\dot{\mathbf{F}}_p^{\alpha q}$ is a linear operator $S_\varphi^* B=S_\varphi\circ B\circ T_\psi$ on $\dot{\mathbf{f}}_p^{\alpha q}$, and B is bounded if and only if $S_\varphi^* B$ is bounded. It is easy to see that for $0<p,q<+\infty$, a bounded linear operator on the sequence space $\dot{\mathbf{f}}_p^{\alpha q}$ corresponds to a matrix $\{a_{QP}\}_{Q,P}$; in particular, $S_\varphi^* B$ then corresponds to the matrix having entries $a_{QP}=\langle B\psi_P,\varphi_Q\rangle$. Thus conditions implying boundedness on $\dot{\mathbf{f}}_p^{\alpha q}$ translate into conditions for operator boundedness on $\dot{\mathbf{F}}_p^{\alpha q}$.

We consider one such matrix condition in detail. We say that a matrix $A=\{a_{QP}\}_{Q,P}$ is almost diagonal if (3.1) is satisfied, which requires $|a_{QP}|$ to decay at a certain rate away from the diagonal (when $Q=P$); i.e., $|a_{QP}|$ must decay as $l(Q)/l(P)$ goes to 0 or ∞, and as P and Q get far away from each other. We then have the following.

THEOREM 3.3. *An almost diagonal operator is bounded.*

In Section 9 we will note that appropriate Calderón–Zygmund operators and certain classes of Fourier multiplier operators correspond to almost diagonal matrices. Thus, the φ-transform simultaneously "almost diagonalizes" these operators. To put this another way, note the obvious fact that if $\{e_i\}_i$ is an orthonormal basis for any Hilbert space, the matrix $\{\langle Ae_i, e_j\rangle\}_{i,j}$ is diagonal if and only if each e_i is an eigenfunction of the operator A. Due to the almost orthogonality of the functions $\{\psi_Q\}_Q$ and $\{\varphi_Q\}_Q$, we regard the functions $\{\psi_Q\}_Q$ as "approximate eigenfunctions" of an operator B if the matrix $\{\langle B\psi_P, \varphi_Q\rangle\}_{Q,P}$ is almost diagonal. Thus the ψ_P's are simultaneous approximate eigenfunctions for the operators mentioned above. This corresponds to the familiar fact that the Fourier characters $\{e^{ix\cdot\xi}\}_{\xi\in\mathbb{R}^n}$ are simultaneous eigenfunctions for translation invariant operators on $\mathbb{R}^n$. The basic trade-off in our approach is that we give up having exact eigenfunctions in order to obtain the norm-characterization in Theorem 2.2.

In Theorem 3.5 we obtain an estimate of the form $\|\sum_Q s_Q m_Q\|_{\dot{F}_p^{\alpha q}} \leqslant c\,\|s\|_{\dot{f}_p^{\alpha q}}$ whenever the m_Q's are "smooth molecules," i.e., when the m_Q's satisfy the smoothness, decay, and cancellation properties (3.3)–(3.6). This generalizes the boundedness of T_ψ in Theorem 2.2 which guarantees that this estimate holds if $m_Q = \psi_Q$. Similarly, Theorem 3.7 generalizes the boundedness of S_φ; we have $\|\{\langle f, b_Q\rangle\}_Q\|_{\dot{f}_p^{\alpha q}} \leqslant c\,\|f\|_{\dot{F}_p^{\alpha q}}$ whenever the b_Q's satisfy (3.7)–(3.10).

This leads us to look for generalized versions of Theorem 2.2. For example, it is useful in Sections 11 and 13 to decompose f into a sum of compactly supported functions. This can be done fairly easily based on Theorem 2.2; in Theorem 4.1 we show that each $f \in \dot{F}_p^{\alpha q}$ has a "smooth atomic" decomposition. By this we mean that we can write $f = \sum_Q s_Q a_Q$ with $\|\{s_Q\}_Q\|_{\dot{f}_p^{\alpha q}} \leqslant c\,\|f\|_{\dot{F}_p^{\alpha q}}$, where the a_Q's are "smooth atoms"—i.e., (4.1)–(4.3) hold. (In fact, we may take $a_Q \in \mathscr{D}$.) However, in this result the a_Q's depend on f, and the coefficients are not linearly determined by f. In Theorems 4.2 and 4.4 we obtain decompositions of the form $f = \sum_Q \langle f, \tau^Q\rangle \sigma^Q$ with appropriate estimates, under fairly general conditions on the families $\{\tau^Q\}_Q$ and $\{\sigma^Q\}_Q$. In particular, we may take either family (though perhaps not both) to consist of translates and dilates to Q of a function in $\mathscr{D}$.

We discuss the spaces $\dot{F}_\infty^{\alpha q}$ in Section 5. The main difficulty is to find the "right" definition of these spaces. The immediate analogue of the definition for $p < +\infty$ is not satisfactory, while certain other approaches have been given which are not computationally explicit. We define

$$\|f\|_{\dot{F}_\infty^{\alpha q}} = \sup_{P\text{ dyadic}} \left(\frac{1}{|P|}\int_P \sum_{\nu=-\log_2 l(P)}^{\infty} (2^{\nu\alpha}\,|\varphi_\nu * f(x)|)^q\,dx\right)^{1/q}.$$

Also, we define a corresponding sequence space $\dot{\mathfrak{f}}_\infty^{\alpha q}$ with the norm

$$\|s\|_{\dot{\mathfrak{f}}_\infty^{\alpha q}} = \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q(x))^q \, dx \right)^{1/q}.$$

We obtain in Theorem 5.2 the analogue for $p = +\infty$ of Theorem 2.2. To show that our definition yields a continuous extension of the $p < +\infty$ case, we consider an operator $m^{\alpha q}$, which is a discrete variant, for our sequence spaces, of the local square function. From this we obtain the operator $A^{\alpha q} = m^{\alpha q} \circ S_\varphi$, on the function space level, which has the property that

$$\|A^{\alpha q} f\|_{L^p} \approx \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}},$$

for all $\alpha \in \mathbb{R}$, and $0 < p, q \leqslant +\infty$ (Corollary 5.8). Further, in Theorem 5.13, we obtain the desired duality

$$(\dot{\mathbf{F}}_1^{\alpha q})^* \approx \dot{\mathbf{F}}_\infty^{-\alpha q'},$$

where $\alpha \in \mathbb{R}$, and q and q' are conjugate indices. This is derived from the corresponding result, Theorem 5.9, for the sequence spaces. In particular, we have $\dot{\mathbf{F}}_\infty^{02} \approx \text{BMO}$. (Other cases of the duality for $p \neq 1$ are easier; we consider these in Remark 5.14, including one case formerly left open.) The theory for $p = +\infty$ is clarified by Corollary 5.6 in which we note that a sequence $s = \{s_Q\}_Q$ belongs to $\dot{\mathfrak{f}}_\infty^{\alpha q}$ if and only if there exists for each Q a subset $E_Q \subset Q$, satisfying $|E_Q|/|Q| \geqslant \frac{1}{2}$, such that

$$\left\| \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^\infty} < +\infty.$$

With this the φ-transform yields another perspective on the H^1-BMO duality, although this perspective is implicit in the second proof of this duality in [Fef-S2].

Real interpolation is considered in Section 6. We define a sequence space $\dot{\mathfrak{f}}_0$ which acts as a common endpoint space for $p = 0$ for the scales $\dot{\mathfrak{f}}_p^{\alpha q}$, $0 < p \leqslant +\infty$, for each fixed α and q. With the norm

$$\|s\|_{\dot{\mathfrak{f}}_0} = \left| \bigcup_{s_Q \neq 0} Q \right|,$$

$\dot{\mathfrak{f}}_0$ is a quasi-normed Abelian group. Then Peetre's K-functional for the pair $(\dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}_\infty^{\alpha q})$ is characterized as follows.

THEOREM 6.4. $K(t, s; \dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}_\infty^{\alpha q}) \approx K(t, m^{\alpha q}(s); L^0, L^\infty)$.

Using this, reiteration, and retraction, we have the following function space result.

COROLLARY 6.7. *$K(t, f; \dot{\mathbf{F}}_{p_0}^{\alpha q}\dot{\mathbf{F}}_{p_1}^{\alpha q}) \approx K(t, A^{\alpha q}f; L^{p_0}, L^{p_1})$ for $0<p_0<p_1\leqslant +\infty$.*

Then the following real interpolation result are easily established:

$$(\dot{\mathbf{f}}_{p_0}^{\alpha q}, \dot{\mathbf{f}}_{p_1}^{\alpha q})_{\theta, p} \approx \dot{\mathbf{f}}_{p}^{\alpha q} \qquad \text{(Corollary 6.6)},$$

and

$$(\dot{\mathbf{F}}_{p_0}^{\alpha q}, \dot{\mathbf{F}}_{p_1}^{\alpha q})_{\theta, p} \approx \dot{\mathbf{F}}_{p}^{\alpha q} \qquad \text{(Corollary 6.7)}$$

for $1/p = (1-\theta)/p_0 + \theta/p_1$, $0<p_0<p_1\leqslant +\infty$. We do not, however, obtain a satisfactory function space analogue of $\dot{\mathbf{f}}_0$. The natural analogue of $\dot{\mathbf{f}}_0$ has the undesirable property that it depends on the choice of the original test function φ.

In Section 7 we obtain an analogue of the traditional "non-smooth" atomic Hardy space decomposition, for the $\dot{\mathbf{F}}_p^{\alpha q}$-spaces, in the range $0<p\leqslant 1$, $p\leqslant q\leqslant +\infty$.

THEOREM 7.4. *Each $f\in\dot{\mathbf{F}}_p^{\alpha q}$ for these p and q can be written in the form $f=\sum_{k\in\mathbb{Z}}\lambda_k A_k$ with $(\sum_{k\in\mathbb{Z}}|\lambda_k|^p)^{1/p}\leqslant c\,\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}$, where each A_k is an "atom for $\dot{\mathbf{F}}_p^{\alpha q}$."*

An "atom for $\dot{\mathbf{F}}_p^{\alpha q}$" is different from a "smooth atom for $\dot{\mathbf{F}}_p^{\alpha q}$" (introduced in Section 4). In particular, each A_k satisfies the usual H^p compact support and vanishing moment conditions, and satisfies $A_k\in\dot{\mathbf{F}}_\infty^{\alpha q}$ (and $\|A_k\|_{\dot{\mathbf{F}}_p^{\alpha q}}\leqslant c$). In the case of $H^p\approx\dot{\mathbf{F}}_p^{02}$, we obtain the usual atomic decomposition with "BMO-atoms." The range of indices above is natural, since this is exactly the range for which $\|\cdot\|_{\dot{\mathbf{F}}_p^{\alpha q}}^p$ is subadditive, yielding the estimate converse to the one in Theorem 7.4. Although there are more direct proofs of Theorem 7.4, similar to certain proofs of the H^p-decomposition (e.g., [Fo-S]), we have given a proof based on the real interpolation results for $\dot{\mathbf{f}}_0$ and $\dot{\mathbf{f}}_\infty^{\alpha q}$ in Section 6. The finite measure condition in the definition of $\dot{\mathbf{f}}_0$ leads to the compact support of the atoms. We give this treatment to stress the reciprocal relation between interpolation and atomic decompositions. It has been a natural conjecture from the time of Coifman's original proof [Co] that the atomic decomposition could be explicitly reduced to the "Fundamental Lemma" of real interpolation (see [Be-L]). Our presentation verifies this conjecture.

We take advantage of the fact that the discrete spaces $\dot{\mathbf{f}}_p^{\alpha q}$ are (quasi-)Banach lattices when we discuss other interpolation methods in Section 8. Because of the lattice structure, the Calderón product of a pair of $\dot{\mathbf{f}}_p^{\alpha q}$-spaces is defined. We show in Theorem 8.2 that this product is the natural intermediate $\dot{\mathbf{f}}_p^{\alpha q}$-space. For lattices, many interpolation methods are known to coincide (under mild conditions) with the Calderón product. Thus we obtain various interpolation results for $\dot{\mathbf{f}}_p^{\alpha q}$, and this yields corresponding results for $\dot{\mathbf{F}}_p^{\alpha q}$ by retraction. For example, for Calderón's complex method of interpolation, we easily derive the following (known) result.

COROLLARY 8.3. *$[\dot{\mathbf{F}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1 q_1}]_\theta \approx \dot{\mathbf{F}}_p^{\alpha q}$ for $\alpha_0, \alpha_1 \in \mathbb{R}$, $1 \leqslant p_0, q_0 < +\infty$, $1 \leqslant p_1, q_1 \leqslant +\infty$, where the indices an related in the usual way.*

We obtain similar results for the $\langle A_0, A_1 \rangle_\theta$ method (Corollary 8.4) and the $\pm$method (Theorem 8.5). This last method yields the interpolation property for the $\dot{\mathbf{F}}_p^{\alpha q}$-spaces in the greatest generality. That is, if T is a linear operator such that T: $\dot{\mathbf{F}}_{p_i}^{\alpha_i q_i} \to \dot{\mathbf{F}}_{t_i}^{\beta_i r_i}$ is bounded, $i = 0, 1$, then T: $\dot{\mathbf{F}}_p^{\alpha q} \to \dot{\mathbf{F}}_t^{\beta r}$ is bounded, where the α's, q's, and p's are related as in Corollary 8.3, and similarly for the β's, r's, and t's, for the full range of possible indices $(0 < p_i, q_i, r_i, t_i \leqslant +\infty)$. We note that we obtain this directly by applying Corollary 5.6 and thus we avoid relying on Wolff's reiteration theorem [Wo].

We discuss the almost diagonality condition from Section 3 further in Section 9.

THEOREM 9.1. *The composition of almost diagonal operators is almost diagonal.*

Thus the collection of almost diagonal operators for $\dot{\mathbf{f}}_p^{\alpha q}$, which we denote $\mathbf{ad}_p^{\alpha q}$, is an algebra under composition.

We say that a family of functions $\{m_P\}_{P \text{ dyadic}}$ is an $\mathbf{Ad}_p^{\alpha q}$-family if the matrix $\{\langle m_P, \varphi_Q \rangle\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$. We noted in Section 3 that a family of smooth molecules is an $\mathbf{Ad}_p^{\alpha q}$-family. For $\alpha = 0$, the converse is true also (Theorem 9.15), but for $\alpha \neq 0$ the smooth molecule conditions are stronger than necessary. In Theorems 9.3–9.4 we give an exact characterization of $\mathbf{Ad}_p^{\alpha q}$-families. We also show the following:

THEOREM 9.9. *If $f = \sum_Q s_Q m_Q$, where $\{m_Q\}_Q$ is an $\mathbf{Ad}_p^{\alpha q}$-family, then*

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c\, \|\{s_Q\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

Hence, by our decomposition results, if T is a linear operator mapping an arbitrary family of smooth atoms $\{a_Q\}_Q$, or the family $\{\psi_Q\}_Q$, into an $\mathbf{Ad}_p^{\alpha q}$-family, then T is bounded on $\dot{\mathbf{F}}_p^{\alpha q}$. Let $\mathbf{Ad}_p^{\alpha q}$ be the set of all linear operators such that $\{\langle T\psi_P, \varphi_Q \rangle\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$. By Section 3 we know that the operators in $\mathbf{Ad}_p^{\alpha q}$ are bounded, and from Theorem 9.1 it follows easily that

$\mathbf{Ad}_p^{\alpha q}$ is closed under composition. It is shown in [Fr-H-J-W; FW; Torr] that certain generalized Calderón–Zygmund operators (of the type considered by David and Journé in [DJ]) map smooth atoms to smooth molecules and hence belong to $\mathbf{Ad}_p^{\alpha q}$. In Example 9.19 we see that Fourier multiplier operators satisfying an L^1-Mihlin condition belong to $\mathbf{Ad}_p^{\alpha q}$ also. Hence, the algebra $\mathbf{Ad}_p^{\alpha q}$ is fairly rich.

In Section 10 we consider conditions, more precise than almost diagonality, under which boundedness on $\dot{\mathbf{f}}_p^{\alpha q}$ or $\dot{\mathbf{F}}_p^{\alpha q}$ is obtained. The general philosophy is that cancellation aspects of the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces are accounted for by the Littlewood–Paley theory implicit in the φ-transform; then, on the sequence space level, it should be possible to obtain key information through estimates depending only on magnitudes. Consistent with this philosophy, we discuss size estimates yielding boundedness of a matrix of $\dot{\mathbf{f}}_p^{\alpha q}$. For this purpose we consider a number of characterizations related to the classical Schur's lemma. For each of the spaces $\dot{\mathbf{f}}_p^{\alpha p}$, $0<p\leqslant 1$, $\dot{\mathbf{f}}_\infty^{\alpha\infty}$, $\dot{\mathbf{f}}_p^{\alpha\infty}$, $0<p\leqslant 1$, and $\dot{\mathbf{f}}_\infty^{\alpha 1}$, we obtain necessary and sufficient conditions for a positive matrix to be bounded. Using this, in Theorem 10.3 we obtain simple conditions under which a matrix is bounded on $\dot{\mathbf{f}}_p^{\alpha q}$ for all $1\leqslant p, q\leqslant +\infty$ (and $\alpha\in\mathbb{R}$ fixed). In Corollary 10.6, we obtain sufficient conditions for general p and q by a reduction to the case $p, q\geqslant 1$.

Passing to the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces, we get as an application a sufficient condition, (10.19), for a Fourier multiplier operator to be bounded. From this and Hölder's inequality, we obtain (Remark 10.9) a relatively straightforward proof of the known fact (see [Tr2]) that for $\alpha\in\mathbb{R}$, $0<p, q\leqslant +\infty$, a function m satisfying

$$\sup_\nu \|m(2^\nu\xi)\,\bar{\varphi}(\xi)\|_{L^2_{J-n/2+\varepsilon}} < +\infty$$

is a bounded multiplier on $\dot{\mathbf{F}}_p^{\alpha q}$. (Here $J=n/\min(1, p, q)$, and $L^2_\beta=\mathbf{F}_2^{\beta 2}$ is the usual Bessel potential space.) This unifies a number of results about Fourier multipliers. When $1<p<+\infty$ and $q=2$ this is the familiar Hörmander multiplier theorem [Hör1]. If $0<p\leqslant 1$, and $q=2$, this gives the H^p-analogue, since here $J=n/p$ (see [Fef-S2; Cal-T]). (Recall that $\dot{\mathbf{F}}_p^{02}\approx L^p$, $1<p<+\infty$, and $\dot{\mathbf{F}}_p^{02}\approx H^p$, $0<p\leqslant 1$.) We remark that in our approach, the H^p-result, $0<p\leqslant 1$, is obtained from the L^p-result, $1<p<+\infty$, by a simple reduction (on the sequence space side), somewhat like the classical reduction in one dimension of H^p to L^p via Blaschke products. This may seem surprising, but it is already implicit in [P2].

In fact, our methods can be sharpened to obtain the following refinement of the Hörmander (Fourier) multiplier theorem.

Corollary 10.10. *Suppose $\Phi(t)$, $t\geqslant 0$, is a nondecreasing function satisfying $\Phi(0)=1$, $\Phi(2t)\leqslant C\Phi(t)$, $t\geqslant 0$, and*

$$\int_2^\infty \left[\int_{|x| \geq t} \frac{1}{\Phi(|x|)}\,dx\right]^{1/2} \frac{dt}{t} < +\infty.$$

Suppose m satisfies

$$\sup_\nu \|\Phi(|x|)^{1/2}\,(m(2^\nu\cdot)\,\bar{\varphi}(\cdot))^\vee\,(x)\|_{L^2} < +\infty.$$

Then m is a bounded Fourier multiplier on $\dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$ *for* $\alpha \in \mathbb{R}$, $1 \leqslant p, q \leqslant +\infty$.

We continue by further discussing boundedness criteria for positive matrices. Using a theorem by Rubio de Francia [RdF2] to reduce to the l^p-case, and then Schur's lemma, we obtain a characterization (Theorem 10.13) of positive matrices bounded on $\dot{\mathbf{f}}_p^{\alpha q}$, $1 < p, q < +\infty$. Applying the ideas of Rubio de Francia (see, e.g., [GC-RdF]) in our context gives some extrapolation results, for instance, the following. (See Section 10 for the relevant definitions.)

THEOREM 10.17. *Suppose* T_m *is a Fourier multiplier operator bounded on* $\dot{\mathbf{F}}_{p_0}^{0 q_0}(w)$, *for all* $w \in A_{p_0}$, *for some fixed* p_0 *and* q_0 *with* $1 < p_0, q_0 < +\infty$. *Then* T_m *is bounded on* $\dot{\mathbf{F}}_p^{0q}(w)$ *for all* $1 < p, q < +\infty$, *and all* $w \in A_p$.

The problem of restricting distributions in a certain $\dot{\mathbf{F}}_p^{\alpha q}$ space on $\mathbb{R}^n$ to the hyperplane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ is considered in Section 11. It has been observed before that the restriction, or trace, results for $\dot{\mathbf{F}}_p^{\alpha q}$ are independent of the index q (e.g., [Ja3]). By considering our sequence spaces $\dot{\mathbf{f}}_p^{\alpha q}$ and exploiting Proposition 2.7 in conjunction with the geometry of the trace problem, we obtain a simple geometric explanation of this fact. From this and the result in the diagonal case $q = p$ in [Fr-J1] ($\dot{\mathbf{F}}_p^{\alpha p}$ coincides with the Besov space $\dot{\mathbf{B}}_p^{\alpha p}$), we obtain complete trace results for $\dot{\mathbf{F}}_p^{\alpha q}$ in Theorem 11.1. This includes the known cases and some that may be new.

Our treatment so far has dealt only with the homogeneous spaces $\dot{\mathbf{F}}_p^{\alpha q}$. In Section 12 we describe the corresponding results for the inhomogeneous spaces $\mathbf{F}_p^{\alpha q}$, which include, for example, the Bessel potential spaces L_α^p. The main difference is that instead of using all dyadic cubes in $\mathbb{R}^n$, we use only cubes Q with sidelength $l(Q) \leqslant 1$, and the functions corresponding to cubes Q with $l(Q) = 1$ are slightly different. Otherwise, everything is essentially the same and all our results for $\dot{\mathbf{F}}_p^{\alpha q}$ have inhomogeneous analogues.

We consider pointwise multipliers for the $\mathbf{F}_p^{\alpha q}$ spaces in Section 13. After some general remarks, we restrict attention to the case of the characteristic function χ_Ω of a domain $\Omega \in \mathbb{R}^n$, and ask when the operator $Tf(x) = \chi_\Omega(x) f(x)$ is bounded on $\mathbf{F}_p^{\alpha q}$. We consider the following condition: we say $\Omega \in D_s$ $(s > 0)$ if

$$\sup_{Q \text{ dyadic}, l(Q) \leqslant 1} l(Q) \left(\frac{1}{|Q|} \int_{\Omega \cap Q} \frac{1}{\delta(x)^s}\,dx\right)^{1/s} < +\infty,$$

where $\delta(x)$ is the distance from x to the complement of Ω. Our main result is Theorem 13.3, which states that if $\Omega \in D_s$, then χ_Ω is a pointwise multiplier for $\mathbf{F}_p^{\alpha q}$ for α in a certain range depending on s, p, and q (see (13.17)–(13.18)). In the proof of this theorem we exploit the smooth atomic decomposition and certain precise conditions from Section 10 for matrix boundedness on the sequence spaces $\dot{\mathbf{f}}_p^{\alpha q}$. By duality and interpolation, we deduce, in Corollaries 13.4–13.5, that χ_Ω is a bounded multiplier for a larger range of indices. To understand the meaning of these results, it is easy to check that if Ω is the upper half space $\mathbb{R}^n_+$, then $\Omega \in D_s$ for $0 < s < 1$ (see (13.7)). Then our results for $\mathbb{R}^n_+$ agree with the known results (e.g., those in [Tr2]). More generally, however, the classes D_s, $0 < s < 1$, include, but are not restricted to, all bounded Lipschitz domains. Thus our pointwise multiplier results for χ_Ω apply to certain general classes of domains Ω, which properly include Lipschitz domains.

Finally, in Section 14 we suggest some possible extensions of our results and make a few concluding comments.

2. The φ-Transform and Some Basic Facts

In this section we shall start by recalling the definition of the φ-transform (cf. [Fr-J1; Fr-J2]). The main result is Theorem 2.2 which shows that the φ-transform allows us to identify the Triebel–Lizorkin spaces $\dot{\mathbf{F}}_p^{\alpha q}$ with subspaces of the analogously defined sequence spaces $\dot{\mathbf{f}}_p^{\alpha q}$ (precise definitions are given below). To prove this theorem we need several basic facts. These are either known or follow by quite elementary arguments from well-known results. In the latter case we have deferred proofs to Appendix A. We conclude the section by showing another basic, geometric property of the $\dot{\mathbf{f}}_p^{\alpha q}$-spaces (Proposition 2.7).

To set notation, let φ and ψ satisfy

$$\varphi, \psi \in \mathcal{S}(\mathbb{R}^n), \tag{2.1}$$

$$\operatorname{supp} \hat{\varphi}, \hat{\psi} \subseteq \{\xi \in \mathbb{R}^n : \tfrac{1}{2} \leqslant |\xi| \leqslant 2\}, \tag{2.2}$$

$$|\hat{\varphi}(\xi)|, |\hat{\psi}(\xi)| \geqslant c > 0 \qquad \text{if} \quad \tfrac{3}{5} \leqslant |\xi| \leqslant \tfrac{5}{3}, \tag{2.3}$$

and

$$\sum_{\nu \in \mathbb{Z}} \overline{\hat{\varphi}(2^\nu \xi)}\, \hat{\psi}(2^\nu \xi) = 1 \qquad \text{if} \quad \xi \neq 0. \tag{2.4}$$

We set $\varphi_\nu(x) = 2^{\nu n} \varphi(2^\nu x)$ and $\psi_\nu(x) = 2^{\nu n} \psi(2^\nu x)$, $\nu \in \mathbb{Z}$.

For $\nu \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, we let $Q_{\nu k}$ be the dyadic cube

$$Q_{\nu k} = \{(x_1, ..., x_n) \in \mathbb{R}^n : k_i \leqslant 2^\nu x_i < k_i + 1,\ i = 1, ..., n\}.$$

We denote the "lower left-corner" $2^{-\nu}k$ of $Q = Q_{\nu k}$ by x_Q and the side length $2^{-\nu}$ by $l(Q)$. Define

$$\varphi_Q(x) = |Q|^{-1/2}\, \varphi(2^\nu x - k) = |Q|^{1/2}\, \varphi_\nu(x - x_Q) \qquad \text{if} \quad Q = Q_{\nu k},$$

and similarly define ψ_Q. Note that $\|\varphi_Q\|_{L^2} = \|\varphi\|_{L^2}$ and $\|\psi_Q\|_{L^2} = \|\psi\|_{L^2}$ for all dyadic Q,

$$\operatorname{supp} \hat{\varphi}_Q, \hat{\psi}_Q \subseteq \{\xi : 2^{\nu-1} \leqslant |\xi| \leqslant 2^{\nu+1}\} \qquad \text{if} \quad l(Q) = 2^{-\nu}, \tag{2.5}$$

and

$$|\partial^\gamma \varphi_Q|, |\partial^\gamma \psi_Q| \leqslant C_{\gamma,L}\, |Q|^{-1/2 - |\gamma|/n}\, (1 + l(Q)^{-1}\, |x - x_Q|)^{-L - |\gamma|}, \tag{2.6}$$

for each $L \in \mathbb{Z}_+$ and multi-index γ of length $|\gamma| \geqslant 0$.

For φ and ψ satisfying (2.1)–(2.4), the *φ-transform* S_φ is the map taking each $f \in \mathscr{S}'/\mathscr{P}$ (the space of tempered distributions modulo polynomials) to the sequence $S_\varphi f = \{(S_\varphi f)_Q\}_Q$ defined by $(S_\varphi f)_Q = \langle f, \varphi_Q \rangle$ for Q dyadic. The *inverse φ-transform* T_ψ is the map taking a sequence $s = \{s_Q\}_Q$ to $T_\psi s = \sum_Q s_Q \psi_Q$. Here and throughout, when Q appears as an index, it is understood that Q runs over the dyadic cubes in $\mathbb{R}^n$.

As in [Fr-J1], the basis for our results concerning the φ-transform is the following lemma.

LEMMA 2.1 [Fr-J1, Lemma 2.1]. *Suppose φ and ψ satisfy* (2.1)–(2.4). *If $f \in \mathscr{S}'/\mathscr{P}(\mathbb{R}^n)$, then*

$$f(\cdot) = \sum_{\nu \in \mathbb{Z}} 2^{-\nu n} \sum_{k \in \mathbb{Z}^n} \tilde{\varphi}_\nu * f(2^{-\nu}k)\, \psi_\nu(\cdot - 2^{-\nu}k) = \sum_Q \langle f, \varphi_Q \rangle\, \psi_Q(\cdot),$$

where $\tilde{\varphi}_\nu(x) = \overline{\varphi_\nu(-x)}$.

Hence, $T_\psi \circ S_\varphi$ equals the identity on $\mathscr{S}'/\mathscr{P}$.

For $\alpha \in \mathbb{R}$, $0 < p < +\infty$, $0 < q \leqslant +\infty$, and any φ satisfying (2.1)–(2.3), the Triebel–Lizorkin space $\dot{\mathbf{F}}_p^{\alpha q}$ is the collection of all $f \in \mathscr{S}'/\mathscr{P}(\mathbb{R}^n)$ such that

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} = \left\| \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha}\, |\varphi_\nu * f|)^q \right)^{1/q} \right\|_{L^p} < +\infty,$$

where the l^q-norm is replaced by the sup on ν if $q = +\infty$. This definition is independent of the choice of φ; see, e.g., Remark 2.6 below. We note that the quantity inside the L^p-norm defining $\dot{\mathbf{F}}_p^{\alpha q}$ is a generalized, discrete Littlewood–Paley expression which corresponds to the usual g-function if $\alpha = 0$ and $q = 2$. Hence, the well-known equivalence $\|g(f)\|_{L^p} \approx \|f\|_{H^p}$, $0 < p < +\infty$ [St, Fef-S2] (here $\approx$ means that the (quasi)-norms are equivalent) suggests the result of Peetre [P1] that $\dot{\mathbf{F}}_p^{02} = H^p$ for $0<p+\infty$, modulo polynomials (see also [Tr1, p. 30; Bui; U2], and Remark 7.8 below).

For $\alpha \in \mathbb{R}$, $0<p<+\infty$, and $0<q\leqslant +\infty$, we let $\dot{\mathbf{f}}_p^{\alpha q}$ be the collection of all complex-valued sequences $s=\{s_Q\}_Q$ such that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}} = \left\| \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p} < +\infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2} \chi_Q$ is the L^2-normalized characteristic function of Q.

From our next theorem follows the fundamental fact that the following diagram is commutative:

$$\begin{tikzcd} & \dot{\mathbf{f}}_p^{\alpha q} \arrow[dr, "T_\psi"] & \\ \dot{\mathbf{F}}_p^{\alpha q} \arrow[ur, "S_\varphi"] \arrow[rr, "\mathrm{Id}"] & & \dot{\mathbf{F}}_p^{\alpha q}. \end{tikzcd}$$

THEOREM 2.2. *Suppose* $\alpha \in \mathbb{R}$, $0<p<+\infty$, $0<q\leqslant+\infty$, *and* φ *and* ψ *satisfy* (2.1)–(2.4). *The operators* $S_\varphi: \dot{\mathbf{F}}_p^{\alpha q} \to \dot{\mathbf{f}}_p^{\alpha q}$ *and* $T_\psi: \dot{\mathbf{f}}_p^{\alpha q} \to \dot{\mathbf{F}}_p^{\alpha q}$ *are bounded. Furthermore,* $T_\psi \circ S_\varphi$ *is the identity on* $\dot{\mathbf{F}}_p^{\alpha q}$. *In particular,* $\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \approx \|S_\varphi f\|_{\dot{\mathbf{f}}_p^{\alpha q}}$, *and* $\dot{\mathbf{F}}_p^{\alpha q}$ *can be identified with a complemented subspace of* $\dot{\mathbf{f}}_p^{\alpha q}$.

More explicitly, S_φ identifies $\dot{\mathbf{F}}_p^{\alpha q}$ with the subspace $S_\varphi(\dot{\mathbf{F}}_p^{\alpha q})$. Note that $\mathrm{Pr} = S_\varphi \circ T_\psi$ is a projection operator from $\dot{\mathbf{f}}_p^{\alpha q}$ onto this subspace. In particular, Pr is the matrix operator $(\mathrm{Pr}(\{s_P\}_P))_Q = \sum_P s_P \langle \psi_P, \varphi_Q \rangle$. Of course, $S_\varphi(\dot{\mathbf{F}}_p^{\alpha q})$ consists exactly of the sequences invariant under Pr; we thus have the criterion that $\{s_P\}_P \in S_\varphi(\dot{\mathbf{F}}_p^{\alpha q})$ if and only if $s_Q = \sum_P s_P \langle \psi_P, \varphi_Q \rangle$ for each Q.

Similarly, since $T_\psi \circ S_\varphi$ is the identity operator, we have

$$\langle f, g \rangle = \left\langle \sum_Q (S_\varphi f)_Q \psi_Q, g \right\rangle = \langle S_\varphi f, S_\psi g \rangle \tag{2.7}$$

for $f \in \mathscr{S}_0$ and $g \in \mathscr{S}'/\mathscr{P}$. Here $\langle s, t \rangle = \sum_Q s_Q \bar{t}_Q$ for sequences s and t. Note that the related identity

$$\langle S_\varphi f, t \rangle = \sum_Q \langle f, \varphi_Q \rangle \bar{t}_Q = \langle f, T_\varphi t \rangle \tag{2.8}$$

is trivial.

If we choose $\psi = \varphi$, which we may, then our representation formula becomes $f = \sum_Q \langle f, \varphi_Q \rangle \varphi_Q$, as if the collection $\{\varphi_Q\}_Q$ forms an orthonormal basis. Similarly, when $\varphi = \psi$, (2.7) takes the form $\langle f, g \rangle = \langle S_\varphi f, S_\varphi g \rangle$, again as in the orthonormal case. Of course, (2.7) is a triviality if $\varphi = \psi$ and the φ_Q's are orthonormal.

Now, to prove Theorem 2.2 we need two additional lemmas; the first is a characterization of $\dot{\mathbf{f}}_p^{\alpha q}$ analogous to the g_λ^* characterization of L^p in classical Littlewood–Paley theory (see, e.g., [St]) and Peetre's φ_v^{**} characterization of the $\dot{\mathbf{F}}_p^{\alpha q}$-spaces [P2].

For a sequence $s = \{s_Q\}_Q$, $0 < r \leqslant +\infty$, and a fixed $\lambda > 0$, we define the sequence $s_r^* = \{(s_r^*)_Q\}_Q$ by

$$(s_r^*)_Q = \left(\sum_{P : l(P) = l(Q)} |s_P|^r / (1 + l(Q)^{-1} |x_P - x_Q|)^\lambda \right)^{1/r}.$$

(We will write $s_{r,\lambda}^*$ when the choice of λ requires emphasis.) Notice that the imbedding $l^r \to l^q$, if $r \leqslant q$, implies that

$$s_{q,\lambda}^* \leqslant s_{r,\mu}^* \tag{2.9}$$

with $\mu = \lambda r / q$.

LEMMA 2.3. *Suppose* $\alpha \in \mathbb{R}$, $0 < p < +\infty$, $0 < q \leqslant +\infty$, *and* $\lambda > n$. *Then*

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}} \approx \|s_{\min(p,q)}^*\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

The proof relies on the Fefferman–Stein vector-valued maximal inequality and is given in Appendix A.

Remark 2.4. The following remark is a reformulation of the remarks in [Fef-S1] regarding the Marcinkiewicz integral; our purpose is to exhibit the geometric content of Lemma 2.3. Let $\Omega \subset \mathbb{R}^n$ be an open set with Whitney decomposition $F = \{Q_i\}_{i=1}^\infty$ (see [St, Chapt. 6]). For $x \in \mathbb{R}^n$, let $\delta(x)$ be the distance from x to $\mathbb{R}^n \backslash \Omega$. The Marcinkiewicz integral of order β is

$$\begin{aligned} J_\beta(x) &= \int_{\mathbb{R}^n} \frac{\delta(x+y)^{n(\beta-1)}}{[|y| + \delta(x+y)]^{n\beta}} \, dy \\ &\approx \sum_{Q \in F} 1/(1 + l(Q)^{-1} |x - x_Q|)^{n\beta}. \end{aligned}$$

Classical results [Cal-Z; Carl; Fef-S1] give

$$\int_{\mathbb{R}^n} J_\beta(x)^a \, dx \leqslant c\, |\Omega| \qquad \text{if} \quad 1/\min(a, 1) < \beta < +\infty.$$

Define a sequence $s=\{s_Q\}_Q$ by $s_Q=|Q|^{1/2}$ if $Q\in F$ and $s_Q=0$ otherwise. Then

$$\|s\|_{\dot{\mathbf{f}}_p^{0q}}=\left\|\left(\sum_{Q\in F}\chi_Q\right)^{1/q}\right\|_{L^p}=\left|\bigcup_{Q\in F}Q\right|^{1/p}=|\Omega|^{1/p}.$$

Also, however,

$$\begin{aligned}\|s^*_{q,\lambda}\|_{\dot{\mathbf{f}}_p^{0q}}&=\left\|\left(\sum_Q\left(\tilde{\chi}_Q\left(\sum_{\substack{l(P)=l(Q)\\ P\in F}}|P|^{q/2}\Big/(1+l(P)^{-1}|x_P-x_Q|)^\lambda\right)^{1/q}\right)^q\right)^{1/q}\right\|_{L^p}\\
&=\left\|\left(\sum_Q\chi_Q\sum_{\substack{l(P)=l(Q)\\ P\in F}}1\Big/(1+l(P)^{-1}|x_P-x_Q|)^\lambda\right)^{1/q}\right\|_{L^p}\\
&\approx\left\|\left(\sum_\nu\sum_{l(Q)=2^{-\nu}}\chi_Q\sum_{\substack{l(P)=l(Q)\\ P\in F}}1\Big/(1+l(P)^{-1}|x-x_P|)^\lambda\right)^{1/q}\right\|_{L^p}\\
&=\left\|\left(\sum_{P\in F}1\Big/(1+l(P)^{-1}|x-x_P|)^\lambda\right)^{1/q}\right\|_{L^p}\approx\|J_{\lambda/n}\|_{L^{p/q}}^{1/q}.\end{aligned}$$

Let $r=\min(p,q)$ and $\mu=r\lambda/q$. Using (2.9), we see that $s^*_{q,\lambda}\leqslant s^*_{r,\mu}$. Thus by Lemma 2.3,

$$\begin{aligned}\|J_{\lambda/n}\|_{L^{p/q}}^{p/q}&\leqslant c\,\|s^*_{q,\lambda}\|_{\dot{\mathbf{f}}_p^{0q}}^p\leqslant c\,\|s^*_{r,\mu}\|_{\dot{\mathbf{f}}_p^{0q}}^p\\
&\leqslant c\,\|s\|_{\dot{\mathbf{f}}_p^{0q}}^p\leqslant c\,|\Omega|,\end{aligned}$$

if $\mu>n$, i.e., if $\lambda/n>1/\min(1,p/q)$. Setting $a=p/q$ and $\beta=\lambda/n$ yields the classical estimates above.

The next lemma is a version of a classical result about entire functions of exponential type which goes back to Plancherel and Pólya [Pl-P]. The underlying idea is that a smooth function cannot oscillate too quickly and, consequently, the supremum and infimum over most sufficiently small cubes must be comparable. The lemma will also be useful later on to relate the norms of $\dot{\mathbf{F}}_p^{\alpha q}$ and $\dot{\mathbf{f}}_p^{\alpha q}$.

Let φ satisfy (2.1)–(2.3). Note that $\tilde{\varphi}(x)=\overline{\varphi(-x)}$ also satisfies (2.1)–(2.3), so that we may take $\tilde{\varphi}$ in place of φ in the definition of $\dot{\mathbf{F}}_p^{\alpha q}$. For $f\in\mathscr{S}'/\mathscr{P}$ and Q dyadic with $l(Q)=2^{-\nu}$, we define the sequence $\sup(f)=\{\sup_Q(f)\}_Q$ by setting

$$\sup_Q(f)=|Q|^{1/2}\sup_{y\in Q}|\tilde{\varphi}_\nu*f(y)|,$$

and, for $\gamma \in \mathbb{Z}$ with $\gamma \geqslant 0$, the sequence $\inf_\gamma(f) = \{\inf_{Q,\gamma}(f)\}_Q$ by

$$\inf_{Q,\gamma}(f) = |Q|^{1/2} \max\{\inf_{y \in \tilde{Q}} |\tilde{\varphi} * f(y)| : l(\tilde{Q}) = 2^{-\gamma} l(Q),\ \tilde{Q} \subseteq Q\}.$$

LEMMA 2.5. *Suppose* $\alpha \in \mathbb{R}$, $0 < p < +\infty$, $0 < q \leqslant +\infty$, *and* $\gamma \in \mathbb{Z}$ *is sufficiently large. Then for* $f \in \mathscr{S}'/\mathscr{P}$,

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \approx \|\sup(f)\|_{\dot{\mathbf{f}}_p^{\alpha q}} \approx \|\inf_\gamma(f)\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

A proof can be found in Appendix A. A more precise version (at least in one dimension) of this lemma may be obtained from the interpolation formula in [Bo, p. 192].

Proof of Theorem 2.2. The boundedness of $S_\varphi : \dot{\mathbf{F}}_p^{\alpha q} \to \dot{\mathbf{f}}_p^{\alpha q}$ follows from Lemma 2.5, since, if $Q = Q_{\nu k}$,

$$|(S_\varphi f)_Q| = |\langle f, \varphi_Q \rangle| = |Q|^{1/2} |\tilde{\varphi}_\nu * f(2^{-\nu} k)| \leqslant \sup_Q(f).$$

To prove the boundedness of $T_\psi : \dot{\mathbf{f}}_p^{\alpha q} \to \dot{\mathbf{F}}_p^{\alpha q}$, suppose $s = \{s_Q\}_Q$ and $f = T_\psi s = \sum_Q s_Q \psi_Q$. By (2.5),

$$\tilde{\varphi}_\nu * f(x) = \sum_{\mu = \nu - 1}^{\nu + 1} \sum_{l(J) = 2^{-\mu}} s_J \tilde{\varphi}_\nu * \psi_J.$$

After a translation and a dilation, the estimate

$$|\tilde{\varphi}_\nu * \psi_J(x)| \leqslant c_{\lambda,r,n} |J|^{-1/2}/(1 + 2^\mu |x - x_J|)^{\lambda/\min(1,r)}$$

if $l(J) = 2^{-\mu}$, $\lambda > n$, and $0 < r < +\infty$, follows from the fact that $\tilde{\varphi} * \psi_{-1}$, $\tilde{\varphi} * \psi$, and $\tilde{\varphi} * \psi_1$ all belong to $\mathscr{S}$. Therefore, if $x \in Q^* \subseteq Q \subseteq Q^{**}$, where Q^*, Q, and Q^{**} are dyadic with $l(Q^*) = 2^{-\nu-1}$, $l(Q) = 2^{-\nu}$, and $l(Q^{**}) = 2^{-\nu+1}$, we have

$$|\tilde{\varphi}_\nu * f(x)| \leqslant c\, |Q|^{-1/2} \sum_{\mu = \nu - 1}^{\nu + 1} \left(\sum_{l(J) = 2^{-\mu}} |s_J|^r \Big/ (1 + 2^\mu |x - x_J|)^\lambda \right)^{1/r},$$

by the r-triangle inequality if $r \leqslant 1$, or by Hölder's inequality and the fact that if $r > 1$ and $\lambda > n$, $\sum_{l(J) = 2^{-\mu}} (1 + 2^\mu |x - x_J|)^{-\lambda} \leqslant c$. Hence for $x \in Q^*$,

$$|\tilde{\varphi}_\nu * f(x)| \leqslant c\, |Q|^{-1/2} ((s_r^*)_{Q^*} + (s_r^*)_Q + (s_r^*)_{Q^{**}}).$$

Taking $r = \min(p, q)$ and applying Lemma 2.3, we obtain

$$\|T_\psi s\|_{\dot{\mathbf{F}}_p^{\alpha q}} = \left\| \sum_J s_J \psi_J \right\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c\, \|s_r^*\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c\, \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

Finally, the fact that $T_\psi \circ S_\varphi$ is the identity on $\dot{\mathbf{F}}_p^{\alpha q}$ follows from Lemma 2.1. ▮

Remark 2.6. Suppose φ^1 and φ^2 each satisfy (2.1)–(2.3). Then it is possible to find ψ^1 and ψ^2 so that (2.1)–(2.4) are satisfied for each pair φ^i, ψ^i, $i=1, 2$. Define $\dot{\mathbf{F}}_p^{\alpha q}(\tilde{\varphi}^1)$ and $\dot{\mathbf{F}}_p^{\alpha q}(\tilde{\varphi}^2)$ as above, using $\tilde{\varphi}^1$ and $\tilde{\varphi}^2$ in place of φ. Note that the proof of the boundedness of $T_\psi: \dot{\mathbf{f}}_p^{\alpha q} \to \dot{\mathbf{F}}_p^{\alpha q}$ above requires only (2.1)–(2.2). Hence,

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}(\tilde{\varphi}^1)} = \left\| \sum_Q (S_{\varphi^2} f)_Q \psi_Q^2 \right\|_{\dot{\mathbf{F}}_p^{\alpha q}(\tilde{\varphi}^1)} \leqslant c \, \|S_{\varphi^2} f\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c \, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}(\tilde{\varphi}^2)}.$$

This shows that the definition of $\dot{\mathbf{F}}_p^{\alpha q}$ is independent of the choice of φ satisfying (2.1)–(2.3) (cf. [P2]).

We also note the following simple fact, which is of considerable use in applications (e.g., Section 3 in [Fr-J2] and Sections 11, 13 below).

PROPOSITION 2.7. *Let $\varepsilon > 0$. Suppose that for each dyadic cube Q, $E_Q \subseteq Q$ is a measurable set with $|E_Q|/|Q| \geqslant \varepsilon$. Then*

$$\|\{s_Q\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \approx \left\| \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p},$$

where $\tilde{\chi}_{E_Q} = |E_Q|^{-1/2} \chi_{E_Q}$.

Proof. Since $\tilde{\chi}_{E_Q} \leqslant \varepsilon^{-1/2} \tilde{\chi}_Q$, one direction is trivial. For the other, note that for all $A > 0$, $\chi_Q \leqslant \varepsilon^{-1/A} (M(\chi_{E_Q}^A))^{1/A}$, where M denotes the Hardy–Littlewood maximal operator (see Appendix A). Select A such that p/A, $q/A > 1$. By Theorem A.1, then,

$$\|\{s_Q\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant \varepsilon^{-1/A} \left\| \left(\sum_Q (M(|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q})^A)^{q/A} \right)^{A/q} \right\|_{L^{p/A}}^{1/A}$$

$$\leqslant c\varepsilon^{-1/A} \left\| \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p}. \quad ▮$$

3. Almost Diagonal Operators and Smooth Molecules

Our purpose in this section is to obtain a sufficient condition for an operator A on $\dot{\mathbf{f}}_p^{\alpha q}$ to be bounded; the condition is simple, yet general enough to include many interesting operators. We then use this condition to generalize Theorem 2.2 in two ways. The first involves decompositions into "smooth molecules" and is an extension of the boundedness of T_ψ. The second generalizes the fact that S_φ is bounded.

The commutative diagram preceding Theorem 2.2 in Section 2 can be extended to the level of operators. For a (quasi-)normed space X, let $\mathscr{L}(X)$ be the space of bounded linear operators on X with the operator norm. For φ and ψ satisfying (2.1)–(2.4), we define maps $S_\varphi^*: \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q}) \to \mathscr{L}(\dot{\mathbf{f}}_p^{\alpha q})$ and $T_\psi^*: \mathscr{L}(\dot{\mathbf{f}}_p^{\alpha q}) \to \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q})$ by

$$S_\varphi^* B = S_\varphi \circ B \circ T_\psi$$

and

$$T_\psi^* A = T_\psi \circ A \circ S_\varphi$$

if $A \in \mathscr{L}(\dot{\mathbf{f}}_p^{\alpha q})$ and $B \in \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q})$. As long as $q \neq +\infty$, any $A \in \mathscr{L}(\dot{\mathbf{f}}_p^{\alpha q})$ is represented by a matrix $\{a_{QP}\}_{Q,P}$. Namely, if $e^P \in \dot{\mathbf{f}}_p^{\alpha q}$ is defined by $(e^P)_Q = 1$ if $Q = P$ and $(e^P)_Q = 0$ otherwise, we set $a_{QP} = (Ae^P)_Q$. Then $(As)_Q = \sum_P a_{QP} s_P$ for $s = \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha q}$. For $f \in \dot{\mathbf{F}}_p^{\alpha q}$, let $s_Q = \langle f, \varphi_Q \rangle$. By Theorem 2.2, then, $f = \sum_Q s_Q \psi_Q$ and hence

$$(T_\psi^* A)(f) = \sum_Q (AS_\varphi f)_Q \psi_Q = \sum_Q (As)_Q \psi_Q.$$

Similarly, if $s \in \dot{\mathbf{f}}_p^{\alpha q}$ and $B \in \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q})$, $q < +\infty$,

$$\begin{aligned}(S_\varphi^* B)(s)_Q &= (S_\varphi B T_\psi s)_Q = \Big\langle B \sum_P s_P \psi_P, \varphi_Q \Big\rangle \\ &= \sum_P s_P \langle B\psi_P, \varphi_Q \rangle,\end{aligned}$$

so $S_\varphi^* B$ is the operator on $\dot{\mathbf{f}}_p^{\alpha q}$ associated with the matrix $a_{QP} = \langle B\psi_P, \varphi_Q \rangle$. If $q = +\infty$ the same representation of $S_\varphi^* B$ holds under weak continuity assumptions on B, e.g., if $g_n \to g$ in $\mathscr{S}'/\mathscr{P}$, $g_n, g \in \dot{\mathbf{F}}_\infty^{\alpha q}$, then $Bg_n \to Bg$ in $\mathscr{S}'/\mathscr{P}$.

PROPOSITION 3.1. *Let $\alpha \in \mathbb{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$. The maps S_φ^* and T_ψ^* are bounded, and $T_\psi^* \circ S_\varphi^*$ is the identity on $\mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q})$. In particular, for $B \in \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q})$,*

$$\|S_\varphi^* B\|_{\mathscr{L}(\dot{\mathbf{f}}_p^{\alpha q})} \approx \|B\|_{\mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q})}.$$

Proof. All conclusions follow immediately from Theorem 2.2 and the definitions of S_φ^* and T_ψ^*. ∎

Thus the following diagram is commutative:

$$\begin{tikzcd} & \mathscr{L}(\dot{\mathbf{f}}_p^{\alpha q}) \arrow[dr, "T_\psi^*"] & \\ \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q}) \arrow[ur, "S_\varphi^*"] \arrow[rr, "\mathrm{Id}"] & & \mathscr{L}(\dot{\mathbf{F}}_p^{\alpha q}). \end{tikzcd}$$

Let $J = n/\min(1, p, q)$. We say that A, with associated matrix $\{a_{QP}\}_{Q,P}$, is *almost diagonal on* $\dot{\mathbf{f}}_p^{\alpha q}$ if there exists $\varepsilon > 0$ such that

$$\sup_{Q,P} |a_{QP}|/\omega_{QP}(\varepsilon) < +\infty, \tag{3.1}$$

where

$$\omega_{QP}(\varepsilon) = \left(\frac{l(Q)}{l(P)}\right)^\alpha \left(1 + \frac{|x_Q - x_P|}{\max(l(P), l(Q))}\right)^{-J-\varepsilon}$$
$$\times \min\left[\left(\frac{l(Q)}{l(P)}\right)^{(n+\varepsilon)/2}, \left(\frac{l(P)}{l(Q)}\right)^{(n+\varepsilon)/2+J-n}\right].$$

Remark 3.2. For $\alpha = 0$ and $p, q \geqslant 1$, the almost diagonality condition has the following interpretation. Let $G = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$ be the group with multiplication

$$(x_0, t_0) \cdot (x_1, t_1) = (t_1 x_0 + x_1, t_0 t_1).$$

Then the map $U: G \to \mathscr{L}(L^2(\mathbb{R}^n))$ defined by

$$U(x_0, t_0) f = t_0^{-n/2} f((x - x_0)/t_0)$$

is a unitary representation of G. In particular, if we associate with the dyadic cube $Q \subseteq \mathbb{R}^n$ the point $(x_Q, l(Q)) \in G$, then $U(x_Q, l(Q))\psi = \psi_Q$. Thus Theorem 2.2 shows that the image of ψ under the subset $\{U(x_Q, l(Q)) : Q \text{ is dyadic}\}$ of $U(G)$ generates $\dot{\mathbf{F}}_p^{\alpha q}$ for each $\alpha \in \mathbb{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$. (See the comments and references in [Fr-J2, Section 4], and [Fei-G] for further discussion of this.) By analogy to the Poincaré metric in the upper half-plane, we define

$$d((x_0, t_0), (x_1, t_1)) = \log\left(\frac{1+\rho}{1-\rho}\right)^{1/2},$$

where

$$\rho = \rho((x_0, t_0), (x_1, t_1)) = \left(\frac{|x_1 - x_0|^2 + (t_1 - t_0)^2}{|x_1 - x_0|^2 + (t_1 + t_0)^2}\right)^{1/2},$$

for $(x_0, t_0), (x_1, t_1) \in G$. Then d is invariant under right multiplication:

$$d((x_0, t_0)\cdot(a, b), (x_1, t_1)\cdot(a, b)) = d((x_0, t_0), (x_1, t_1)).$$

We have the elementary observations that

$$\begin{aligned}\left(\frac{1+\rho}{1-\rho}\right)^{1/2} &= \frac{1+\rho}{(1-\rho^2)^{1/2}}\\ &= \frac{1+\rho}{2}\left(\frac{(t_1+t_0)^2}{t_1 t_0}\right)^{1/2}\left(1+\frac{|x_1-x_0|^2}{|t_1+t_0|^2}\right)^{1/2}.\end{aligned}$$

Since $0 \leqslant \rho < 1$, we obtain

$$\left(\frac{1+\rho}{1-\rho}\right)^{1/2} \approx \max\left(\sqrt{\frac{t_1}{t_0}}, \sqrt{\frac{t_0}{t_1}}\right)\left(1+\frac{|x_1-x_0|}{\max(t_1, t_0)}\right).$$

For P and Q dyadic cubes, we obtain a "distance" by setting

$$d(P, Q) = d((x_P, l(P)), (x_Q, l(Q))).$$

Then for $\alpha = 0$ and $p, q \geqslant 1$, (3.1) reduces to the condition

$$|a_{QP}| \leqslant c_\varepsilon e^{-(n+\varepsilon)\, d(P,Q)}. \tag{3.2}$$

THEOREM 3.3. *Suppose* $\alpha \in \mathbb{R}$, $0 < p < +\infty$, *and* $0 < q \leqslant +\infty$. *An almost diagonal operator on* $\dot{\mathfrak{f}}_p^{\alpha q}$ *is bounded.*

Proof. We assume $\alpha = 0$, since this case implies the general case, and put $r = \min(p, q)$. We shall consider the case $r > 1$ first.

Let A be an almost diagonal operator on $\dot{\mathfrak{f}}_p^{\alpha q}$ with matrix $\{a_{QP}\}_{Q,P}$, satisfying (3.2). We write $A = A_0 + A_1$ with

$$(A_0 s)_Q = \sum_{l(P) \geqslant l(Q)} a_{QP} s_P \qquad \text{and} \qquad (A_1 s)_Q = \sum_{l(P) < l(Q)} a_{QP} s_P,$$

for $s = \{s_Q\}_Q \in \dot{\mathfrak{f}}_p^{0q}$. If $l(Q) = 2^{-\nu}$, our assumptions and Lemma A.2 with $\lambda = n + \varepsilon$ and $a = r = 1$ yield

$$\begin{aligned}|(A_0 s)_Q| &\leqslant c_\varepsilon \sum_{l(P) \geqslant l(Q)} \omega_{QP}(\varepsilon)\, |s_P|\\ &= c_\varepsilon \sum_{l(P) \geqslant l(Q)} \left(\frac{l(Q)}{l(P)}\right)^{(n+\varepsilon)/2} |s_P|/(1 + l(P)^{-1}\, |x_P - x_Q|)^{n+\varepsilon}\\ &\leqslant c_\varepsilon \sum_{\mu \leqslant \nu} 2^{(\mu-\nu)(n+\varepsilon)/2} M\left(\sum_{l(P)=2^{-\mu}} |s_P \chi_P|\right)(x),\end{aligned}$$

for $x \in Q$. Hence, since $|Q|^{-1/2} = 2^{(\nu-\mu)n/2}|P|^{-1/2}$ if $l(P) = 2^{-\mu}$,

$$\begin{aligned}
\|A_0 s\|_{\dot{f}_p^{0q}} &= \left\| \left(\sum_Q |(A_0 s)_Q \tilde{\chi}_Q|^q \right)^{1/q} \right\|_{L^p} \\
&\leqslant c_\varepsilon \left\| \left(\sum_{\nu \in \mathbb{Z}} \left(\sum_{\mu \leqslant \nu} 2^{(\mu-\nu)\varepsilon/2} M \left(\sum_{l(P)=2^{-\mu}} |s_P \tilde{\chi}_P| \right) \right)^q \right)^{1/q} \right\|_{L^p} \\
&\leqslant c_\varepsilon \left\| \left(\sum_{\mu \in \mathbb{Z}} \left(M \left(\sum_{l(P)=2^{-\mu}} |s_P \tilde{\chi}_P| \right) \right)^q \right)^{1/q} \right\|_{L^p},
\end{aligned}$$

by Minkowski's inequality. Since $p, q > 1$, applying the Fefferman–Stein vector-valued maximal inequality (Theorem A.1), we find that

$$\|A_0 s\|_{\dot{f}_p^{0q}} \leqslant c\, \|s\|_{\dot{f}_p^{0q}}.$$

It is easy to see (see Remark 5.11) that the dual of $\dot{f}_p^{0q}$ is $\dot{f}_{p'}^{0q'}$, where $1/p' + 1/p = 1$, and similarly for q'. Also, notice that the adjoint of A_1, A_1^*, has the same form as A_0. Hence, by using this duality and the argument we just gave applied to A_1^*, we obtain that A_1 is also bounded on $\dot{f}_p^{0q}$. This proves the theorem in the case $r > 1$ and $q < +\infty$.

The case $r \leqslant 1$ and $q < +\infty$ is in fact a simple consequence of the case $r > 1$. We pick an $\tilde{r} < r$ so close to r so that (3.1) is still satisfied with $r = \min(p, q)$ replaced by $\tilde{r}$. This means that $p/\tilde{r} > 1$ and $q/\tilde{r} > 1$, and that the matrix $\tilde{A} = \{\tilde{a}_{QP}\} \equiv \{|a_{QP}|^{\tilde{r}} (|Q|/|P|)^{1/2 - \tilde{r}/2}\}$ satisfies (3.2) for a different value of ε. Define $t = \{t_Q\}_Q$ by $t_Q = |Q|^{1/2 - \tilde{r}/2} |s_Q|^{\tilde{r}}$. Then $\|t\|_{\dot{f}_{p/\tilde{r}}^{0,q/\tilde{r}}}^{1/\tilde{r}} = \|s\|_{\dot{f}_p^{0q}}$. By the $\tilde{r}$-triangle inequality, we have

$$|(As)_Q| \leqslant \left(\sum_P |a_{QP}|^{\tilde{r}} |s_P|^{\tilde{r}} \right)^{1/\tilde{r}}.$$

Hence, $\|As\|_{\dot{f}_p^{0q}} \leqslant \|\tilde{A}t\|_{\dot{f}_{p/\tilde{r}}^{0,q/\tilde{r}}}^{1/\tilde{r}}$. Therefore the boundedness of A on $\dot{f}_p^{0q}$ follows from the boundedness of $\tilde{A}$ on $\dot{f}_{p/\tilde{r}}^{0,q/\tilde{r}}$.

Now, the case $q = +\infty$ and $p > 1$ follows by duality from the result for $q = 1$ which we have just obtained. Finally, for $p \leqslant 1$ and $q = +\infty$ we reduce to the case $p > 1$ as before. ∎

Remark 3.4. Another, perhaps more direct, proof of this theorem, avoiding the duality and the reduction, can be given by using Remark A.3 instead of Lemma A.2.

Lemma 3.1 and Theorem 3.3 yield that a linear operator B corresponding to an almost diagonal matrix is bounded on $\dot{F}_p^{\alpha q}$. Using this, we can generalize the estimate $\|\sum_Q s_Q \psi_Q\|_{\dot{F}_p^{\alpha q}} \leqslant c\, \|\{s_Q\}_Q\|_{\dot{f}_p^{\alpha q}}$ in Theorem 2.2.

For α, q, and p as above regarded as fixed, let $J=n/\min(1, p, q)$, $N=\max([J-n-\alpha], -1)$, and $\alpha^*=\alpha-[\alpha]$. We say that $\{m_Q\}_Q$ is a family of *smooth molecules for* $\dot{\mathbf{F}}_p^{\alpha q}$ if there exist δ and M with $\alpha^*<\delta\leqslant 1$, $M>J$, such that for each dyadic cube Q,

$$\int x^\gamma m_Q(x)\,dx=0 \qquad \text{if} \quad |\gamma|\leqslant N, \tag{3.3}$$

$$|m_Q(x)|\leqslant |Q|^{-1/2}\,(1+l(Q)^{-1}\,|x-x_Q|)^{-\max(M, M-\alpha)}, \tag{3.4}$$

$$|\partial^\gamma m_Q(x)|\leqslant |Q|^{-1/2-|\gamma|/n}\,(1+l(Q)^{-1}\,|x-x_Q|)^{-M} \tag{3.5}$$

if $|\gamma|\leqslant[\alpha]$, and

$$\begin{aligned}&|\partial^\gamma m_Q(x)-\partial^\gamma m_Q(y)|\\ &\quad\leqslant |Q|^{-1/2-|\gamma|/n-\delta/n}\,|x-y|^\delta \sup_{|z|\leqslant|y-x|}(1+l(Q)^{-1}\,|x-z-x_Q|)^{-M}\end{aligned} \tag{3.6}$$

if $|\gamma|=[\alpha]$. We shall call a function m_Q satisfying (3.3)–(3.6), for some fixed δ and M, a smooth (δ, M)-molecule.

To be explicit, let us make the following comments. If $\alpha<0$, then (3.5)–(3.6) are void. If $\alpha\geqslant 0$, (3.4) follows from (3.5). If also $\alpha>J-n$, then $N=-1$, and (3.3) is void. If $\alpha=0$, then $N=[J-n]$; if also $0<p\leqslant 1$ and $q\geqslant p$, then $N=[n(1/p-1)]$, and (3.3) is the usual vanishing moment condition for H^p-atoms. In the case $\alpha=0$ and $\min(p,q)\geqslant 1$ (e.g., H^1 and L^p for $1<p<+\infty$), the conditions are merely (3.3), (3.5), and (3.6) with $\gamma=0$ for some $\delta>0$ and $M>n$. For $\alpha\geqslant 0$, (3.6) and the assumption $\delta>\alpha^*$ show that $m_Q\in C^\beta$ for some $\beta>\alpha$.

Clearly, $\{\psi_Q\}_Q$ is a family of molecules for all $\dot{\mathbf{F}}_p^{\alpha q}$. We should also remark that the assumptions (3.3)–(3.6) are weaker than in the definition of "smooth molecule" in [Fr-J1, Fr-J2].

THEOREM 3.5. *If* $f=\sum_Q s_Q m_Q$, *where* $\{m_Q\}_Q$ *is a family of smooth molecules for* $\dot{\mathbf{F}}_p^{\alpha q}$, *then*

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}\leqslant c\,\|\{s_Q\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

Proof. By Lemma 2.1, we can write $m_P=\sum_Q\langle m_P, \varphi_Q\rangle\psi_Q$. If A is the operator on $\dot{\mathbf{f}}_p^{\alpha q}$ with matrix $a_{QP}=\langle m_P, \varphi_Q\rangle$, and $s=\{s_Q\}_Q$, then

$$T_\psi As=\sum_Q\sum_P a_{QP}s_P\psi_Q=\sum_P s_P\sum_Q\langle m_P, \varphi_Q\rangle\psi_Q=\sum_P s_Pm_P=f.$$

Lemma 3.6 below will show that the operator A is almost diagonal. Then, by Theorem 3.3, A is bounded, and by Theorem 2.2, T_ψ is also bounded. This yields the conclusion. ∎

LEMMA 3.6. *If $\{m_Q\}_Q$ is a family of smooth molecules for $\dot{\mathbf{F}}_p^{\alpha q}$, then the operator A on $\dot{\mathbf{f}}_p^{\alpha q}$ with matrix $a_{QP} = \langle m_P, \varphi_Q \rangle$ is almost diagonal.*

The proof is completely elementary, but quite tedious; it is given in Appendix B.

Since $\{\psi_Q\}_Q$ is a family of smooth molecules, Theorem 3.5 is a generalization of the result in Theorem 2.2 that $T_\psi : \dot{\mathbf{f}}_p^{\alpha q} \to \dot{\mathbf{F}}_p^{\alpha q}$ is bounded. Similarly, replacing the φ_Q's by more general functions $\{b_Q\}_Q$, Theorem 3.3 also leads to a generalization of the boundedness of $S_\varphi : \dot{\mathbf{F}}_p^{\alpha q} \to \dot{\mathbf{f}}_p^{\alpha q}$.

Let J be defined as above, and let $(J-\alpha)^* = J - \alpha - [J-\alpha]$. In what follows, we consider a sequence $\{b_Q\}_Q$ of functions such that for some ρ and M with $(J-\alpha^*) < \rho \leqslant 1$, and $M > J$,

$$\int x^\gamma b_Q(x)\,dx = 0 \qquad \text{if} \quad |\gamma| \leqslant [\alpha], \tag{3.7}$$

$$|b_Q(x)| \leqslant |Q|^{-1/2} (1 + l(Q)^{-1} |x - x_Q|)^{-\max(M, M+n+\alpha-J)}, \tag{3.8}$$

$$|\partial^\gamma b_Q(x)| \leqslant |Q|^{-1/2 - |\gamma|/n} (1 + l(Q)^{-1} |x - x_Q|)^{-M} \tag{3.9}$$

if $|\gamma| \leqslant N$, and

$$\begin{aligned} &|\partial^\gamma b_Q(x) - \partial^\gamma b_Q(y)| \\ &\quad \leqslant |Q|^{-1/2 - |\gamma|/n - \rho/n} |x-y|^\rho \sup_{|z| \leqslant |y-x|} (1 + l(Q)^{-1} |x - z - x_Q|)^{-M} \end{aligned} \tag{3.10}$$

if $|\gamma| = N$. (Note the inversion of the roles of N and $[\alpha]$ here as compared to (3.3)–(3.6).

We can now state the dual analogue of Theorem 3.5.

THEOREM 3.7. *If $f \in \dot{\mathbf{F}}_p^{\alpha q}$ and $\{b_Q\}_Q$ is a family of functions satisfying* (3.7)–(3.10), *then*

$$\|\{\langle f, b_Q \rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c\, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}.$$

For this we need another elementary lemma, analogous to Lemma 3.6; the proof is in Appendix B.

LEMMA 3.8. *If $\{b_Q\}_Q$ is a family of functions satisfying* (3.7)–(3.10) *then the operator A on $\dot{\mathbf{f}}_p^{\alpha q}$ with matrix $a_{QP} = \langle \psi_P, b_Q \rangle$ is almost diagonal.*

Proof of Theorem 3.7. By Theorem 2.2,

$$\langle f, b_Q\rangle = \sum_P s_P \langle \psi_P, b_Q\rangle,$$

where $s = \{s_P\}_P$ satisfies $\|s\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|f\|_{\dot{F}_p^{\alpha q}}$. Set $a_{QP} = \langle \psi_P, b_Q\rangle$; we then have $\langle f, b_Q\rangle = \sum_P a_{QP} s_P = (As)_Q$, where A is the operator associated with the matrix $\{a_{QP}\}_{Q,P}$. According to Lemma 3.8, A is almost diagonal on $\dot{f}_p^{\alpha q}$, and, by Theorem 3.3, we have the estimate $\|As\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|s\|_{\dot{f}_p^{\alpha q}}$. This completes the proof. ∎

Remark 3.9. Note that since $\varphi_Q \in \mathscr{S}$ and $0 \notin \operatorname{supp} \hat{\varphi}_Q$, $\langle f, \varphi_Q\rangle$ is well defined for $f \in \mathscr{S}'/\mathscr{P}$. However, under weaker assumptions on b_Q, care must be taken in interpreting the expression $\langle f, b_Q\rangle$. For a discussion of this technical point, we refer to Remark B.4 in Appendix B.

Remark 3.10. In Remark 2.6 we saw that $\dot{F}_p^{\alpha q}$ is independent of choice φ satisfying (2.1)–(2.3). More generally, suppose b satisfies

$$\int x^\gamma b(x)\, dx = 0 \qquad \text{if} \quad |\gamma| \leqslant [\alpha], \tag{3.11}$$

$$|b(x)| \leqslant (1 + |x|)^{-\max(M, M+n+\alpha-J)}, \tag{3.12}$$

$$|\partial^\gamma b(x)| \leqslant (1 + |x|)^{-M} \tag{3.13}$$

if $|\gamma| \leqslant N$, and

$$|\partial^\gamma b(x) - \partial^\gamma b(y)| \leqslant |x-y|^\rho \sup_{|z| \leqslant |y-x|} (1 + |x-z|)^{-M} \tag{3.14}$$

if $|\gamma| = N$. Let $b_\nu(x) = 2^{\nu n} b(2^\nu x)$, for $\nu \in \mathbb{Z}$. Let $\{x^Q\}_Q$ be any sequence of points with $x^Q \in Q$ for each dyadic Q. Let

$$b_Q(x) = |Q|^{-1/2}\, b(2^\nu(x - x^Q)) = |Q|^{1/2}\, b_\nu(x - x^Q),$$

if $l(Q) = 2^{-\nu}$. Since $|x^Q - x_Q| \leqslant \sqrt{n}\, l(Q)$, the b_Q's satisfy (3.7)–(3.10) up to a constant factor. Note that $\langle f, b_Q\rangle = |Q|^{1/2}\, \tilde{b}_\nu * f(x^Q)$, where $\tilde{b}_\nu(x) = \overline{b_\nu(-x)}$. Theorem 3.7 yields the estimate

$$\left\| \left(\sum_{\nu \in \mathbb{Z}} \sum_{l(Q) = 2^{-\nu}} (2^{\nu\alpha}\, |\tilde{b}_\nu * f(x^Q)|\, \chi_Q)^q \right)^{1/q} \right\|_{L^p} \leqslant c\, \|f\|_{\dot{F}_p^{\alpha q}},$$

with c independent of the selection of the points $x^Q \in Q$. Setting $\sup_{Q,b}(f) = \sup_{x \in Q} |Q|^{1/2}\, |\tilde{b}_\nu * f(x)|$ if $l(Q) = 2^{-\nu}$, we have

$$\|\{\sup_{Q,b}(f)\}_Q\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|f\|_{\dot{F}_p^{\alpha q}} \tag{3.15}$$

(cf. Lemma 2.5). In particular, we have

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}(b)} \equiv \left\| \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha} |\tilde{b}_\nu * f|)^q \right)^{1/q} \right\|_{L^p} \leqslant c \, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}, \tag{3.16}$$

for any b satisfying (3.11)–(3.14). In other words, replacing φ by any reasonable kernel (one satisfying (3.11)–(3.14)) in the definition of $\dot{\mathbf{F}}_p^{\alpha q}$ yields a "norm" dominated by the $\dot{\mathbf{F}}_p^{\alpha q}$-norm.

The converse to (3.15)–(3.16) will, in general, require some non-degeneracy condition. We state one such result, taking b in $\mathscr{S}$, although this condition can certainly be weakened. However, the following is sufficient to allow us to take b of compact support, which is sometimes useful. The method of proof is derived from [P3, Chap. 8], and the references given there. Let $AV_b(f)$ be the sequence defined for Q dyadic by

$$AV_{Q,b}(f) = |Q|^{-1/2} \int_Q |\tilde{b}_\nu * f(y)| \, dy.$$

PROPOSITION 3.11. *Suppose* $b \in \mathscr{S}$ *satisfies* (3.11), *and*

$$|\hat{b}(\xi)| \geqslant c > 0 \qquad \text{if} \quad \tfrac{3}{5} \leqslant |\xi| \leqslant \tfrac{5}{3}.$$

(i) *If* $\alpha \in \mathbb{R}$, $0 < p < +\infty$ *and* $0 < q \leqslant +\infty$, *then*

$$\|AV_b(f)\|_{\dot{\mathbf{f}}_p^{\alpha q}} \approx \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}.$$

(ii) *If* $\alpha \in \mathbb{R}$, $1 < p < +\infty$ *and* $1 < q \leqslant +\infty$, *then*

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}(b)} \approx \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}.$$

Proof. From (3.15)–(3.16) and the fact that $AV_{Q,b}(f) \leqslant \sup_{Q,b}(f)$, we see that the right-hand side of (i) and (ii) dominates the left. For the other direction, our assumptions guarantee that there exists $\eta \in \mathscr{S}$ such that $\varphi = \eta * \tilde{b}$. For f and b fixed, we let $t = \{t_Q\}_Q = \{AV_{Q,b}(f)\}_Q$ as defined earlier. We write out the convolution $\varphi_\nu * f = \eta_\nu * \tilde{b}_\nu * f$, break the integral up over cubes of sidelength $2^{-\nu} = l(Q)$, and use the rapid decay of η. We obtain

$$\sup_Q (f) \leqslant c_L (t^*_{1,L})_Q \leqslant c_\lambda (t^*_{r,\lambda})_Q$$

for L sufficiently large, $r = \min(p, q)$, and $\lambda > n$, using Hölder's inequality if $r > 1$ and (2.9) if not. Hence, by Lemma 2.3,

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c \, \|\sup(f)\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c \, \|t^*_{r,\lambda}\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c \, \|t\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

This yields (i). To complete (ii), observe that

$$\sum_{l(Q)=2^{-\nu}} AV_{Q,b}(f)\chi_Q \leqslant M(\tilde{b}_\nu * f).$$

Inserting this in the definition of $\|t\|_{\dot{f}_p^{\alpha q}}$ and using the vector-valued maximal inequality, Theorem A.1, yields (ii). ∎

4. The Generalized φ-Transform

Under the assumptions (2.1)–(2.4), we obtained in Theorem 2.2 the representation $f = \sum_Q \langle f, \varphi_Q \rangle \psi_Q$ with the estimate $\|\{\langle f, \varphi_Q \rangle\}\|_{\dot{f}_p^{\alpha q}} \leqslant c\,\|f\|_{\dot{F}_p^{\alpha q}}$, as well as the estimate $\|\sum_Q s_Q \psi_Q\|_{\dot{F}_p^{\alpha q}} \leqslant c\,\|s\|_{\dot{f}_p^{\alpha q}}$ for any sequence $s = \{s_Q\}_Q$. In this section we consider other possibilities for representing f and obtaining one or the other of these estimates under less restrictive conditions on the functions involved. For example, it is sometimes convenient to be able to represent f as a sum of functions of compact support, as in the traditional Hardy space atomic decomposition.

We say that $\{a_Q\}_Q$ is a family of *smooth atoms for* $\dot{\mathbf{F}}_p^{\alpha q}$ if there exist $\tilde{K}$ and $\tilde{N}$ with $\tilde{K} \geqslant [\alpha + 1]_+$ and $\tilde{N} \geqslant N$ (for N as above) such that for each dyadic cube Q,

$$\operatorname{supp} a_Q \subset 3Q, \tag{4.1}$$

$$\int x^\gamma a_Q(x)\, dx = 0 \qquad \text{if} \quad |\gamma| \leqslant \tilde{N}, \tag{4.2}$$

and

$$|\partial^\gamma a_Q(x)| \leqslant |Q|^{-1/2 - |\gamma|/n} \qquad \text{if} \quad |\gamma| \leqslant \tilde{K}. \tag{4.3}$$

When emphasis is required, we call a function a_Q satisfying (4.1)–(4.3) a $(\tilde{\mathbf{K}}, \tilde{\mathbf{N}})$-*smooth atom*. Note that a smooth atom is also a smooth molecule. The following result appears in [Fr-J2, Theorem II]; here we present another proof, which illustrates that this result can be considered a consequence of Theorem 2.2.

THEOREM 4.1. *Let* $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, *and* $0 < p < +\infty$. *For each* $f \in \dot{\mathbf{F}}_p^{\alpha q}$, *there exists a family of smooth atoms* $\{a_Q\}_Q$ *and a sequence of coefficients* $s = \{s_Q\}_Q$ *such that* $f = \sum_Q s_Q a_Q$ *(in* $\mathcal{S}'/\mathcal{P}$*), and* $\|s\|_{\dot{f}_p^{\alpha q}} \leqslant c\,\|f\|_{\dot{F}_p^{\alpha q}}$. *Conversely,* $\|\sum_Q s_Q a_Q\|_{\dot{F}_p^{\alpha q}} \leqslant c\,\|s\|_{\dot{f}_p^{\alpha q}}$ *for any family of smooth atoms* $\{a_Q\}_Q$.

Proof. Pick φ and ψ satisfying (2.1)–(2.4). By Theorem 2.2, we can write $f = \sum_Q t_Q \psi_Q$, where $t = \{t_Q\}_Q = \{\langle f, \varphi_Q \rangle\}_Q$ satisfies

$\|t\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|f\|_{\dot{F}_p^{\alpha q}}$. Select $\theta \in \mathscr{S}$ satisfying $\operatorname{supp} \theta \subseteq \{x \in \mathbb{R}^n : |x| \leqslant 1\}$, $\int x^\gamma \theta(x)\, dx = 0$ if $|\gamma| \leqslant \tilde{N}$, and $|\hat{\theta}(\xi)| \geqslant c > 0$ if $\frac{1}{2} \leqslant |\xi| \leqslant 2$ (see [Fr-J1, p. 783], for a construction of θ). By (2.2), $\hat{\psi}/\hat{\theta} \in \mathscr{S}$, so $\psi = \theta * \eta$ for some $\eta \in \mathscr{S}$. Setting $g_k(x) = \int_{Q_{0k}} \theta(x-y)\, \eta(y)\, dy$ for $k \in \mathbb{Z}^n$, we have $\psi = \sum_{k \in \mathbb{Z}^n} g_k$, and, hence, for $\nu \in \mathbb{Z}$ and $l \in \mathbb{Z}^n$,

$$\psi_{Q_{\nu l}}(x) = |Q_{\nu l}|^{-1/2} \sum_{k \in \mathbb{Z}^n} g_k(2^\nu x - l).$$

Note that $\operatorname{supp} g_k \subseteq 3Q_{0k}$, $\int x^\gamma g_k(x)\, dx = 0$ if $|\gamma| \leqslant \tilde{N}$, and $|\partial^\gamma g_k(x)| \leqslant c_{M,\gamma}(1+|k|)^{-M}$ for any $M > 0$. For $Q = Q_{\nu k}$, set $s_Q = C(t_r^*)_Q$ and

$$a_Q(x) = |Q|^{-1/2} \sum_{l \in \mathbb{Z}^n} t_{Q_{\nu l}}\, g_{k-l}(2^\nu x - l)/s_Q,$$

where $r = \min(p, q)$ and C will be determined shortly. From the representations above of f and ψ_Q, we have $f = \sum_Q s_Q a_Q$. Since $\operatorname{supp} g_{k-l}(2^\nu x - l) \subseteq 3Q_{\nu k}$, (4.1) holds. Clearly (4.2) holds. Letting M be greater than $J = n/\min(1, r)$, the estimate for $\partial^\gamma g_l$ above yields

$$\begin{aligned} |\partial^\gamma a_Q(x)| &\leqslant c\, |Q|^{-1/2 - |\gamma|/n} \sum_{l(P) = l(Q)} |t_P|\, (1 + l(Q)^{-1}\, |x_P - x_Q|)^{-M}/C(t_r^*)_Q \\ &\leqslant c\, |Q|^{-1/2 - |\gamma|/n}/C, \end{aligned}$$

by Hölder's inequality if $r > 1$, or by the imbedding $l^r \to l^1$ if $r \leqslant 1$. Taking C large enough yields (4.3). Finally, by Lemma 2.3,

$$\|s\|_{\dot{f}_p^{\alpha q}} = C\, \|t_r^*\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|t\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|f\|_{\dot{F}_p^{\alpha q}}.$$

The converse follows from Theorem 3.5. ∎

Notice that the proof in fact shows that we may take the a_Q's in $\mathscr{D}$.

Although the smooth atomic decomposition in Theorem 4.1 is useful in applications (see, e.g., Sections 11, 13 below), it suffers from two disadvantages. First, the functions $\{a_Q\}_Q$ are not canonical, in the sense that different a_Q's appear in the representations of different distributions. This is unlike the case of Theorem 2.2, where there is one fixed ψ_Q for each cube Q, for all f. Second, the coefficients $\{s_Q\}_Q$ in Theorem 4.1 are not determined linearly by f, as they are in the case of Theorem 2.2.

We will consider below families of distributions $\{\sigma^Q\}_Q$, which may be used to represent distributions in $\dot{F}_p^{\alpha q}$, and families of distributions $\{\tau^Q\}_Q$, which linearly determine coefficients of the form $\langle f, \tau^Q \rangle$ in the representation of f. In this section, we reserve the subindex notation σ_Q (and τ_Q) for the case where there exists a function σ (or τ) such that $\sigma_Q(x) = |Q|^{-1/2}\, \sigma(2^\nu x - k)$ (similarly τ_Q) if $Q = Q_{\nu k}$. We say that a family

$\{\sigma^Q\}_Q$ of distributions *represents* $\dot{\mathbf{F}}_p^{\alpha q}$, if there exists a family $\{\tau^Q\}_Q$ such that

(i) for all $f \in \dot{\mathbf{F}}_p^{\alpha q}$, we have

$$f = \sum_Q \langle f, \tau^Q \rangle \sigma^Q, \tag{4.4}$$

with

$$\| \{ \langle f, \tau^Q \rangle \}_Q \|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c \, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}, \tag{4.5}$$

and

(ii) for any sequence $s = \{s_Q\}_Q$, we have

$$\left\| \sum_Q s_Q \sigma^Q \right\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c \, \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}. \tag{4.6}$$

If, instead, $\{\tau^Q\}_Q$ is a sequence of distributions such that there exists a family $\{\sigma^Q\}_Q$ so that (i) and (ii) hold, we say that $\{\tau^Q\}_Q$ *norms* $\dot{\mathbf{F}}_p^{\alpha q}$. In either case, we call the map sending f to $\{\langle f, \tau^Q \rangle\}_Q$ the *generalized φ-transform*, and the map sending $\{s_Q\}_Q$ to $\sum_Q s_Q \sigma^Q$ the *generalized inverse φ-transform.*

We single out the following result (similarly Theorem 4.4) to see that the conditions on $\{\sigma^Q\}_Q$ required to obtain (i) and (ii) above are slightly different. Recall the definitions of N and J above. Throughout the remainder of this section, $\{x^Q\}_Q$ represents any sequence of points satisfying $x^Q \in Q$ for each dyadic cube Q.

THEOREM 4.2. *Let* $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, *and* $0 < p < +\infty$. *Suppose* δ *satisfies* $\alpha^* = \alpha - [\alpha] < \delta \leqslant 1$, *and suppose* $M > J$. *Let* u *be a function satisfying*

$$|\hat{u}(\xi)| \geqslant c > 0 \qquad \textit{if} \quad \tfrac{1}{2} \leqslant |\xi| \leqslant 2, \tag{4.7}$$

$$\int x^\gamma u(x)\, dx = 0 \qquad \textit{if} \quad |\gamma| \leqslant N - 1, \tag{4.8}$$

$$|u(x) - u(y)| \leqslant |x-y|^\delta \sup_{|z| \leqslant |y-x|} (1 + |x-z|)^{-\max(M, M-\alpha)}, \tag{4.9}$$

$$|\partial^\gamma u(x)| \leqslant c_\gamma (1 + |x|)^{-M} \qquad \textit{if} \quad |\gamma| \leqslant [\alpha + 1]_+, \tag{4.10}$$

and

$$|\partial^\gamma u(x) - \partial^\gamma u(y)| \leqslant |x-y|^\delta \sup_{|z| \leqslant |y-x|} (1 + |x-z|)^{-M} \tag{4.11}$$

if $|\gamma| = [\alpha + 1]_+$. *For* $\mu \in \mathbb{Z}$, *let* $u_\mu(x) = 2^{\mu n} u(2^\mu x)$, *and set*

$$\sigma^Q(x) = |Q|^{-1/2} u_\mu(2^\nu(x - x^Q)) \tag{4.12}$$

for $Q = Q_{\nu k}$. *Then there is a* $\mu_0 \leqslant 0$ *with the property that for each* $\mu \leqslant \mu_0$, *there exists a family of functions* $\{\tau^Q\}_Q$ *such that* $\{\sigma^Q\}_Q$ *and* $\{\tau^Q\}_Q$ *satisfy* (4.4)–(4.5).

Proof. By (4.7), there exists φ satisfying (2.1)–(2.3) such that

$$\sum_{\nu \in \mathbb{Z}} \hat{u}(2^\nu \xi)\, \overline{\hat{\varphi}(2^\nu \xi)} = 1 \qquad \text{if} \quad \xi \neq 0. \tag{4.13}$$

For $\mu \in \mathbb{Z}$, define

$$\eta(x) = \int_{Q_{00}} \varphi_\mu(x - y)\, dy \tag{4.14}$$

and

$$\eta_Q(x) = |Q|^{-1/2}\, \eta(2^\nu x - k) \qquad \text{if} \quad Q = Q_{\nu k}. \tag{4.15}$$

Note that

$$\eta \in \mathscr{S}, \qquad \int x^\gamma \eta(x)\, dx = 0 \tag{4.16}$$

and

$$|\partial^\gamma \eta(x)| \leqslant c_{M,\gamma} 2^{\mu(n + |\gamma|)} (1 + 2^\mu |x|)^{-M}, \tag{4.17}$$

for all multi-indices γ. Define the operator T_μ on $\dot{\mathbf{F}}_p^{\alpha q}$ by

$$\begin{aligned} T_\mu f(x) &= \sum_Q \langle f, \eta_Q \rangle\, \sigma^Q(x) \\ &= \sum_{\beta \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^n} \int_{Q_{\beta l}} \tilde{\varphi}_{\beta + \mu} * f(y)\, dy u_{\beta + \mu}(x - x^{Q_{\beta l}}), \end{aligned}$$

where $\tilde{\varphi}_\beta(x) = \overline{\varphi_\beta(-x)}$, as usual.

We claim that

$$\|(\mathrm{I} - T_\mu) f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c\, 2^{\mu\delta}\, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}, \tag{4.18}$$

for $\mu \leqslant 0$, where I is the identity operator. To see this, note that by the choice of φ,

$$\begin{aligned} f(x) &= \sum_{\beta \in \mathbb{Z}} \tilde{\varphi}_{\beta + \mu} * u_{\beta + \mu} * f(x) \\ &= \sum_{\beta \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^n} \int_{Q_{\beta l}} \tilde{\varphi}_{\beta + \mu} * f(y)\, u_{\beta + \mu}(x - y)\, dy. \end{aligned}$$

Hence, replacing β by $\nu-\mu$ and collecting terms, we can write

$$(\mathrm{I}-T_\mu)\, f(x)=\sum_{\nu\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\sum_{l:Q_{\nu-\mu,l}\subseteq Q_{\nu k}}\int_{Q_{\nu-\mu,l}}\tilde{\varphi}_\nu * f(y)\\ \times(u_\nu(x-y)-u_\nu(x-x^{Q_{\nu-\mu,l}}))\,dy.$$

Let $s_{Q_{\nu k}}=|Q_{\nu k}|^{-1/2}\int_{Q_{\nu k}}|\tilde{\varphi}_\nu * f(y)|\,dy$ and

$$m_{Q_{\nu k}}(x)=\frac{2^{-\mu\delta}}{Cs_{Q_{\nu k}}}\sum_{l:Q_{\nu-\mu,l}\subseteq Q_{\nu k}}\int_{Q_{\nu-\mu,l}}\tilde{\varphi}_\nu * f(y)\\ \times(u_\nu(x-y)-u_\nu(x-x^{Q_{\nu-\mu,l}}))\,dy$$

for C sufficiently large, to be chosen later. By (4.8), each m_Q satisfies (3.3). If $|\gamma|\leqslant[\alpha]$ and $y\in Q_{\nu-\mu,l}\subseteq Q_{\nu k}$, (4.10) yields

$$\begin{aligned}
&|\partial_x^\gamma(u_\nu(x-y)-u_\nu(x-x^{Q_{\nu-\mu,l}}))|\\
&\quad=c2^{\nu(|\gamma|+1)}2^{\nu n}2^{-(\nu-\mu)}\sup_{z\in Q_{\nu-\mu,l}}|\nabla_x\,\partial_x^\gamma u(2^\nu x-2^\nu z)|\\
&\quad\leqslant c2^\mu 2^{\nu(n+|\gamma|)}\sup_{z\in Q_{\nu k}}(1+2^\nu|x-z|)^{-M}\\
&\quad\leqslant c2^\mu|Q_{\nu k}|^{-1-|\gamma|/n}(1+2^\nu|x-x_{Q_{\nu k}}|)^{-M}.
\end{aligned}\tag{4.19}$$

Hence for $|\gamma|\leqslant[\alpha]$ and C large enough,

$$\begin{aligned}
|\partial^\gamma m_{Q_{\nu k}}(x)|&\leqslant c\frac{2^{\mu(1-\delta)}}{s_{Q_{\nu k}}}|Q_{\nu k}|^{-1-|\gamma|/n}(1+2^\nu|x-x_{Q_{\nu k}}|)^{-M}\\
&\quad\times\sum_{l:Q_{\nu-\mu,l}\subseteq Q_{\nu k}}\int_{Q_{\nu-\mu,l}}|\tilde{\varphi}_\nu * f(y)|\,dy\\
&\leqslant|Q_{\nu k}|^{-1/2-|\gamma|/n}(1+2^\nu|x-x_{Q_{\nu k}}|)^{-M},
\end{aligned}$$

since $\delta\leqslant 1$ and $\mu<0$. Thus, (3.5) holds. Similarly, (4.9) yields (3.4). Also, applying the mean value theorem to each pair of closest points, followed by (4.10)–(4.11), leads to (3.6). Thus, $\{m_Q\}_Q$ is a family of smooth molecules. Since $(\mathrm{I}-T_\mu)f=C2^{\mu\delta}\sum_Q s_Q m_Q$, Theorem 3.5 yields

$$\|(\mathrm{I}-T_\mu)f\|_{\dot{F}_p^{\alpha q}}\leqslant c2^{\mu\delta}\,\|\{s_Q\}_Q\|_{\dot{f}_p^{\alpha q}}.$$

Noting, however, that $|s_Q|\leqslant|Q|^{1/2}\sup_{y\in Q}|\tilde{\varphi}_\nu * f(y)|=\sup_Q(f)$, Lemma 2.5 yields (4.18).

Therefore, there exists $\mu_0 \leqslant 0$ such that for $\mu \leqslant \mu_0$, $\|I - T_\mu\| < 1$, so T_μ is invertible. Hence for $f \in \dot{\mathbf{F}}_p^{\alpha q}$,

$$f = T_\mu T_\mu^{-1} f = \sum_Q \langle T_\mu^{-1} f, \eta_Q \rangle \sigma^Q.$$

Thus (4.4) holds with $\tau^Q = (T_\mu^{-1})^* \eta_Q$. To obtain (4.5), note that if $\eta_{-\mu}(x) = 2^{-\mu n} \eta(2^{-\mu} x)$, then $\eta_{-\mu}$ satisfies (3.11), and, up to a constant, (3.12)–(3.14). For any $g \in \dot{\mathbf{F}}_p^{\alpha q}$, write

$$\|\{\langle g, \eta_Q \rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} = \left\| \left(\sum_{v \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} (2^{v\alpha} |g * \tilde{\eta}_v(2^{-v}k)| \chi_{Q_{vk}})^q \right)^{1/q} \right\|_{L^p}.$$

Replacing v by $v - \mu$ and noting that for $\mu \leqslant 0$,

$$\sum_{Q_{v-\mu,k} \subseteq Q_{v,l}} |g * \tilde{\eta}_{v-\mu}(2^{-v+\mu}k)| \chi_{Q_{v-\mu,k}} \leqslant \sup_{x \in Q_{vl}} |g * (\tilde{\eta}_{-\mu})_v (x)| \chi_{Q_{vl}},$$

we obtain

$$\|\{\langle g, \eta_Q \rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant 2^{-\mu\alpha} \|\{\sup\nolimits_{Q,\eta_{-\mu}}(g)\}\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c 2^{-\mu\alpha} \|g\|_{\dot{\mathbf{F}}_p^{\alpha q}},$$

by (3.15). Hence, we have

$$\begin{aligned} \|\{\langle f, \tau^Q \rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} &= \|\{\langle T_\mu^{-1} f, \eta_Q \rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \\ &\leqslant c 2^{-\mu\alpha} \|T_\mu^{-1} f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c 2^{-\mu\alpha} \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}, \end{aligned}$$

yielding (4.5). ∎

COROLLARY 4.3. *Suppose u is a function satisfying* (4.7),

$$\int x^\gamma u(x)\, dx = 0 \qquad \text{if} \quad |\gamma| \leqslant N, \tag{4.8'}$$

and (4.9)–(4.11), *and define $\{\sigma^Q\}_Q$ by* (4.12). *Then there exists $\mu_0 \leqslant 0$ such that for all $\mu \leqslant \mu_0$, the family $\{\sigma^Q\}_Q$ represents $\dot{\mathbf{F}}_p^{\alpha q}$.*

Proof. By Theorem 4.2, only (4.6) requires proof. However, note that for $|\gamma| \leqslant [\alpha + 1]_+$ and $Q = Q_{vk}$,

$$\begin{aligned} |\partial^\gamma \sigma^Q(x)| &= |Q|^{-1/2 - |\gamma|/n}\, 2^{\mu(n + |\gamma|)}\, |(\partial^\gamma u)(2^{\mu + v}(x - x^Q))| \\ &\leqslant c 2^{\mu(n + |\gamma|)} |Q|^{-1/2 - |\gamma|/n} (1 + 2^{\mu + v} |x - x^Q|)^{-M} \\ &\leqslant c 2^{-\mu(M - n)} |Q|^{-1/2 - |\gamma|/n} (1 + 2^v |x - x^Q|)^{-M}, \end{aligned}$$

which yields (3.5) for σ^Q. Similar calculations yield (3.4) and (3.6). Since, clearly, $\int x^\gamma \sigma^Q(x)\,dx = 0$ if $|\gamma| \leqslant N$, $\{\sigma^Q\}_Q$ is, up to a factor of $c2^{-\mu(M-n)}$, a family of smooth molecules. By Theorem 3.5, then,

$$\left\| \sum_Q s_Q \sigma^Q \right\|_{\dot{F}_p^{\alpha q}} \leqslant c2^{-\mu(M-n)} \, \|\{s_Q\}_Q\|_{\dot{f}_p^{\alpha q}}. \quad \blacksquare$$

In particular, it is possible to select $u \in \mathscr{S}$ with $\operatorname{supp} u \subseteq \{x: |x| \leqslant 1\}$ satisfying (4.7), (4.8′), and (4.9)–(4.11). Then $\operatorname{supp} u_\mu \subseteq \{x: |x| \leqslant 2^{-\mu}\}$ and, hence, for σ^Q as in Corollary 4.3, $\operatorname{supp} \sigma^Q \subseteq (2^{-\mu}+1)Q$, $\int x^\gamma \sigma^Q(x)\,dx = 0$ if $|\gamma| \leqslant N$, and $|\partial^\gamma \sigma^Q(x)| \leqslant c2^{\mu(n+|\gamma|)}\,|Q|^{-1/2-|\gamma|/n}$. Thus, for μ sufficiently small, we obtain a linear, canonical version of the smooth atomic decomposition in Theorem 4.1, except for the fact that the constant in (4.1) is 3.

The following result is dual to Theorem 4.2 and has an analogous proof.

THEOREM 4.4. *Let $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$ and $0 < p < +\infty$. Suppose ρ satisfies $(J-\alpha)^* < \rho \leqslant 1$ and suppose $M > J$. Let u be a function satisfying* (4.7),

$$\int x^\gamma u(x)\,dx = 0 \qquad \textit{if} \quad |\gamma| \leqslant [\alpha - 1], \tag{4.20}$$

$$|u(x) - u(y)| \leqslant |x-y|^\rho \sup_{|z| \leqslant |y-x|} (1 + |x-z|)^{-\max(M, M+n+\alpha-J)}, \tag{4.21}$$

$$|\partial^\gamma u(x)| \leqslant c_\gamma (1+|x|)^{-M} \qquad \textit{if} \quad |\gamma| \leqslant N+1, \tag{4.22}$$

and

$$|\partial^\gamma u(x) - \partial^\gamma u(y)| \leqslant |x-y|^\rho \sup_{|z| \leqslant |y-x|} (1+|x-z|)^{-M} \tag{4.23}$$

if $|\gamma| = N+1$. Set

$$\tau^Q(x) = |Q|^{-1/2}\, u_\mu(2^\nu(x - x^Q)) \tag{4.24}$$

for $Q = Q_{\nu k}$. Then there is a $\mu_0 \leqslant 0$ with the property that for each $\mu \leqslant \mu_0$, there exists a family of functions $\{\sigma^Q\}_Q$ such that $\{\sigma^Q\}_Q$ and $\{\tau^Q\}_Q$ satisfy (4.4) *and* (4.6).

Proof. As in the proof of Theorem 4.2, (4.7) implies the existence of φ satisfying (2.1)–(2.3) such that (4.13) holds. For $\mu \in \mathbb{Z}$, define η by (4.14), thereby obtaining (4.16)–(4.17). Define the operator T_μ on $\dot{F}_p^{\alpha q}$ by

$$\begin{aligned} T_\mu f(x) &= \sum_Q \langle f, \tau^Q \rangle \eta_Q \\ &= \sum_{\beta \in \mathbb{Z}} \sum_{l \in \mathbb{Z}^n} \int_{Q_{\beta l}} \tilde{u}_{\beta+\mu} * f(x^{Q_{\beta l}})\, \varphi_{\beta+\mu}(x-y)\,dy. \end{aligned}$$

Taking the complex conjugate of (4.13) yields

$$f(x)=\sum_{\beta\in\mathbb{Z}}\tilde{u}_{\beta+\mu}*\varphi_{\beta+\mu}*f(x)$$

$$=\sum_{\beta\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^n}\int_{Q_{\beta l}}\tilde{u}_{\beta+\mu}*f(y)\,\varphi_{\beta+\mu}(x-y)\,dy.$$

Hence, replacing β by $v-\mu$, for $\mu\leqslant 0$, similarly to the proof of Theorem 4.2, we obtain

$$(\mathrm{I}-T_\mu)f(x)=\sum_{v\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}\sum_{l:Q_{v-\mu,l}\subseteq Q_{vk}}$$
$$\times\int_{Q_{v-\mu,l}}(\tilde{u}_v*f(y)-\tilde{u}_v*f(x^{Q_{v-\mu,l}}))\,\varphi_v(x-y)\,dy.$$

Let

$$s_{Q_{v,k}}=|Q_{vk}|^{-1/2}\sum_{l:Q_{v-\mu,l}\subseteq Q_{vk}}\int_{Q_{v-\mu,l}}|\tilde{u}_v*f(y)-\tilde{u}_v*f(x^{Q_{v-\mu,l}})|\,dy,$$

and

$$m_{Q_{vk}}(x)=\frac{1}{Cs_{Q_{vk}}}\sum_{l:Q_{v-\mu,l}\subseteq Q_{vk}}\int_{Q_{v-\mu,l}}(\tilde{u}_v*f(y)$$
$$-\tilde{u}_v*f(x^{Q_{v-\mu,l}}))\,\varphi_v(x-y)\,dy,$$

for C a constant to be chosen later. By (2.2), m_Q satisfies (3.3). If $|\gamma|\leqslant[\alpha+1]_+$ and $y\in Q_{v-\mu,l}\subseteq Q_{vk}$,

$$|\partial^\gamma\varphi_v(x-y)|\leqslant c_\gamma 2^{v(n+|\gamma|)}(1+2^v\,|x-x_{Q_{vk}}|)^{-M},$$

which easily yields (3.5) and (3.6) with $\delta=1$, if C is chosen large enough. Similarly, (3.4) follows. Therefore $\{m_Q\}_Q$ is a family of smooth molecules. Since $(\mathrm{I}-T_\mu)f=C\sum_Q s_Q m_Q$, Theorem 3.5 yields

$$\|(\mathrm{I}-T_\mu)f\|_{\dot{\mathbf{F}}_p^{\alpha q}}\leqslant c\,\|\{s_Q\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}}. \tag{4.25}$$

By Theorem 2.2, we can write $f=\sum_P t_P\psi_P$, where $t=\{t_P\}_P$ satisfies $\|t\|_{\dot{\mathbf{f}}_p^{\alpha q}}\leqslant c\,\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}$. If we set

$$a_{QP}=|Q_{vk}|^{-1/2}\sum_{l:Q_{v-\mu,l}\subseteq Q_{vk}}\int_{Q_{v-\mu,l}}|\psi_P*\tilde{u}_v(y)-\psi_P*\tilde{u}_v(x^{Q_{v-\mu,l}})|\,dy,$$

then $|s_Q| \leqslant \sum_P a_{QP} |t_P|$. We will obtain the estimate

$$a_{QP} \leqslant c 2^{\mu\rho} \omega_{QP}(\varepsilon), \tag{4.26}$$

for some $\varepsilon > 0$, with c independent of Q, P, and μ. Then (4.25) and Theorem 3.3 imply that

$$\|(\mathrm{I} - T_\mu) f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c \, \|\{s_Q\}\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c 2^{\mu\rho} \, \|t\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c 2^{\mu\rho} \, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}. \tag{4.27}$$

To prove (4.26), fix ν, k, and l with $Q_{\nu-\mu,l} \subseteq Q_{\nu k}$, and fix $y \in Q_{\nu-\mu,l}$. Let $h(x) = u_\nu(x-y) - u_\nu(x - x^{Q_{\nu-\mu,l}})$. Then

$$\psi_P * \tilde{u}_\nu(y) - \psi_p * \tilde{u}_\nu(x^{Q_{\nu-\mu,l}}) = \langle \psi_P, h \rangle. \tag{4.28}$$

By (4.20), h satisfies $\int x^\gamma h(x)\, dx = 0$ if $|\gamma| \leqslant [\alpha]$. Also, as in (4.19) above, (4.21), (4.22), and (4.23), respectively, imply that

$$|Q_{\nu k}|^{1/2} \, |h(x)| \leqslant C 2^{\mu\rho} \, |Q_{\nu k}|^{-1/2} \, (1 + 2^\nu \, |x - x_{Q_{\nu k}}|)^{-M},$$
$$|Q_{\nu k}|^{1/2} \, |\partial^\gamma h(x)| \leqslant C 2^{\mu} \, |Q_{\nu k}|^{-1/2 - |\gamma|/n} \, (1 + 2^\nu \, |x - x_{Q_{\nu k}}|)^{-M},$$

if $|\gamma| \leqslant N$, and

$$\begin{aligned} &|Q_{\nu k}|^{1/2} \, |\partial^\gamma h(x) - \partial^\gamma h(y)| \\ &\quad \leqslant C 2^{\mu\rho} \, |Q_{\nu k}|^{-1/2 - |\gamma|/n - \rho/n} \, |x - y|^\rho \sup_{|z| \leqslant |y-x|} (1 + |x - z - x_{Q_{\nu k}}|)^{-M} \end{aligned}$$

if $|\gamma| = N$, with C independent of our fixed quantities. Therefore $2^{-\mu\rho} \, |Q_{\nu k}|^{1/2} \, h/C$ satisfies the assumptions of Corollary B.3; so

$$|Q_{\nu k}|^{1/2} \, |\langle \psi_P, h \rangle| \leqslant c 2^{\mu\rho} \omega_{QP}(\varepsilon),$$

for some $\varepsilon > 0$ and c again independent of our choices. Replacing this and (4.28) in the definition of a_{QP} easily yields (4.26).

By (4.27), then, there exists $\mu_0 \leqslant 0$ such that for all $\mu \leqslant \mu_0$, we have $\|\mathrm{I} - T_\mu\| < 1$, so that T_μ is invertible. Therefore, for $f \in \dot{\mathbf{F}}_p^{\alpha q}$,

$$f = T_\mu^{-1} T_\mu f = T_\mu^{-1} \sum_Q \langle f, \tau^Q \rangle \eta_Q = \sum_Q \langle f, \tau^Q \rangle \sigma^Q,$$

if $\sigma^Q = T_\mu^{-1} \eta_Q$. So we have (4.4).

To obtain (4.6), let $s = \{s_Q\}_Q \in \dot{\mathbf{f}}_p^{\alpha q}$. Then

$$\Big\| \sum_Q s_Q \sigma^Q \Big\|_{\dot{\mathbf{F}}_p^{\alpha q}} = \Big\| T_\mu^{-1} \sum_Q s_Q \eta_Q \Big\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c \, \Big\| \sum_Q s_Q \eta^Q \Big\|_{\dot{\mathbf{F}}_p^{\alpha q}},$$

by the boundedness of T_μ^{-1}. However, (4.16)–(4.17) show that, up to a factor of $c2^{-\mu(M-n)}$ (as in the proof of Corollary 4.3), $\{n_Q\}_Q$ is a family of smooth molecules. Hence, Theorem 3.5 gives

$$\left\|\sum_Q s_Q \sigma^Q\right\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c2^{-\mu(M-n)} \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}. \quad \blacksquare$$

Corollary 4.5. *Suppose u is a function satisfying* (4.7),

$$\int x^\gamma u(x)\,dx = 0 \qquad \textit{if} \quad |\gamma| \leqslant [\alpha], \tag{4.20'}$$

(4.21)–(4.23), *and*

$$|u(x)| \leqslant c(1+|x|)^{-\max(M, M+n+\alpha-J)}. \tag{4.29}$$

Define $\{\tau^Q\}_Q$ by (4.24). *Then there exists $\mu_0 \leqslant 0$ such that for all $\mu \leqslant \mu_0$, the family $\{\tau^Q\}_Q$ norms $\dot{\mathbf{F}}_p^{\alpha q}$.*

Proof. By Theorem 4.4, only (4.5) requires proof. Note that

$$\|\{\langle f, \tau^Q\rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} = \left\|\left(\sum_{\nu\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} (2^{\nu\alpha}\, |f * u_{\mu+\nu}(x^{Q_{\nu k}})|\, \chi_{Q_{\nu k}})^q\right)^{1/q}\right\|_{L^p}.$$

Replacing ν by $\nu-\mu$ and noting that for $\mu \leqslant 0$,

$$\sum_{k : Q_{\nu-\mu,k} \subseteq Q_{\nu l}} |f * \tilde{u}_\nu(x^{Q_{\nu-\mu,k}})|\, \chi_{Q_{\nu-\mu,k}} \leqslant \sup_{x\in Q_{\nu l}} |f * \tilde{u}_\nu(x)|\, \chi_{Q_{\nu l}},$$

we obtain

$$\|\{\langle f, \tau^Q\rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant 2^{-\mu\alpha}\, \|\{\sup_{Q,\tilde{u}}(f)\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

However, u satisfies the conditions (3.11)–(3.14). Therefore (3.15) gives

$$\|\{\langle f, \tau^Q\rangle\}_Q\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c2^{-\mu\alpha}\, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}. \quad \blacksquare$$

As in the remark following Corollary 4.3, it is possible to obtain u of compact support satisfying the assumptions of Corollary 4.5. Thus we obtain a linear, canonical decomposition in which the coefficients are "locally" determined. For remarks regarding the dilation factor 2^μ in these results see [Fr-J2, Section 4].

We have seen that in our decomposition (4.4), we can take one of the two families, i.e., either $\{\sigma^Q\}_Q$ or $\{\tau^Q\}_Q$, to consist of translates and dilates of a fixed nice function. In doing so, explicit information on the other family seems to be lost (we have either $\tau^Q = (T_\mu^{-1})^* \eta_Q$ or $\sigma^Q = (T_\mu)^{-1} \eta_Q$ for η_Q defined by (4.15)). In Remark 9.17, however, we will obtain certain specific information regarding the second family. In particular, in the case $\alpha = 0$, $1 \leqslant p, q \leqslant +\infty$, the second family can be taken to be a family of smooth molecules for $\dot{\mathbf{F}}_p^{0q}$.

5. The Case $p = +\infty$

If we replace L^p by L^∞ in the definition of $\dot{\mathbf{F}}_p^{\alpha q}$, we do not obtain a satisfactory definition of $\dot{\mathbf{F}}_\infty^{\alpha q}$ (unless $q = +\infty$, in which case $\dot{\mathbf{F}}_\infty^{\alpha\infty} = \dot{\mathbf{B}}_\infty^{\alpha\infty}$). Triebel remarks [Tr2, p. 46] that this norm is no longer independent of the choice of the function φ. Furthermore, we should certainly have $\dot{\mathbf{F}}_\infty^{02} \approx (\dot{\mathbf{F}}_1^{02})^* \approx (H^1)^* \approx \mathrm{BMO}$, but, as pointed out in [P4], this fails for the naive definition of $\dot{\mathbf{F}}_\infty^{\alpha q}$. A result in [Ch-W-W] states that if the square function Sf of f (similar to the expression $(\sum_{\nu \in \mathbb{Z}} |\varphi_\nu * f|^2)^{1/2}$) is bounded, then f is locally square exponentially integrable. However, a BMO function is, in general, only locally exponentially integrable.

To define $\dot{\mathbf{F}}_\infty^{\alpha q}$ as a natural extension of the scale of $\dot{\mathbf{F}}_p^{\alpha q}$ spaces, $0 < p < +\infty$, it becomes clear from [Fef-S2, pp. 148–149; Fr-J1, Section 4; Ja-T1] that the norm should be localized appropriately. For $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, and φ satisfying (2.1)–(2.3), we define $\dot{\mathbf{F}}_\infty^{\alpha q}$ to be the set of all $f \in \mathscr{S}'/\mathscr{P}$ such that

$$\|f\|_{\dot{\mathbf{F}}_\infty^{\alpha q}} = \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \sum_{\nu = -\log_2 l(P)}^{\infty} (2^{\nu\alpha} |\varphi_\nu * f(x)|)^q \, dx \right)^{1/q} < +\infty. \tag{5.1}$$

We will show that this definition is independent of the choice of φ satisfying (2.1)–(2.3), and that

$$(\dot{\mathbf{F}}_1^{\alpha q})^* \approx \dot{\mathbf{F}}_\infty^{-\alpha q'} \tag{5.2}$$

for $\alpha \in \mathbb{R}$, $1 \leqslant q < +\infty$, and $1/q + 1/q' = 1$. We will also show that there exists an operator $A^{\alpha q}$ such that $\|A^{\alpha q} f\|_{L_p} \approx \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}$ for $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant +\infty$. These facts indicate that our definition of $\dot{\mathbf{F}}_\infty^{\alpha q}$ is appropriate. Later we will see that interpolation with $\dot{\mathbf{F}}_\infty^{\alpha q}$ as an endpoint space behaves as it should. We will also obtain analogues of the results in Sections 3–4 for the case $p = +\infty$.

We note that in [Tr2, p. 239], Triebel gives a definition of $\dot{\mathbf{F}}_\infty^{\alpha q}$ for $1 < q < +\infty$ that yields (5.2), almost by definition, but this definition is not effectively computable in the way that (5.1) is. Obviously, however, by (5.2), the two definitions agree.

We make some elementary remarks regarding the definition (5.1). First, an equivalent norm is obtained if we take the sup with respect to all cubes with sides parallel to the axes, since every such cube is contained in the union of at most 2^n dyadic cubes with side lengths at most double the original. Second, using this, if $k \geqslant 0$ we obtain

$$\sup_{P \text{ dyadic}} \frac{1}{|P|} \int_P \sum_{v=-\log_2 l(P)-k}^{\infty} (2^{v\alpha} |\varphi_v * f(x)|)^q \, dx \leqslant c 2^{kn} \|f\|_{\dot{\mathbf{F}}_\infty^{\alpha q}}^q, \quad (5.3)$$

since $|2^k P| = 2^{kn} |P|$. Finally, we note that our definition is "localized" in the sense that if we took the sum in (5.1) over all $v \in \mathbb{Z}$, the result would be equivalent to $\|(\sum_{v \in \mathbb{Z}} (2^{v\alpha} |\varphi_v * f|)^q)^{1/q}\|_{L^\infty}$, the naive but incorrect definition of the $\dot{\mathbf{F}}_\infty^{\alpha q}$-norm.

We define $\dot{\mathbf{f}}_\infty^{\alpha q}$, the sequence space corresponding to $\dot{\mathbf{F}}_\infty^{\alpha q}$, to be the set of all sequences $s = \{s_Q\}_{Q \text{ dyadic}}$ such that

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha q}} = \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \, \tilde{\chi}_Q(x))^q \, dx \right)^{1/q} < +\infty. \quad (5.4)$$

For $0 < q < +\infty$, we can carry out the integration in (5.4) to obtain that

$$\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha q}} = \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \sum_{Q \subset P} (|Q|^{-\alpha/n - 1/2} |s_Q|)^q \, |Q| \right)^{1/q}; \quad (5.5)$$

i.e., $\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha q}}^q$ is equivalent to the Carleson norm of the measure

$$\sum_Q (|Q|^{-\alpha/n - 1/2} |s_Q|)^q \, |Q| \, \delta_{(x_Q, l(Q))},$$

where $\delta_{(x,t)}$ is the point mass at $(x, t) \in \mathbb{R}_+^{n+1}$.

To prove the analogue of Theorem 2.2 for $p = +\infty$, we first require the following analogue of Lemma 2.3.

Lemma 5.1. *Suppose* $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, *and* $\lambda > n$. *Then*

$$\|s_q^*\|_{\dot{\mathbf{f}}_\infty^{\alpha q}} \approx \|s\|_{\dot{\mathbf{f}}_\infty^{\alpha q}}.$$

Proof. One direction is trivial, since $|s_Q| \leqslant (s_q^*)_Q$ for all Q. The other direction is not very much harder, given Lemma 2.3. Let us fix a dyadic cube P. Let $r_Q = s_Q$ if $Q \subseteq 3P$ and $r_Q = 0$ otherwise, and let $t_Q = s_Q - r_Q$. With $r = \{r_Q\}_Q$ and $t = \{t_Q\}_Q$, we then have $(s_q^*)_Q^q = (r_q^*)_Q^q + (t_q^*)_Q^q$ for each Q.

By Lemma 2.3,

$$\frac{1}{|P|}\int_P \sum_{Q\subset P} (|Q|^{-\alpha/n}(r_q^*)_Q\,\tilde{\chi}_Q(x))^q\,dx$$

$$\leqslant \frac{1}{|P|}\int \sum_Q (|Q|^{-\alpha/n}(r_q^*)_Q\,\tilde{\chi}_Q(x))^q\,dx$$

$$=\frac{1}{|P|}\|r_q^*\|^q_{\dot{\mathrm{f}}_q^{\alpha q}} \leqslant \frac{c}{|P|}\|r\|^q_{\dot{\mathrm{f}}_q^{\alpha q}}$$

$$=\frac{c}{|P|}\int_{3P}\sum_{Q\subset 3P}(|Q|^{-\alpha/n}|s_Q|\,\tilde{\chi}_Q(x))^q\,dx \leqslant c\,\|s\|^q_{\dot{\mathrm{f}}_\infty^{\alpha q}}.$$

On the other hand, given P and Q with $Q\subset P$, suppose $\tilde{Q}$ is dyadic with $l(Q)=l(\tilde{Q})=2^{-k}l(P)$ and $\tilde{Q}\subset P+jl(P)\nsubseteq 3P$ for some $j\in\mathbb{Z}^n$; then $1+l(\tilde{Q})^{-1}|x_Q-x_{\tilde{Q}}|\approx 2^k|j|$. Hence, using (5.5),

$$\frac{1}{|P|}\int_P\sum_{Q\subset P}(|Q|^{-\alpha/n}(t_q^*)_Q\,\tilde{\chi}_Q(x))^q\,dx$$

$$=\frac{1}{|P|}\sum_{Q\subset P}\sum_{l(\tilde{Q})=l(Q)}|Q|\,(|Q|^{-\alpha/n-1/2}|t_{\tilde{Q}}|)^q/(1+l(\tilde{Q})^{-1}|x_Q-x_{\tilde{Q}}|)^\lambda$$

$$\leqslant c\sum_{\substack{j\in\mathbb{Z}^n\\|j|\geqslant 2}}|j|^{-\lambda}\sum_{k=0}^{\infty}2^{k(n-\lambda)}\frac{1}{|P|}\sum_{\substack{\tilde{Q}\subset P+jl(P)\\ l(\tilde{Q})=2^{-k}l(P)}}|\tilde{Q}|$$

$$\times(|\tilde{Q}|^{-\alpha/n-1/2}|t_{\tilde{Q}}|)^q$$

$$\leqslant c\,\|t\|^q_{\dot{\mathrm{f}}_\infty^{\alpha q}}\leqslant c\,\|s\|^q_{\dot{\mathrm{f}}_\infty^{\alpha q}},$$

since $\lambda>n$. This yields the result. ▮

THEOREM 5.2. *Let $\alpha\in\mathbb{R}$ and $0<q\leqslant+\infty$. Then $S_\varphi:\dot{F}_\infty^{\alpha q}\to\dot{\mathrm{f}}_\infty^{\alpha q}$ and $T_\psi:\dot{\mathrm{f}}_\infty^{\alpha q}\to\dot{F}_\infty^{\alpha q}$ are bounded operators. Also, $T_\psi\circ S_\varphi$ is the identity on $\dot{F}_\infty^{\alpha q}$.*

Proof. The estimates

$$\|f\|_{\dot{F}_\infty^{\alpha q}}\approx\|\sup(f)\|_{\dot{\mathrm{f}}_\infty^{\alpha q}}\approx\|\inf_\gamma(f)\|_{\dot{\mathrm{f}}_\infty^{\alpha q}} \tag{5.6}$$

are obtained essentially as in Lemma 2.5, except for the occasional use of (5.3). Similarly, the proof of Theorem 2.2 carries over without significant change, using Lemma 5.1 instead of Lemma 2.3. ▮

COROLLARY 5.3. *The definition of $\dot{F}_\infty^{\alpha q}$ is independent of the choice of φ satisfying* (2.1)–(2.3).

Proof. See Remark 2.6. ▮

It is worth noting at this point that the following analogue of Proposition 2.7 is trivial for $p = +\infty$.

PROPOSITION 5.4. *Let $\varepsilon > 0$. Suppose that for each dyadic cube Q there is a set $E_Q \subseteq Q$ with $|E_Q|/|Q| > \varepsilon$. Then*

$$\|\{s_Q\}_Q\|_{\dot{f}_\infty^{\alpha q}} \approx \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q}(x))^q \, dx \right)^{1/q}.$$

Proof. Immediate by (5.5). ∎

We now consider an operator $m^{\alpha q}$ on sequences which will be useful in proving (5.2) and which will lead us to the operator $A^{\alpha q}$ promised above. First, for a sequence $s = \{s_Q\}_{Q \text{dyadic}}$, we define

$$G^{\alpha q}(s)(x) = \left(\sum_P (|P|^{-\alpha/n} |s_P| \tilde{\chi}_P(x))^q \right)^{1/q}$$

and

$$G_Q^{\alpha q}(s)(x) = \left(\sum_{P \subset Q} (|P|^{-\alpha/n} |s_P| \tilde{\chi}_P(x))^q \right)^{1/q}.$$

We let $m_Q^{\alpha q}(s)$ denote the "$\frac{1}{4}$-median" of $G_Q^{\alpha q}(s)$ on Q, i.e.,

$$m_Q^{\alpha q}(s) = \inf\{\varepsilon : |\{x \in Q : G_Q^{\alpha q}(s)(x) > \varepsilon\}| < |Q|/4\}. \tag{5.7}$$

We also set

$$m^{\alpha q}(s)(x) = \sup_Q m_Q^{\alpha q}(s) \chi_Q(x).$$

PROPOSITION 5.5. *Let $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant +\infty$. Then*

$$\|s\|_{\dot{f}_p^{\alpha q}} \approx \|m^{\alpha q}(s)\|_{L^p}.$$

Proof. We observe that

$$\{x : m^{\alpha q}(s)(x) > t\} \subset \{x : M(\chi_{\{y : G^{\alpha q}(s)(y) > t\}})(x) \geqslant \tfrac{1}{4}\}.$$

Since M is of weak-type (1, 1), we obtain

$$|\{x : m^{\alpha q}(s)(x) > t\}| \leqslant c \, |\{x : G^{\alpha q}(x) > t\}|$$

for $t > 0$, and, hence,

$$\|m^{\alpha q}(s)\|_{L^p} \leqslant c \, \|G^{\alpha q}(s)\|_{L^p} = c \, \|s\|_{\dot{f}_p^{\alpha q}},$$

for $0<p<+\infty$. When $p=+\infty$, we use Chebyshev's inequality to see that

$$|\{x \in Q\colon G_Q^{\alpha q}(x)>\varepsilon\}| \leqslant \frac{1}{\varepsilon^q}\int_Q (G_Q^{\alpha q}(x))^q\,dx \leqslant \frac{|Q|}{\varepsilon^q}\,\|s\|_{\dot{\mathfrak{f}}_\infty^{\alpha q}}^q < \frac{1}{4}\,|Q|, \tag{5.8}$$

if $\varepsilon > 4^{1/q}\,\|s\|_{\dot{\mathfrak{f}}_\infty^{\alpha q}}$. Hence, $\|m^{\alpha q}(s)\|_{L^\infty} \leqslant c\,\|s\|_{\dot{\mathfrak{f}}_\infty^{\alpha q}}$.

The converse inequalities are deeper. Using a discrete version of the argument in [Fef-S2], we define the extended integer-valued stopping time $v(x)$, for $x \in \mathbb{R}^n$, by

$$v(x)=\inf\left\{v \in \mathbb{Z}\colon \left(\sum_{l(Q)\leqslant 2^{-v}} (|Q|^{-\alpha/n}\,|s_Q|\,\tilde{\chi}_Q(x))^q\right)^{1/q} \leqslant m^{\alpha q}(s)(x)\right\}. \tag{5.9}$$

Also, set

$$E_Q=\{x \in Q\colon 2^{-v(x)} \geqslant l(Q)\}=\{x \in Q\colon G_Q^{\alpha q}(s)(x) \leqslant m^{\alpha q}(s)(x)\},$$

for each Q. By (5.7), $|E_Q|/|Q| \geqslant \frac{3}{4}$, and

$$\left(\sum_Q (|Q|^{-\alpha/n}\,|s_Q|\,\tilde{\chi}_{E_Q}(x))^q\right)^{1/q} \leqslant c m^{\alpha q}(s)(x), \tag{5.10}$$

for each $x \in R^n$. By Proposition 2.7, then, for $0<p<+\infty$, $\|s\|_{\dot{\mathfrak{f}}_p^{\alpha q}} \leqslant \|m^{\alpha q}(s)\|_{L^p}$. Similarly, (5.10) and Proposition 5.4 yield $\|s\|_{\dot{\mathfrak{f}}_\infty^{\alpha q}} \leqslant c\,\|m^{\alpha q}(s)\|_{L^\infty}$. ∎

Notice that this proposition and its proof provide us with another equivalent definition of $\dot{\mathfrak{f}}_p^{\alpha q}$ for all $0<p\leqslant+\infty$.

COROLLARY 5.6. *Let $\alpha \in \mathbb{R}$ and $0<p, q\leqslant+\infty$. Then $s=\{s_Q\}_Q \in \dot{\mathfrak{f}}_p^{\alpha q}$ if and only if for each Q there is a subset $E_Q \subset Q$ with $|E_Q|/|Q|>\frac{1}{2}$ (or any other, fixed, number $0<\varepsilon<1$) such that*

$$\left\|\left(\sum_Q (|Q|^{-\alpha/n}\,|s_Q|\,\tilde{\chi}_{E_Q})^q\right)^{1/q}\right\|_{L^p} < +\infty. \tag{5.11}$$

Moreover, the infimum of this expression over all such collections $\{E_Q\}_Q$ is equivalent to $\|s\|_{\dot{\mathfrak{f}}_p^{\alpha q}}$.

Proof. For $p<+\infty$ this follows at once from Proposition 2.7. If $p=+\infty$ and $s \in \dot{\mathfrak{f}}_\infty^{\alpha q}$, the E_Q's chosen in the proof of Proposition 5.5 above yield (5.11). The converse follows from Proposition 5.4. ∎

Corollary 5.6 is in a natural way the limiting case $r=+\infty$ of the following.

COROLLARY 5.7. *Let $\alpha \in \mathbb{R}$ and $0 < q \leqslant +\infty$. For each $0 < r < +\infty$,*

$$\sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \left(\sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q(x))^q \right)^{r/q} dx \right)^{1/r} \approx \|s\|_{\dot{f}_\infty^{\alpha q}}. \quad (5.12)$$

Proof. Let us first consider the case $r \geqslant q$. Then, by Hölder's inequality, the right-hand side of (5.12) is dominated by the left. On the other hand, if P is fixed cube and E_Q are the subsets given by Corollary 5.6, then by Proposition 2.7 we have

$$\frac{1}{|P|} \int_P \left(\sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q(x))^q \right)^{r/q} dx$$
$$\leqslant \frac{c}{|P|} \int_P \left(\sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q}(x))^q \right)^{r/q} dx.$$

Now this is clearly less than

$$c \left\| \left(\sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q}(x))^q \right)^{1/q} \right\|_{L^\infty}^r,$$

and by Corollary 5.6 this can be estimated by $c \|s\|_{\dot{f}_\infty^{\alpha q}}^r$.

If $r < q$, then Hölder's inequality shows that the left-hand side of (5.12) is dominated by the expression on the right. To prove the converse inequality we can repeat the argument in the proof of Proposition 5.5 involving Chebyshev's inequality; the only difference is that we need to replace q by r in (5.8). It follows that $\|m^{\alpha q}(s)\|_{L^\infty}$ can be estimated by the left-hand side of (5.12) and this completes the proof. Alternately, let $H = \sum_{Q \subset P} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q})^q$. By Corollary 5.6,

$$H \leqslant c H^{r/q} \|s\|_{\dot{f}_\infty^{\alpha q}}^{q-r}.$$

This readily yields the result when $s \in \dot{f}_\infty^{\alpha q}$, and the general case follows by using the monotone convergence theorem. ∎

Corollary 5.7 is an analogue of the John–Nirenberg lemma (cf. [Joh-N]) on the sequence space level.

For $f \in \mathcal{S}'/\mathcal{P}$, we define

$$A^{\alpha q} f = m^{\alpha q}(S_\varphi f).$$

We remark that $A^{\alpha q}$ is an analogue of the local square function whose study goes back to Fefferman and Stein [Fef-S2] and Strömberg [Strö1].

COROLLARY 5.8. *Let $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant +\infty$. Then*

$$\|A^{\alpha q} f\|_{L^p} \approx \|f\|_{\dot{F}_p^{\alpha q}}.$$

Proof. By Proposition 5.5, Theorem 2.2, and Theorem 5.2,

$$\|m^{\alpha q}(S_\varphi f)\|_{L^p} \approx \|S_\varphi f\|_{\dot{f}_p^{\alpha q}} \approx \|f\|_{\dot{F}_p^{\alpha q}}. \quad \blacksquare$$

We remark that we could use the operator

$$B^{\alpha q} f(x) = \sup_{Q: x \in Q} \inf \left\{ \varepsilon : \left| \left\{ y \in Q : \left(\sum_{\nu = -\log_2 l(Q)}^{\infty} (2^{\nu\alpha} |\varphi_\nu * f(y)|)^q \right)^{1/q} > \varepsilon \right\} \right| < \delta\, |Q| \right\},$$

for δ sufficiently small, in place of $A^{\alpha q} f$ in Corollary 5.8; $B^{\alpha q} f$ is the supremum of the δ-medians of the truncated Littlewood–Paley function. However, owing to the quantity γ in Lemma 2.5, the proof of the equivalence $\|B^{\alpha q} f\|_{L^p} \approx \|f\|_{\dot{F}_p^{\alpha q}}$ involves a few technicalities, which we omit.

Next we shall consider duality. Let q' denote the conjugate of q, so that $1/q + 1/q' = 1$ when $1 \leqslant q \leqslant +\infty$; if $0 < q \leqslant 1$ it is also convenient to let $q' = +\infty$.

THEOREM 5.9. *Suppose $\alpha \in \mathbb{R}$ and $0 < q < +\infty$. Then $(\dot{f}_1^{\alpha q})^* \approx \dot{f}_\infty^{-\alpha q'}$. In particular, if $t = \{t_Q\}_Q \in \dot{f}_\infty^{-\alpha q'}$, then the map $s = \{s_Q\}_Q \to \langle s, t \rangle \equiv \sum_Q s_Q \bar{t}_Q$ defines a continuous linear functional on $\dot{f}_1^{\alpha q}$ with operator norm $\|t\|_{(\dot{f}_1^{\alpha q})^*}$ equivalent to $\|t\|_{\dot{f}_\infty^{-\alpha q'}}$, and every $l \in (\dot{f}_1^{\alpha q})^*$ is of this form for some $t \in \dot{f}_\infty^{-\alpha q'}$.*

Proof. Suppose first that $1 \leqslant q < +\infty$. Similarly to the proof of Proposition 5.5, let

$$E_Q = \{x \in Q : G_Q^{-\alpha q'}(t)(x) \leqslant m^{-\alpha q'}(t)(x)\},$$

for each dyadic Q. Then $|E_Q|/|Q| \geqslant \frac{3}{4}$, so

$$\begin{aligned}
\left| \sum_Q s_Q \bar{t}_Q \right| &\leqslant c \int \sum_Q |Q|^{-\alpha/n} |s_Q|\, \tilde{\chi}_Q\, |Q|^{\alpha/n} |t_Q|\, \tilde{\chi}_{E_Q} \\
&\leqslant c \int \left(\sum_Q (|Q|^{-\alpha/n} |s_Q|\, \tilde{\chi}_Q(x))^q \right)^{1/q} \\
&\quad \times \left(\sum_Q (|Q|^{\alpha/n} |t_Q|\, \tilde{\chi}_{E_Q}(x))^{q'} \right)^{1/q'} dx \\
&\leqslant c\, \|s\|_{\dot{f}_1^{\alpha q}} \|m^{-\alpha q'}(t)\|_{L^\infty} \leqslant c\, \|s\|_{\dot{f}_1^{\alpha q}} \|t\|_{\dot{f}_\infty^{-\alpha q'}},
\end{aligned}$$

where we have used the analogue of (5.10) for t, $-\alpha$, and q', and the conclusion of Proposition 5.5. This yields $\|t\|_{(\dot{\mathbf{f}}_1^{\alpha q})^*} \leqslant c\,\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}}$ if $1 \leqslant q < +\infty$. The case $0 < q < 1$ then follows from the trivial imbedding $\dot{\mathbf{f}}_1^{\alpha q} \to \dot{\mathbf{f}}_1^{\alpha 1}$.

The converse is elementary: Clearly every $l \in (\dot{\mathbf{f}}_1^{\alpha q})^*$ is of the form $s \to \sum_Q s_Q \bar{t}_Q$ for some $t = \{t_Q\}_Q$. Now fix P dyadic and assume first that $1 \leqslant q < +\infty$. Let X be the sequence space of all dyadic cubes Q such that $Q \subseteq P$, and let μ be a measure on X such that the μ-measure of the "point" Q is $|Q|/|P|$. Then, with (5.5) in mind,

$$\begin{aligned}
&\left(\frac{1}{|P|} \sum_{Q \subset P} (|Q|^{\alpha/n - 1/2}\, |t_Q|)^{q'}\, |Q|\right)^{1/q'} \\
&\quad = \|\{|Q|^{\alpha/n - 1/2}\, t_Q\}_Q\|_{l^{q'}(X, d\mu)} \\
&\quad = \sup_{|s|_{l^q(X, d\mu)} \leqslant 1} \left|\frac{1}{|P|} \sum_{Q \subset P} s_Q\, |Q|^{\alpha/n - 1/2}\, \bar{t}_Q\, |Q|\right| \\
&\quad \leqslant \|t\|_{(\dot{\mathbf{f}}_1^{\alpha q})^*} \sup_{|s|_{l^q(X, d\mu)} \leqslant 1} \|\{s_Q\, |Q|^{\alpha/n + 1/2}/|P|\}_Q\|_{\dot{\mathbf{f}}_1^{\alpha q}}.
\end{aligned}$$

However, by Hölder's inequality,

$$\begin{aligned}
&\|\{s_Q\, |Q|^{\alpha/n + 1/2}/|P|\}_Q\|_{\dot{\mathbf{f}}_1^{\alpha q}} \\
&\quad = \frac{1}{|P|} \int_P \left(\sum_{Q \subset P} (|s_Q|\, \chi_Q)^q\right)^{1/q} \\
&\quad \leqslant \left(\frac{1}{|P|} \int_P \sum_{Q \subset P} (|s_Q|\, \chi_Q)^q\right)^{1/q} = \|s\|_{l^q(X, d\mu)} \leqslant 1.
\end{aligned}$$

Hence $\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}} \leqslant \|t\|_{(\dot{\mathbf{f}}_1^{\alpha q})^*}$ if $1 \leqslant q < +\infty$. For $0 < q < 1$ we have $q' = +\infty$ and the extremal sequence s in the above has only one non-zero element. The argument then simply reduces to the following. Given a dyadic cube R, we set $(s^R)_Q = |R|^{\alpha/n - 1/2}$ for $Q = R$ and 0 otherwise. Clearly, $\|s^R\|_{\dot{\mathbf{f}}_1^{\alpha q}} = 1$ and, hence,

$$\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha\infty}} \leqslant \sup_R |\langle s^R, t\rangle| \leqslant \|t\|_{(\dot{\mathbf{f}}_1^{\alpha q})^*}. \quad \blacksquare$$

Remark 5.10. We can modify the first half of the proof of Theorem 5.9 as follows. Let $0 < q \leqslant +\infty$ and set

$$\begin{aligned}
\tilde{v}(x) = \inf\Bigg\{ v \in \mathbb{Z} :& \left(\sum_{l(Q) \leqslant 2^{-v}} (|Q|^{-\alpha/n}\, |s_Q|\, \tilde{\chi}_Q(x))^q\right)^{1/q} \\
&\times \left(\sum_{l(Q) \leqslant 2^{-v}} (|Q|^{\alpha/n}\, |t_Q|\, \tilde{\chi}_Q(x))^{q'}\right)^{1/q'} \leqslant m^{\alpha q}(s)(x)\, m^{-\alpha q'}(t)(x) \Bigg\}.
\end{aligned}$$

Let

$$\begin{aligned}\tilde{E}_Q &= \{x \in Q: 2^{-\nu(x)} \geqslant l(Q)\}\\ &= \{x \in Q: G^{\alpha q}(s)(x)\, G^{-\alpha q'}(t)(x) \leqslant m^{\alpha q}(s)(x)\, m^{-\alpha q'}(t)(x)\}.\end{aligned}$$

Since we have taken the $\frac{1}{4}$-medians, we have $|\tilde{E}_Q| > |Q|/2$. If $1 \leqslant q < +\infty$, Hölder's inequality and (5.10) yield

$$\left|\sum_Q s_Q \bar{t}_Q\right| \leqslant c \int m^{\alpha q}(s)(x)\, m^{-\alpha q'}(t)(x)\, dx;$$

if $0 < q < 1$ this estimate still holds, since $m^{\alpha 1}(s)(x) \leqslant m^{\alpha q}(s)(x)$.

Remark 5.11. The dual of $\dot{\mathbf{f}}_p^{\alpha q}$ when $p \neq 1$ (and $p \neq +\infty$) is more elementary to characterize. If $1 < p < +\infty$ and $0 < q < +\infty$ we have

$$(\dot{\mathbf{f}}_p^{\alpha q})^* \approx \dot{\mathbf{f}}_{p'}^{-\alpha q'}. \tag{5.13}$$

Let us outline a proof of this: we have

$$\begin{aligned}\left|\sum_Q s_Q \bar{t}_Q\right| &\leqslant \left|\int \sum_Q |Q|^{-\alpha/n} s_Q \tilde{\chi}_Q\, |Q|^{\alpha/n} \bar{t}_Q \tilde{\chi}_Q\right|\\ &\leqslant \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}} \|t\|_{\dot{\mathbf{f}}_{p'}^{-\alpha q'}}.\end{aligned}$$

For $q \geqslant 1$ this follows by applying Hölder's inequality twice, while for $0 < q < 1$, we use this with $q = 1$ and the imbedding $\dot{\mathbf{f}}_p^{\alpha q} \to \dot{\mathbf{f}}_p^{\alpha 1}$. Conversely, every $l \in (\dot{\mathbf{f}}_p^{\alpha q})^*$ is of the form $l(s) = \sum_Q s_Q \bar{t}_Q$ for some sequence $t = \{t_Q\}_Q$. Now, we shall take for granted the result that $(L^p(l^q))^* = L^{p'}(l^{q'})$ if $1 < p < +\infty$ and $0 < q < +\infty$, where

$$L^p(l^q) = \left\{f = \{f_\nu\}: \|f\|_{L^p(l^q)} \equiv \left\|\left(\sum_{\nu \in \mathbb{Z}} |f_\nu|^q\right)^{1/q}\right\|_{L^p} < +\infty\right\},$$

with the obvious pairing, namely $f \to \int \sum_{\nu \in \mathbb{Z}} f_\nu \bar{g}_\nu$ for $g = \{g_\nu\}_\nu \in L^{p'}(l^{q'})$ (for this fact see, e.g., [Tr2, p. 177]). Note that the map $\mathrm{In}: \dot{\mathbf{f}}_p^{\alpha q} \to L^p(l^q)$ defined by $\mathrm{In}(s) = \{f_\nu(s)\}_{\nu \in \mathbb{Z}}$, where $f_\nu(s) = \sum_{l(Q) = 2^{-\nu}} |Q|^{-\alpha/n} s_Q \tilde{\chi}_Q$ is a linear isometry onto a subspace of $L^p(l^q)$. By the Hahn–Banach theorem, there exists $\tilde{l} \in (L^p(l^q))^*$ with $\|\tilde{l}\| = \|l\|$ such that $\tilde{l} \circ \mathrm{In} = l$. In other words, there exists $g = \{g_\nu\}_\nu \in L^{p'}(l^{q'})$ with $\|g\|_{L^{p'}(l^{q'})} \leqslant \|l\|$ such that

$$\sum_Q s_Q \bar{t}_Q = \int \sum_{\nu \in \mathbb{Z}} f_\nu(s)\, \bar{g}_\nu,$$

for all $s \in \dot{\mathbf{f}}_p^{\alpha q}$. By taking $s_Q = 0$ for all but one cube, we see that $t_Q = \int_Q g_\nu / |Q|^{\alpha/n + 1/2}$ for any dyadic cube Q with $l(Q) = 2^{-\nu}$. Hence, estimating the average over a cube by the Hardy–Littlewood maximal operator and using the vector-valued maximal inequality (Theorem A.1),

$$\|t\|_{\dot{\mathbf{f}}_{p'}^{-\alpha q'}} \leqslant \|\{Mg_\nu\}_{\nu \in \mathbb{Z}}\|_{L^{p'}(l^{q'})} \leqslant c\, \|g\|_{L^{p'}(l^{q'})} \leqslant c\, \|l\|.$$

This completes the proof of (5.13). Notice that we may restate this last step in the proof by saying that the operator $\Pr(\{g_\nu\}_\nu) = \{t_Q\}_Q = \{\int_Q g_\nu / |Q|^{\alpha/n + 1/2} \colon l(Q) = 2^{-\nu}\}_{Q\,\text{dyadic}}$ is bounded from $L^{p'}(l^{q'})$ to $\dot{\mathbf{f}}_{p'}^{-\alpha q'}$. Since clearly $\Pr \circ \mathrm{In} = \text{identity}$, we have, in particular, that $\dot{\mathbf{f}}_p^{\alpha q}$ is a retract of $L^p(l^q)$ as long as $1 < p < +\infty$, $1 < q \leqslant +\infty$.

It is not difficult to treat the remaining cases. We have

$$(\dot{\mathbf{f}}_p^{\alpha q})^* \approx \dot{\mathbf{f}}_\infty^{\beta \infty} \tag{5.14}$$

for $0 < p < 1$ and $0 < q < +\infty$, where $\beta = \beta(\alpha, p) = -\alpha + n(1/p - 1)$. As in [Fr-J2, Theorem 3.8; Ja3], we have the imbedding $\dot{\mathbf{f}}_p^{\alpha q} \to \dot{\mathbf{f}}_1^{-\beta 1}$ $(0 < p < 1)$. Using this and the duality $(\dot{\mathbf{f}}_1^{\beta 1})^* = \dot{\mathbf{f}}_\infty^{-\beta \infty}$ yields one direction of (5.14). The other is similar to the case $p = 1$ and $0 < q < 1$ in the proof of Theorem 5.9.

The spaces $\dot{\mathbf{f}}_1^{\alpha q}$ are not reflexive, but, similar to the situation for l^1 and l^∞, finite sequences in $\dot{\mathbf{f}}_\infty^{-\alpha q'}$ norm $\dot{\mathbf{f}}_1^{\alpha q}$ in the following sense:

COROLLARY 5.12. *Suppose $\alpha \in \mathbb{R}$ and $1 \leqslant q < +\infty$. If $s = \{s_Q\}_Q \in \dot{\mathbf{f}}_1^{\alpha q}$, then $\|s\|_{\dot{\mathbf{f}}_1^{\alpha q}} \approx \sup\{|\sum_Q s_Q \bar{t}_Q| \colon t$ finite with $\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}} \leqslant 1\}$.*

Proof. Theorem 5.9 shows that $\|s\|_{\dot{\mathbf{f}}_1^{\alpha q}}$ dominates the supremum above. Except for the restriction that t be finite, the converse follows from the Hahn–Banach theorem. Approximating s (in $\dot{\mathbf{f}}_1^{\alpha q}$-norm) by truncation allows us to assume that t is finite. ∎

We can derive the duality (5.2) from the sequence space case in Theorem 5.9. Let $\mathscr{S}_0 = \{f \in \mathscr{S} \colon \hat{f} = 0$ in a neighborhood of the origin$\}$. Then it is easy to see that $\mathscr{S}_0$ is dense in $\dot{\mathbf{F}}_p^{\alpha q}$ if $0 < p, q < +\infty$ (e.g., by using Theorem 2.2 or see [Tr2]).

THEOREM 5.13. *Suppose $\alpha \in \mathbb{R}$ and $0 < q < +\infty$. Then $(\dot{\mathbf{F}}_1^{\alpha q})^* \approx \dot{\mathbf{F}}_\infty^{-\alpha q'}$. Namely, if $g \in \dot{\mathbf{F}}_\infty^{-\alpha q'}$, the map l_g, given by $l_g(f) = \langle f, g \rangle$, defined initially for $f \in \mathscr{S}_0$, extends to a continuous linear functional on $\dot{\mathbf{F}}_1^{\alpha q}$ with $\|l_g\| \approx \|g\|_{\dot{\mathbf{F}}_\infty^{-\alpha q'}}$. Conversely, every $l \in (\dot{\mathbf{F}}_1^{\alpha q})^*$ satisfies $l = l_g$ for some $g \in \dot{\mathbf{F}}_\infty^{-\alpha q'}$.*

Proof. As we noted in Section 2, we may choose $\psi = \varphi$ in (2.1)–(2.4).

In this case, if $g \in \dot{\mathbf{F}}_{\infty}^{-\alpha q'}$ and $f \in \mathscr{S}_0$, Theorem 5.9, Theorem 5.2, and the identity $\langle f, g \rangle = \langle S_\varphi f, S_\varphi g \rangle$ (i.e., (2.7)) imply that

$$|\langle f, g \rangle| \leqslant c \, \|S_\varphi f\|_{\dot{\mathbf{f}}_1^{\alpha q}} \, \|S_\varphi g\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}} \leqslant c \, \|f\|_{\dot{\mathbf{F}}_1^{\alpha q}} \, \|g\|_{\dot{\mathbf{F}}_\infty^{-\alpha q'}}.$$

This proves that $\|l_g\| \leqslant c \, \|g\|_{\dot{\mathbf{F}}_\infty^{-\alpha q'}}$.

Conversely, suppose $l \in (\dot{\mathbf{F}}_1^{\alpha q})^*$. Then $l_1 \equiv l \circ T_\psi \in (\dot{\mathbf{f}}_1^{\alpha q})^*$, so by Theorem 5.9, there exists $t = \{t_Q\}_Q \in \dot{\mathbf{f}}_\infty^{-\alpha q'}$ such that $l_1(s) = \sum_Q s_Q \bar{t}_Q$ for $s = \{s_Q\}_Q \in \dot{\mathbf{f}}_1^{\alpha q}$, and $\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}} \approx \|l_1\| \leqslant c \, \|l\|$, since T_ψ is bounded. Now $l_1 \circ S_\varphi = l \circ T_\psi \circ S_\varphi = l$ by Theorem 5.2. Hence, with $f \in \mathscr{S}_0$ and letting $g = T_\psi(t) = \sum_Q t_Q \psi_Q$,

$$l(f) = l_1(S_\varphi f) = \langle S_\varphi f, t \rangle = \langle f, g \rangle,$$

by (2.8), with $\varphi = \psi$, still. Then $l = l_g$, and, by Theorem 5.2,

$$\|g\|_{\dot{\mathbf{F}}_\infty^{-\alpha q'}} \leqslant c \, \|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}} \leqslant c \, \|l_g\|. \quad \blacksquare$$

Remark 5.14. Using (5.13) instead of Theorem 5.9 and Theorem 2.2 instead of Theorem 5.2, we could also obtain the result $(\dot{\mathbf{F}}_p^{\alpha q})^* \approx \dot{\mathbf{F}}_{p'}^{-\alpha q'}$ for $\alpha \in \mathbb{R}$, $0 < q < +\infty$, and $1 < p < +\infty$, by the same method as in Theorem 5.13. For $1 < q < +\infty$, this is well known (see [Tr2]), but for $0 < q \leqslant 1$ it seems to be new (cf. [Tr2, p. 180]). (Alternatively, one could obtain the result for $0 < q \leqslant 1$ by the methods in [Tr2] by using Theorem A.1 to obtain Proposition 1, p. 50 of [Tr2] for $q = +\infty$.)

Similarly, (5.14) gives

$$(\dot{\mathbf{F}}_p^{\alpha q})^* \approx \dot{\mathbf{F}}_\infty^{\beta \infty} \tag{5.15}$$

for $0 < p < 1$ and $0 < q \leqslant +\infty$, where $\beta = \beta(\alpha, p) = -\alpha + n(1/p - 1)$. This is known (cf. [Tr2, pp. 177–182]). This includes the well-known result that $(H^p)^* \approx \dot{B}_\infty^{n(1/p-1), \infty}$ for $0 < p < 1$ [DRS; Wa].

Corollary 5.12 translates into the following:

COROLLARY 5.15. *Suppose* $\alpha \in \mathbb{R}$ *and* $1 \leqslant q < +\infty$. *If* $f \in \dot{\mathbf{F}}_1^{\alpha q}$, *then* $\|f\|_{\dot{\mathbf{F}}_1^{\alpha q}} \approx \sup\{|\langle f, g \rangle| : g \in \mathscr{S}_0$ *with* $\|g\|_{\dot{\mathbf{F}}_\infty^{-\alpha q'}} \leqslant 1\}$.

Proof. One direction is of course an immediate consequence of Theorem 5.13. For the other, we fix $f \in \dot{\mathbf{F}}_1^{\alpha q}$ and assume again that $\varphi = \psi$ in (2.1)–(2.4). Corollary 5.12 provides us with a finite sequence $t = \{t_Q\}_Q$ such that $\|t\|_{\dot{\mathbf{f}}_\infty^{-\alpha q'}} \leqslant 1$ and

$$|\langle S_\varphi f, t \rangle| \approx \|S_\varphi f\|_{\dot{\mathbf{f}}_1^{\alpha q}}.$$

By Theorem 2.2, $\|S_\varphi f\|_{\dot{\mathbf{f}}_1^{\alpha q}} \approx \|f\|_{\dot{\mathbf{F}}_1^{\alpha q}}$. If we let $g = T_\psi(t)$, then we have $|\langle f, g\rangle| = |\langle S_\varphi f, t\rangle|$ by (2.8). Also, $g \in \mathscr{S}_0$ since it is a finite sum of functions in $\mathscr{S}_0$. Hence, (a multiple of) this g satisfies all the requirements. ∎

We now turn to the analogues for $p = +\infty$ of the results in Sections 3–4. The results on almost diagonality can be extended easily; for $1 < q \leqslant +\infty$ by duality and then reducing the case $0 < q \leqslant 1$ to $q > 1$ as in the proof of Theorem 3.3. We say that $\{m_Q\}_Q$ is a family of smooth molecules (for $\dot{\mathbf{F}}_\infty^{\alpha q}$) if (3.3)–(3.6) hold with N as above and $J = n/\min(1, q)$. With this, Theorem 3.5 holds with $p = +\infty$. Similarly, the analogues of Theorem 3.7 and Remark 3.10 can be proved as before. Then all the conclusions of Section 4 can also be extended, virtually verbatim.

6. The Case $p = 0$ and Real Interpolation

In the previous section we discussed the limiting case $p = +\infty$. Here we will show that the extension to the other limit value, $p = 0$, is also possible at least on the sequence space level. These results are closely connected with real interpolation and, as we shall see in Section 7, with the John–Nirenberg lemma and atomic decompositions.

The space $L^0(\mathbb{R}^n, dx)$ is defined to be the collection of all measurable functions f such that

$$\|f\|_{L^0} = |\{x \in \mathbb{R}^n : f(x) \neq 0\}| < +\infty.$$

Peetre and Sparr [PS] studied L^0 in the context of interpolation. As $\|f\|_{L^\infty}$ is the "height" of a function, $\|f\|_{L^0}$ is the "width." If we define the best approximation functional for a pair of spaces (X_0, X_1) by

$$E(t) = E(t, x; X_0, X_1) = \inf_{\|x_1\|_{X_1} \leqslant t} \|x - x_1\|_{X_0}, \qquad 0 < t < +\infty,$$

then it is easy to see that

$$E(t, f; L^0, L^\infty) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|. \tag{6.1}$$

We also have the usual definitions of other functionals such as

$$K(t) = K(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} (\|x_0\|_{X_0} + t\,\|x_1\|_{X_1})$$

and

$$K_\infty(t) = K_\infty(t, x; X_0, X_1) = \inf_{x = x_0 + x_1} \max(\|x_0\|_{X_0}, t\,\|x_1\|_{X_1}).$$

Obviously $K(t) \approx K_\infty(t)$. It follows from the definitions that

$$K_\infty(t)/t \geqslant s \qquad \text{if and only if} \quad E(s)/s \geqslant t. \tag{6.2}$$

In other words, K_∞ is the right continuous inverse of E. From this it follows from the definitions that an inequality of the form $K_\infty(t, b; X_0, X_1) \leqslant cK_\infty(t, a; X_0, X_1)$, $t > 0$, is equivalent to $E(cs, b; X_0, X_1) \leqslant cE(s, a; X_0, X_1)$, $s > 0$. We also recall the definition of the real interpolation spaces: for $0 < \theta < 1$, and $0 < q \leqslant +\infty$, $(X_0, X_1)_{\theta,q}$ is the set of all $x \in X_0 + X_1$ such that

$$\|x\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t, x; X_0, X_1))^q \frac{dt}{t}\right)^{1/q}$$

is finite. If in general we set $\|\cdot\|_{X^\gamma} = \|\cdot\|_X^\gamma$ for $\gamma > 0$, then we have the following well-known fact (cf. [PS] and also [Ja-T2; Ja-R-W]).

LEMMA 6.1. *Suppose* $0 < \theta < 1$ *and* $p = \theta/(1-\theta)$. *Then*

$$(L^0, L^\infty)^{1/\theta}_{\theta, 1/(1-\theta)} \approx L^p.$$

Proof. By calculation we have

$$\begin{aligned}\|f\|^{1/\theta}_{(L^0, L^\infty)_{\theta, 1/(1-\theta)}} &= \|K(t)/t\|^{(1+p)/p}_{L^{1+p}(0,\infty)} \approx \|K_\infty(t)/t\|^{(1+p)/p}_{L^{1+p}(0,\infty)} \\ &= (1+p) \int_0^\infty s^p \,|\{t > 0 \colon K_\infty(t)/t \geqslant s\}|\, ds.\end{aligned}$$

By (6.2), $|\{t > 0 \colon K_\infty(t)/t \geqslant s\}| = |(0, E(s)/s)| = E(s)/s$. Thus, by (6.1),

$$\|f\|^{1/\theta}_{(L^0, L^\infty)_{\theta, 1/(1-\theta)}} \approx \left(\int_0^\infty s^{p-1} \,|\{x \colon |f(x)| > s\}|\, ds\right)^{1/p} = c\,\|f\|_{L^p}. \quad \blacksquare$$

We also note that if a function $f \in L^0$ satisfies $f \in L^p$ for all sufficiently small p's, then $\|f\|_{L^0} = \lim_{p \to 0} \|f\|^p_{L^p}$.

We now define a sequence space that corresponds to L^0. We let $\mathfrak{k}_0$ be the collection of all sequences $s = \{s_Q\}$, Q dyadic, such that

$$\|s\|_{\mathfrak{k}_0} = \left|\bigcup_{s_Q \neq 0} Q\right| < +\infty.$$

Like L^0, $\mathfrak{k}_0$ is not a normed vector space, since $\|\lambda s\|_{\mathfrak{k}_0} = \|s\|_{\mathfrak{k}_0}$ for $\lambda \in \mathbb{C} \setminus \{0\}$, but $\mathfrak{k}_0$ is a "normed Abelian group" (see, e.g., [Be-L, Sections 3.10–3.11]), since we have the triangle inequality $\|s+t\|_{\mathfrak{k}_0} \leqslant \|s\|_{\mathfrak{k}_0} + \|t\|_{\mathfrak{k}_0}$. One can also check that L^0 and $\dot{\mathbf{f}}_0$ are complete. We note that $\dot{\mathbf{f}}_0$ is indeed a continuous extension of the scale of $\dot{\mathbf{f}}_p^{\alpha q}$-spaces since, for $\alpha \in \mathbb{R}$ and $0 < q \leqslant +\infty$,

$$\|s\|_{\dot{\mathbf{f}}_0} = \left\| \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^0}. \tag{6.3}$$

It is interesting to note that the analogues of Propositions 2.7 and 5.5 hold for $\dot{\mathbf{f}}_0$.

LEMMA 6.2. *Let $\varepsilon > 0$. Suppose that for each dyadic cube Q there is a set $E_Q \subset Q$ such that $|E_Q|/|Q| > \varepsilon$. Then*

$$\|s\|_{\dot{\mathbf{f}}_0} \approx \left| \bigcup_{s_Q \neq 0} E_Q \right|.$$

Proof. Since M is weak-type $(1, 1)$, we have

$$\left| \bigcup_{s_Q \neq 0} Q \right| \leqslant |\{x: M(\chi_{\bigcup_{s_Q \neq 0} E_Q})(x) > \varepsilon\}| \leqslant C_\varepsilon \left| \bigcup_{s_Q \neq 0} E_Q \right|.$$

The other direction is trivial. ▮

LEMMA 6.3. *Suppose $\alpha \in \mathbb{R}$ and $0 < q \leqslant +\infty$. Then*

$$\|s\|_{\dot{\mathbf{f}}_0} \approx \|m^{\alpha q}(s)\|_{L^0}.$$

Proof. As in Proposition 5.5, we have

$$\{x: m^{\alpha q}(s)(x) > 0\} \subset \{x: M(\chi_{\{y: G^{\alpha q}(s)(y) > 0\}})(x) > \tfrac{1}{4}\}.$$

By the weak-type $(1, 1)$ inequality for M, and (6.3), we obtain $\|m^{\alpha q}(s)\|_{L^0} \leqslant C \|s\|_{\dot{\mathbf{f}}_0}$.

For the converse inequality, we could argue as in Proposition 5.5, but it is simpler to just note that if $s_Q \neq 0$, then $m_Q^{\alpha q}(s) \neq 0$, and hence $m^{\alpha q}(s)(x) \neq 0$ for all $x \in Q$. ▮

Lemmas 6.2–6.3 are exactly what is needed to carry through the analogue for $\dot{\mathbf{f}}_0$ and $\dot{\mathbf{f}}_\infty^{\alpha q}$ of the real interpolation argument in [Fr-J2, Section 3].

THEOREM 6.4. *Suppose $\alpha \in \mathbb{R}$ and $0 < q \leqslant +\infty$. Then*

$$K(t, s; \dot{\mathbf{f}}_0, \dot{\mathbf{f}}_\infty^{\alpha q}) \approx K(t, m^{\alpha q}(s); L^0, L^\infty),$$

with constants in the equivalence independent of t and s.

Proof. The direction

$$K(t, m^{\alpha q}(s); L^0, L^\infty) \leqslant CK(t, s; \dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}^{\alpha q}_\infty) \tag{6.4}$$

is very easy. We define the "$\frac{1}{8}$-medians" $m^{\alpha q}_{Q,1/8}(s)$ as in (5.7) except with $|Q|/8$ in place of $|Q|/4$, and we set

$$m^{\alpha q}_{1/8}(s)(x) = \sup_Q m^{\alpha q}_{Q,1/8}(s)\, \chi_Q(x).$$

Then Proposition 5.5 and Lemma 6.3 still hold with $m^{\alpha q}_{1/8}(s)$ in place of $m^{\alpha q}(s) = m^{\alpha q}_{1/4}(s)$. Also, it is elementary to verify the subadditivity property

$$m^{\alpha q}_Q(s_0 + s_1) \leqslant C_q(m^{\alpha q}_{1/8}(s_0) + m^{\alpha q}_{1/8}(s_1)).$$

Letting $f_0 = \min(m^{\alpha q}(s), C_q m^{\alpha q}_{1/8}(s_0))$ and $f_1 = m^{\alpha q}(s) - f_0$ readily gives (6.4).

To prove the converse inequality, by (6.2) it suffices to prove that

$$E(Ct, s; \dot{\mathfrak{f}}_0, \dot{\mathfrak{f}}^{\alpha q}_\infty) \leqslant CE(t, m^{\alpha q}(s); L^0, L^\infty)$$

for some C. By (6.1), then, it is sufficient to show that there exists a splitting $s = s^0 + s^1$ such that

$$\|s^0\|_{\dot{\mathfrak{f}}_0} \leqslant C\, |\{x : m^{\alpha q}(s)(x) > t\}| \tag{6.5}$$

and

$$\|s^1\|_{\dot{\mathfrak{f}}^{\alpha q}_\infty} \leqslant Ct. \tag{6.6}$$

Let $Q_t^+ = \{x \in Q : m^{\alpha q}(s)(x) > t\}$ and $Q_t^- = Q \backslash Q^t_+$. Let $A_t = \{Q : |Q_t^+| > |Q|/2\}$ and $A_t^c = \{Q : |Q_t^-| \geqslant |Q|/2\}$. Define s^0 and s^1 by setting $s^0_Q = s_Q$ if $Q \in A_t$, $s^0_Q = 0$ if $Q \in A_t^c$, and $s^1_Q = s_Q - s^0_Q$. As in Proposition 5.5, define $E_Q = \{x \in Q : G^{\alpha q}_Q(s)(x) \leqslant m^{\alpha q}(s)(x)\}$. Then $|E_Q|/|Q| \geqslant \frac{3}{4}$. Let $\tilde{E}_Q = E_Q \cap Q_t^+$ if $Q \in A_t$ and $\tilde{E}_Q = E_Q \cap Q_t^-$ if $Q \in A_t^c$. Then $|\tilde{E}_Q|/|Q| \geqslant \frac{1}{4}$. By Lemma 6.2 with $\tilde{E}_Q$ in place of E_Q,

$$\|s^0\|_{\dot{\mathfrak{f}}_0} = \Big| \bigcup_{\substack{s_Q \neq 0 \\ Q \in A_t}} Q \Big| \leqslant C \Big| \bigcup_{\substack{s_Q \neq 0 \\ Q \in A_t}} \tilde{E}_Q \Big|,$$

which yields (6.5), since $\tilde{E}_Q \subset Q_t^+$ for $Q \in A_t$. Also, by Proposition 5.4 and (5.10),

$$\begin{aligned}\|s^1\|_{\dot{\mathfrak{f}}^{\alpha q}_\infty} &\leqslant C\left(\sup_P \frac{1}{|P|} \int_P \sum_{Q \subset P} (|Q|^{-\alpha/n}\, |s^1_Q|\, \tilde{\chi}_{\tilde{E}_Q})^q\right)^{1/q} \\ &\leqslant C\left(\sup_P \frac{1}{|P|} \int_{P_t^-} (m^{\alpha q}(s))^q\right)^{1/q} \leqslant Ct,\end{aligned}$$

since $\tilde{E}_Q \subset Q_t^- \subset P_t^-$ for $Q \subset P$ and $Q \in A_t^c$. ∎

Notice that another way to state the theorem would be to say that

$$|\{x \in \mathbb{R}^n : m^{\alpha q}(s)(x) > c_1 t\}|/c_1 \leqslant E(t, s; \dot{\mathbf{f}}_0, \dot{\mathbf{f}}_\infty^{\alpha q}) \leqslant |\{x \in \mathbb{R}^n : m^{\alpha q}(s)(x) > c_2 t\}|/c_2.$$

for some constants c_1 and c_2. This means that where we are used to seeing the distribution function of f in the L^p context, we should expect the distribution function of $m^{\alpha q}(s)$ for our sequence spaces. For instance, the theorem has the following immediate corollary.

COROLLARY 6.5. *Suppose* $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, *and* $0 < \theta < 1$. *Let* $p = \theta/(1-\theta)$. *Then*

$$(\dot{\mathbf{f}}_0, \dot{\mathbf{f}}_\infty^{\alpha q})_{\theta, 1/(1-\theta)}^{1/\theta} \approx \dot{\mathbf{f}}_p^{\alpha q}.$$

Proof. This follows from the definitions, Theorem 6.4, Lemma 6.1, and Proposition 5.5. ∎

We now use some standard facts from interpolation theory such as reiteration (see [Be-L, pp. 67–68]), the remarks following (6.2), the fact that $E(t, a; X_0^\alpha, X_1^\beta) = E(t^{1/\beta}, a; X_0, X_1)^\alpha$, $\alpha, \beta > 0$, and Holmstedt's formula (see [Be-L, pp. 52–53]), which show that an equivalence between K-functionals persists after mutual reiteration. The endpoint results in Theorem 6.4 and its corollary then have a number of immediate consequences.

COROLLARY 6.6. *Suppose* $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, *and* $0 < p_0 < p < p_1 \leqslant +\infty$. *Then*

$$K(t, s; \dot{\mathbf{f}}_0, \dot{\mathbf{f}}_{p_1}^{\alpha q}) \approx K(t, m^{\alpha q}(s); L^0, L^{p_1}), \tag{6.7}$$

$$K(t, s; \dot{\mathbf{f}}_{p_0}^{\alpha q}, \dot{\mathbf{f}}_{p_1}^{\alpha q}) \approx K(t, m^{\alpha q}(s); L^{p_0}, L^{p_1}), \tag{6.8}$$

$$(\dot{\mathbf{f}}_0, \dot{\mathbf{f}}_{p_1}^{\alpha q})_{\theta, p/\theta}^{1/\theta} = \dot{\mathbf{f}}_p^{\alpha q} \qquad \text{if } \frac{1}{\theta} = 1 + \frac{1}{p} - \frac{1}{p_1}, \tag{6.9}$$

and

$$(\dot{\mathbf{f}}_{p_0}^{\alpha q}, \dot{\mathbf{f}}_{p_1}^{\alpha q})_{\theta, p} = \dot{\mathbf{f}}_p^{\alpha q} \qquad \text{if } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{6.10}$$

Of course, corresponding results for the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces follow immediately.

COROLLARY 6.7. *Suppose* $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, *and* $0 < p_0 < p < p_1 \leqslant +\infty$. *Then*

$$K(t, f; \dot{\mathbf{F}}_{p_0}^{\alpha q}, \dot{\mathbf{F}}_{p_1}^{\alpha q}) \approx K(t, S_\varphi f; \dot{\mathbf{f}}_{p_0}^{\alpha q}, \dot{\mathbf{f}}_{p_1}^{\alpha q}) \approx K(t, A^{\alpha q} f; L^{p_0}, L^{p_1}) \tag{6.11}$$

and

$$(\dot{\mathbf{F}}_{p_0}^{\alpha q}, \dot{\mathbf{F}}_{p_1}^{\alpha q})_{\theta, p} = \dot{\mathbf{F}}_p^{\alpha q} \qquad \text{if } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{6.12}$$

Proof. The first equivalence in (6.11) follows from the $\dot{\mathbf{f}}_p^{\alpha q} - \dot{\mathbf{F}}_p^{\alpha q}$ retraction diagram (Theorems 2.2 and 5.2), while the second follows from Corollary 6.6. Then (6.12) follows either by (6.10) and retraction, or by (6.11), the usual results for L^p-spaces, and Corollary 5.8. ∎

We remark that in [Fr-J2, Section 3], we obtain (6.8) with $G^{\alpha q}(s)$ in place of $m^{\alpha q}(s)$, and (6.11) with $G^{\alpha q}(S_\varphi f)$ in place of $A^{\alpha q} f$, both in the case $p_1 < +\infty$, by direct arguments analogous to those above. These yield (6.10) and (6.12) for $p_1 < +\infty$. However, the argument above is easier than the one in [Fr-J2] and, still, it proves more. We could also prove (6.8) and (6.10)–(6.12) for $p_1 = +\infty$ by direct arguments like those above. In particular, (6.11)–(6.12) for $p_1 = +\infty$ generalize and simplify the results in [Ja1].

We have not defined any space $\dot{\mathbf{F}}_0$ corresponding to $\dot{\mathbf{f}}_0$. If, for example, we define $\|f\|_{\dot{\mathbf{F}}_0} = \inf\{\|s\|_{\dot{\mathbf{f}}_0} : f = \sum_Q s_Q \psi_Q\}$, the resulting space would not be independent of the test function ψ chosen. This is closely related to the fact that the analogues of Lemmas 2.3 and 5.1 fail for $\dot{\mathbf{f}}_0$.

We now briefly discuss another possibility for defining a sequence space corresponding to $p = 0$. Suppose we (temporarily) let α be a function of p via the relation $\alpha(p) = n[(1-\beta)/p - \frac{1}{2}]$ for some fixed $\beta \in \mathbb{R}$. Let ω_β be the measure on the sequence space $\{Q : Q \text{ is dyadic}\}$ with weight $\omega_\beta(Q) = |Q|^\beta$. Then it follows easily that

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha(p) p}} = \left(\sum_Q |s_Q|^p \, |Q|^\beta \right)^{1/p} = \|s\|_{l^p(\omega_\beta)},$$

for $0 < p \leqslant +\infty$. Considering $(\dot{\mathbf{f}}_p^{\alpha(p) p})^p$ as $p \to 0$, define $\dot{\mathbf{f}}_0^\beta$ to be the collection of all sequences $s = \{s_Q\}_Q$ dyadic such that

$$\|s\|_{\dot{\mathbf{f}}_0^\beta} = \sum_{s_Q \neq 0} |Q|^\beta = \|s\|_{l^0(\omega_\beta)},$$

for l^0 defined analogously to L^0 above; that is, the l^0-"norm" is the ω_β-measure of the set of "points" Q such that $s(Q) = s_Q \neq 0$. We then have the following.

PROPOSITION 6.8. *Suppose* $0 < p_0 < p < p_1 \leqslant +\infty$, $\beta \in \mathbb{R}$, *and* $\alpha(p)$ *is as above. Then*

$$(\dot{\mathbf{f}}_0^\beta, \dot{\mathbf{f}}_{p_1}^{\alpha(p_1) p_1})_{\theta, p/\theta}^{1/\theta} = \dot{\mathbf{f}}_p^{\alpha(p) p} \qquad \text{if } \frac{1}{\theta} = 1 + \frac{1}{p} - \frac{1}{p_1}.$$

Proof. This follows from Lemma 6.1 applied to $l^0(\omega_\beta)$ and $l^\infty(\omega_\beta)$ for $p_1 = +\infty$, and by reiteration for $p_1 < +\infty$. ∎

7. Atomic Decompositions

Using the interpolation results in the previous section, we will obtain "atomic" decompositions of the elements in $\dot{\mathbf{f}}_p^{\alpha q}$ for $0 < p \leqslant 1$ and $p \leqslant q \leqslant +\infty$. This will yield corresponding results for the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces. In particular, we recover the traditional atomic decomposition of the Hardy spaces. We will take a slightly roundabout approach in order to clarify the connection between real interpolation and atomic decompositions. The point we wish to make is that we (essentially) get atomic decompositions as soon as we use a standard, alternative, definition of the real interpolation method and the spaces $\dot{\mathbf{f}}_0$ and $\dot{\mathbf{f}}_p^{\alpha q}$.

Let us first recall this alternative description of the real method. Let $\bar{X} = (X_0, X_1)$ be a pair of (quasi-)Banach spaces. Following [Ja-R-W] we define the e-functional for $t > 0$ by

$$e(t, x; \bar{X}) = \begin{cases} \|X\|_{X_0} & \text{if} \quad \|x\|_{X_1} \leqslant t \\ +\infty & \text{otherwise.} \end{cases}$$

(Then e corresponds to E as the J-functional corresponds to the K-functional.) We define

$$\|x\|_{\theta, q; e} = \inf_{x = \sum_{v \in \mathbb{Z}} x_v} \left(\sum_{v \in \mathbb{Z}} (2^{v\theta/(1-\theta)} e(2^v, x_v; \bar{X}))^{(1-\theta)q} \right)^{1/q}.$$

The following proposition is known.

Proposition 7.1. *Let $0 < \theta < 1$ and $0 < q \leqslant +\infty$. Then*

$$\|x\|_{\theta, q} \approx \|x\|_{\theta, q; e}.$$

Proof. See Appendix C. ∎

Combining Corollary 6.5 with Proposition 7.1, and noting $p = \theta/(1-\theta)$, yields

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}^p \approx \inf_{s = \sum s_k} \sum_{k \in \mathbb{Z}} 2^{kp} e(2^k, s_k; \dot{\mathbf{f}}_0, \dot{\mathbf{f}}_\infty^{\alpha q}) \tag{7.1}$$

for $\alpha \in \mathbb{R}$, $0 < q \leqslant +\infty$, and $0 < p < +\infty$. Since

$$\inf_t t^p e(t, x; \bar{X}) = \|x\|_{X_0} \|x\|_{X_1}^p,$$

(7.1) implies

$$\|s\|^p_{\dot{\mathbf{f}}^{\alpha q}_p} \geqslant c \inf_{s=\sum s_k} \sum_{k\in\mathbb{Z}} \|s_k\|_{\dot{\mathbf{f}}_0} \|s_k\|^p_{\dot{\mathbf{f}}^{\alpha q}_\infty}. \tag{7.2}$$

The converse of (7.2) holds if $0<p\leqslant 1$ and $p\leqslant q\leqslant +\infty$. First, by (7.1) we have

$$\|s\|^p_{\dot{\mathbf{f}}^{\alpha q}_p} \leqslant c \inf_{k\in\mathbb{Z}} 2^{k\theta/(1-\theta)} e(2^k, s; \dot{\mathbf{f}}_0, \dot{\mathbf{f}}^{\alpha q}_\infty) \leqslant c\, \|s\|_{\dot{\mathbf{f}}_0} \|s\|^p_{\dot{\mathbf{f}}^{\alpha q}_\infty} \tag{7.3}$$

for $\alpha\in\mathbb{R}$ and $0<p, q\leqslant +\infty$. (This also follows directly; for $p\leqslant q$ trivially by Hölder's inequality and, in general, by Corollary 5.6.) If $0<p\leqslant 1$ and $p\leqslant q\leqslant +\infty$, the p-triangle inequality and Minkowski's inequality with exponent q/p yield

$$\|s+t\|^p_{\dot{\mathbf{f}}^{\alpha q}_p} \leqslant \|s\|^p_{\dot{\mathbf{f}}^{\alpha q}_p} + \|t\|^p_{\dot{\mathbf{f}}^{\alpha q}_p}. \tag{7.4}$$

It follows that we have equivalence in (7.2) if $0<p\leqslant 1$ and $p\leqslant q\leqslant +\infty$. If we normalize by setting $\lambda_k = \|s_k\|^{1/p}_{\dot{\mathbf{f}}_0} \|s_k\|_{\dot{\mathbf{f}}^{\alpha q}_\infty}$, then $\|s_k/\lambda_k\|^{1/p}_{\dot{\mathbf{f}}_0} \|s_k/\lambda_k\|_{\dot{\mathbf{f}}^{\alpha q}_\infty} = 1$, since $\|\lambda s\|_{\dot{\mathbf{f}}_0} = \|s\|_{\dot{\mathbf{f}}_0}$. By a renaming, then, we obtain, for $0<p\leqslant 1$, $p\leqslant q\leqslant +\infty$, and $\alpha\in\mathbb{R}$,

$$\|s\|_{\dot{\mathbf{f}}^{\alpha q}_p} \approx \inf\left\{\left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right)^{1/p} : s=\sum_{k\in\mathbb{Z}} \lambda_k s_k \text{ and } \|s_k\|^{1/p}_{\dot{\mathbf{f}}_0} \|s_k\|_{\dot{\mathbf{f}}^{\alpha q}_\infty} \leqslant 1 \text{ for all } k\right\}. \tag{7.5}$$

Because of the particular properties of the dyadic cubes, we can develop (7.5) one step further. We say that a sequence $r=\{r_Q\}_{Q\,\text{dyadic}}$ is an *atom* for $\dot{\mathbf{f}}^{\alpha q}_p$, $0<p\leqslant 1$, $p\leqslant q\leqslant +\infty$, and $\alpha\in\mathbb{R}$, if there exists a dyadic cube $\bar{Q}$ such that $r_Q=0$ if $Q\nsubseteq\bar{Q}$, and $\|r\|_{\dot{\mathbf{f}}^{\alpha q}_\infty} \leqslant |\bar{Q}|^{-1/p}$. Note that by (7.3), $\|r\|_{\dot{\mathbf{f}}^{\alpha q}_p} \leqslant c$ for any atom r for $\dot{\mathbf{f}}^{\alpha q}_p$.

THEOREM 7.2. *Suppose* $\alpha\in\mathbb{R}$, $0<p\leqslant 1$, *and* $p\leqslant q\leqslant +\infty$. *Then*

$$\|s\|_{\dot{\mathbf{f}}^{\alpha q}_p} \approx \inf\left\{\left(\sum_{k\in\mathbb{Z}} |\lambda_k|^p\right)^{1/p} : s=\sum_{k\in\mathbb{Z}} \lambda_k r_k \textit{ and each } r_k \textit{ is an atom for } \dot{\mathbf{f}}^{\alpha q}_p\right\}.$$

Proof. By (7.5), if $s\in\dot{\mathbf{f}}^{\alpha q}_p$, select $\{\gamma_k\}_{k\in\mathbb{Z}}$ and $\{s_k\}_{k\in\mathbb{Z}}$ such that $s=\sum_{k\in\mathbb{Z}}\gamma_k s_k$, $\|s_k\|^{1/p}_{\dot{\mathbf{f}}_0} \|s_k\|_{\dot{\mathbf{f}}^{\alpha q}_\infty} \leqslant 1$ for all k, and $(\sum_{k\in\mathbb{Z}} |\gamma_k|^p)^{1/p} \leqslant c\,\|s\|_{\dot{\mathbf{f}}^{\alpha q}_p}$. Let $\Omega_k = \bigcup\{Q: (s_k)_Q\neq 0\}$. Then $|\Omega_k| = \|s_k\|_{\dot{\mathbf{f}}_0} < +\infty$. Let $\{\bar{Q}_{kj}\}_j$ be the (unique) collection of maximal pairwise disjoint dyadic cubes such that $(s_k)_{\bar{Q}_{kj}}\neq 0$ and $\Omega_k=\bigcup_j \bar{Q}_{kj}$. Let $t_{kj} = \gamma_k(|\bar{Q}_{kj}|/|\Omega_k|)^{1/p}$ and let r_{kj} be the sequence defined by $(r_{kj})_Q = \gamma_k (s_k)_Q/t_{kj}$ if $Q\subset\bar{Q}_{kj}$ and $(r_{kj})_Q=0$ otherwise.

Then

$$\|r_{kj}\|_{\dot{f}_\infty^{\alpha q}} \leqslant \|s_k\|_{\dot{f}_\infty^{\alpha q}} |\Omega_k|^{1/p} |\bar{Q}_{kj}|^{-1/p} \leqslant |\bar{Q}_{kj}|^{-1/p}$$

and $(r_{kj})_Q = 0$ if $Q \nsubseteq \bar{Q}_{kj}$. Thus r_{kj} is an atom for $\dot{f}_p^{\alpha q}$. Also,

$$s = \sum_{k \in \mathbb{Z}} \gamma_k s_k = \sum_{k,j} t_{kj} r_{kj},$$

with

$$\sum_{k,j} |t_{kj}|^p = \sum_{k \in \mathbb{Z}} |\gamma_k|^p |\Omega_k|^{-1} \sum_{j \in \mathbb{Z}} |\bar{Q}_{kj}| = \sum_{k \in \mathbb{Z}} |\gamma_k|^p.$$

We thus have that $\|s\|_{\dot{f}_p^{\alpha q}}$ dominates the infimum in the statement of the theorem.

The converse estimate follows easily from (7.4) and the remark above that $\|r\|_{\dot{f}_p^{\alpha q}} \leqslant c$ if r is an atom for $\dot{f}_p^{\alpha q}$. ∎

There is a somewhat more direct proof of Theorem 7.2 following the ideas of Calderón [Cal2] and Chang and Fefferman [Ch-F2] (cf. also [Fo-S; Ja-T2]). Our approach emphasizes the close connection between atomic decompositions and real interpolation. Frequently, atomic decompositions have been used to obtain interpolation results (see the survey in [Jon]). The proof above makes the reverse connection explicit by exploiting the $\dot{f}_0$-spaces to obtain the compact support (cf. [P7, Co, Fef-R-S]).

Remark 7.3. We say that a sequence $r = \{r_Q\}_{Q \text{ dyadic}}$ is a p_1-*atom* for $\dot{f}_p^{\alpha q}$, $p < p_1 < +\infty$, if there exists a dyadic cube $\bar{Q}$ such that $r_Q = 0$ for $Q \nsubseteq \bar{Q}$ and $\|r\|_{\dot{f}_{p_1}^{\alpha q}} \leqslant |\bar{Q}|^{1/p_1 - 1/p}$. By Hölder's inequality and (7.3), we have

$$\|s\|_{\dot{f}_p^{\alpha q}} \leqslant \|s\|_{\dot{f}_0}^{1/p - 1/p_1} \|s\|_{\dot{f}_{p_1}^{\alpha q}} \leqslant c\, \|s\|_{\dot{f}_0}^{1/p} \|s\|_{\dot{f}_\infty^{\alpha q}}$$

for $p < p_1 < +\infty$. It follows easily that the modification of Theorem 7.3 in which each r_k is a p_1-atom for $\dot{f}_p^{\alpha q}$, $p < p_1 < +\infty$, holds also. This could also be derived from (6.9), using the technique of Theorem 7.2.

Naturally, Theorem 7.2 leads to corresponding result for the $\dot{F}_p^{\alpha q}$-spaces. Let $\{\sigma^Q\}_{Q \text{ dyadic}}$ be a given family of distributions representing $\dot{F}_p^{\alpha q}$ (see Section 4 for definitions). We say that $\Psi \in \mathscr{S}'/\mathscr{P}$ is a *wave-cluster* for $\dot{F}_p^{\alpha q}$ if $\Psi = \sum_{Q \subset \bar{Q}} r_Q \sigma^Q$ (in $\mathscr{S}'/\mathscr{P}$), where $r = \{r_Q\}_{Q \text{ dyadic}}$ is an atom for $\dot{f}_p^{\alpha q}$ associated with the dyadic cube $\bar{Q}$. Note that by (4.6) and (7.3),

$$\|\Psi\|_{\dot{F}_p^{\alpha q}} \leqslant c\, \|r\|_{\dot{f}_p^{\alpha q}} \leqslant c\, |\bar{Q}|^{1/p} \|r\|_{\dot{f}_\infty^{\alpha q}} \leqslant c. \tag{7.6}$$

We say A is an *atom* for $\dot{\mathbf{F}}_p^{\alpha q}$ if $A = \sum_{Q \subset \bar{Q}} r_Q a_Q$, where $r = \{r_Q\}_{Q \text{ dyadic}}$ is an atom for $\dot{\mathbf{f}}_p^{\alpha q}$ for the cube $\bar{Q}$, and the a_Q's are smooth atoms for $\dot{\mathbf{F}}_p^{\alpha q}$ (i.e., (4.1)–(4.3) hold for some $\tilde{N} \geqslant N$ and $\tilde{K} \geqslant [\alpha + 1]_+$). Observe that

$$\operatorname{supp} A \subset 3\bar{Q}, \int x^\gamma A(x)\,dx = 0 \qquad \text{if} \quad |\gamma| \leqslant \tilde{N},$$

and, by Theorem 4.1, $\|A\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c$, as in (7.6).

THEOREM 7.4. *Suppose* $\alpha \in \mathbb{R}$, $0 < p \leqslant 1$, *and* $p \leqslant q \leqslant +\infty$. *Then*

(i) $\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \approx \inf\{(\sum_{k \in \mathbb{Z}} |\lambda_k|^p)^{1/p} : f = \sum_{k \in \mathbb{Z}} \lambda_k \Psi_k$ *and each* Ψ_k *is a wave-cluster for* $\dot{\mathbf{F}}_p^{\alpha q}\}$, *and*

(ii) $\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \approx \inf\{(\sum_{k \in \mathbb{Z}} |\lambda_k|^p)^{1/p} : f = \sum_{k \in \mathbb{Z}} \lambda_k A_k$ *and each* A_k *is an atom for* $\dot{\mathbf{F}}_p^{\alpha q}\}$.

Proof. One direction follows trivially from (7.6) and its analogue for atoms A and the inequality, for $0 < p \leqslant 1$ and $p \leqslant q \leqslant +\infty$,

$$\|f + g\|^p_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant \|f\|^p_{\dot{\mathbf{F}}_p^{\alpha q}} + \|g\|^p_{\dot{\mathbf{F}}_p^{\alpha q}}, \tag{7.7}$$

as in (7.4). The other direction follows readily from the definitions, Theorem 4.1, and Theorem 7.2. ∎

Remark 7.5. Recall that $\dot{\mathbf{F}}_p^{02} \approx H^p$ if $0 < p < +\infty$ and $\dot{\mathbf{F}}_\infty^{02} \approx \mathrm{BMO}$. Note that an atom $A = \sum_{Q \subset \bar{Q}} r_Q a_Q$ for $\dot{\mathbf{F}}_p^{02}$ $(0 < p \leqslant 1)$ satisfies

$$\begin{aligned} &\operatorname{supp} A \subseteq 3\bar{Q}, \\ &\int x^\gamma A(x)\,dx = 0 \qquad \text{for} \quad |\gamma| \leqslant [n(1/p - 1)], \end{aligned} \tag{7.8}$$

and

$$\|A\|_{\mathrm{BMO}} \leqslant c\,\|r\|_{\dot{\mathbf{f}}_\infty^{02}} \leqslant c\,|\bar{Q}|^{-1/p}.$$

Thus, Theorem 7.4 yields a decomposition of H^p $(0 < p \leqslant 1)$ into "BMO-atoms." Also, using Remark 7.3 and the corresponding analogue of Theorem 7.4 for p_1-atoms, we obtain the familiar decomposition of H^p, $0 < p \leqslant 1$, into L^{p_1}-atoms, $1 < p_1 < +\infty$, which by definition satisfy (7.8) and $\|A\|_{L^{p_1}} \leqslant c\,|\bar{Q}|^{1/p_1 - 1/p}$ (see [Co, La, Cal2, Wi]).

Remark 7.6. There is no difficulty in obtaining

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \geqslant c \inf\left\{\left(\sum_{k \in \mathbb{Z}} |\lambda_k|^p\right)^{1/p} : f = \sum_{k \in \mathbb{Z}} \lambda_k A_k \text{ and each } A_k \text{ is an atom for } \dot{\mathbf{F}}_p^{\alpha q}\right\},$$

and similarly for the wave clusters, for all $\alpha \in \mathbb{R}$, $0<p<+\infty$, and $0<q\leqslant+\infty$, since (7.2) holds in general. We do not obtain equivalence (Theorem 7.4) unless $0<p\leqslant 1$ and $p\leqslant q\leqslant+\infty$ simply because (7.7) fails. This indicates why there is a distinct break between the cases $p<1$ and $p\geqslant 1$ for the traditional H^p atomic decomposition, but not for the decompositions in Theorems 2.2 and 4.1.

Remark 7.7. For $f\in\mathscr{S}'/\mathscr{P}$, let $f_{\bar{Q}}=\sum_{Q\subseteq\bar{Q}}(S_\varphi f)_Q\,\psi_Q$, where $\bar{Q}$ is dyadic. By (7.3), Theorem 2.2, and Theorem 5.2,

$$\|f_{\bar{Q}}\|_{\dot{\mathbf{F}}_p^{\alpha q}}\leqslant c\,|\bar{Q}|^{1/p}\,\|f_{\bar{Q}}\|_{\dot{\mathbf{F}}_\infty^{\alpha q}}$$

for $\alpha\in\mathbb{R}$, $0<p, q\leqslant+\infty$. Taking $\alpha=0$, $q=2$, and $1<p<+\infty$ gives

$$\|f_{\bar{Q}}\|_{L^p}\leqslant c\,|\bar{Q}|^{1/p}\,\|f\|_{\mathrm{BMO}}.$$

This is analogous to the John–Nirenberg inequality ([Joh-N]). Thus we see that Corollary 5.7 and (7.3) are analogous to the John–Nirenberg inequality for our sequence spaces.

Remark 7.8. Instead of taking the equivalence between H^p and $\dot{\mathbf{F}}_p^{02}$ $(0<p<+\infty)$, and between BMO and $\dot{\mathbf{F}}_\infty^{02}$, for granted and obtaining the traditional atomic decomposition, as in Remark 7.5, we can take the standard results for H^p for granted and obtain these equivalences as consequences of the results above. First, (2.1)–(2.3) and Plancherel's theorem easily imply that $\dot{\mathbf{F}}_2^{02}\approx L^2$. (See Appendix B for a discussion of the identification.) By Remark 7.3 and as in Remark 7.5, we obtain a decomposition of $\dot{\mathbf{F}}_p^{02}$ $(0<p\leqslant 1)$ into, say, L^2-atoms. This yields the continuous imbedding $\dot{\mathbf{F}}_p^{02}\to H^p$, $0<p\leqslant 1$.

The converse imbedding follows in a familiar way (see, e.g., [Torch, pp. 341–342); it suffices to show that an L^2-atom $a(x)$ for H^p, $0<p\leqslant 1$, satisfies $\|a\|_{\dot{\mathbf{F}}_p^{02}}\leqslant c$. Hence we have $\dot{\mathbf{F}}_p^{02}\approx H^p$ for $0<p\leqslant 1$ and for $p=2$. Then real interpolation ((6.12) and [Fef-S2]) between $p=1$ and $p=2$ yields $\dot{\mathbf{F}}_p^{02}\approx L^p$ for $1<p<2$. Duality yields the remaining cases.

8. The Calderón Product and the Interpolation Property

Given a pair $\bar{X}=(X_0, X_1)$ of (compatible) Banach spaces, there are many ways to construct intermediate spaces. For the special case of a pair of quasi-Banach lattices, there is a particularly simple construction, motivated by Hölder's inequality, known as the Calderón product (see [Cal1]). Although the distribution spaces $\dot{\mathbf{F}}_p^{\alpha q}$ are not necessarily lattices, the sequence spaces $\dot{\mathbf{f}}_p^{\alpha q}$ are. In this section we will see that Calderón product for these sequence spaces is easily computed. Further, it is well known that, under mild conditions, the Calderón product coincides with the interpolation spaces obtained by several different interpolation methods. Using this, we will also obtain certain interpolation results for the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces immediately via the retract diagrams (Theorems 2.2 and 5.2). In particular, we will obtain the interpolation property for the $\dot{\mathbf{f}}$- and $\dot{\mathbf{F}}$-spaces in the greatest generality.

Suppose (M, μ) is a measure space and X is a quasi-Banach space of μ-measurable functions (identified if equal μ-a.e.). Then X is said to be a *quasi-Banach lattice* on M if the conditions $f \in X$ and $|g(x)| \leqslant |f(x)|$ μ-a.e. imply that $g \in X$ and $\|g\|_X \leqslant \|f\|_X$. Now suppose X_0 and X_1 are quasi-Banach lattices on M. If $0<\theta<1$, the *Calderón product* $X_0^{1-\theta} X_1^\theta$ of X_0 and X_1 is defined to be the set of μ-measurable functions u on M such that there exists $v \in X_0$ with $\|v\|_{X_0} \leqslant 1$, $w \in X_1$ with $\|w\|_{X_1} \leqslant 1$, and $\lambda>0$ such that

$$|u(x)| \leqslant \lambda\, |v(x)|^{1-\theta}\, |w(x)|^\theta \qquad \text{for} \quad \mu\text{-a.e. } x. \tag{8.1}$$

We set

$$\|u\|_{X_0^{1-\theta} X_1^\theta} = \inf\{\lambda>0\colon (8.1) \text{ holds with } \|v\|_{X_0} \leqslant 1 \text{ and } \|w\|_{X_1} \leqslant 1\}.$$

Although restricted to the case of a lattice, the Calderón product has the advantage of being defined in the quasi-Banach case, and, frequently, of being easy to compute. It has the disadvantage that the interpolation property (i.e., the property that a linear transformation T bounded on X_0 and X_1 should be bounded on the space in between) is not clear, in general. However, we have an elementary substitute; recall that a linear operator T on a quasi-Banach lattice X is called *positive* if $T(f) \geqslant 0$ whenever $f \geqslant 0$.

PROPOSITION 8.1. *Let X_i and Y_i be quasi-Banach lattices, and let T be a positive linear operator bounded from X_i to Y_i with operator norm $\|T\|_i$, $i=0,1$. Then T is bounded from the Calderón product $X_0^{1-\theta} X_1^\theta$ to the Calderón product $Y_0^{1-\theta} Y_1^\theta$, $0<\theta<1$, with operator norm $\|T\|_\theta$ satisfying*

$$\|T\|_\theta \leqslant \|T\|_0^{1-\theta}\, \|T\|_1^\theta.$$

Proof. This is just Hölder's inequality: the usual proof of this inequality, replacing the integral by the positive operator T, shows that

$$T(|f_0|^{1-\theta}\, |f_1|^\theta) \leqslant T(|f_0|)^{1-\theta}\, T(|f_1|)^\theta.$$

The rest follows now by the definition of the Calderón product. Namely, suppose $f \in X_0^{1-\theta} X_1^\theta$ and $M=(1+\varepsilon)\, \|f\|_{X_0^{1-\theta} X_1^\theta}$, for an arbitrary $\varepsilon>0$.

Then there exist $f_0 \in X_0$ and $f_1 \in X_1$ such that $\|f_0\|_{X_0} \leqslant 1$, $\|f_1\|_{X_1} \leqslant 1$, and $|f| \leqslant M\, |f_0|^{1-\theta}\, |f_1|^{\theta}$. Hence,

$$|Tf| \leqslant \|T\|_0^{1-\theta}\, \|T\|_1^{\theta}\, M(T(|f_0|)/\|T\|_0)^{1-\theta}\, (T(|f_1|)/\|T\|_1)^{\theta},$$

and, consequently, $Tf \in Y_0^{1-\theta} Y_1^{\theta}$ with $\|Tf\|_{Y_0^{1-\theta}Y_1^{\theta}} \leqslant \|T\|_0^{1-\theta}\, \|T\|_1^{\theta}\, M$. Here we have used the facts that $|Tf| \leqslant T(|f|)$ for any positive operator T and that g and $|g|$ have the same norm in a quasi-Banach lattice. Letting $\varepsilon \to 0$ gives the conclusion. ∎

As we pointed out above, the $\dot{\mathbf{F}}_p^{\alpha q}$-spaces are not Banach lattices in general, but the sequence spaces $\dot{\mathbf{f}}_p^{\alpha q}$ are. Let M be the sequence space indexed by the dyadic cubes $Q \subseteq \mathbb{R}^n$ and let μ be counting measure on M. For sequences $s = \{s_Q\}$, $t = \{t_Q\}$, and $r = \{r_Q\}$, (8.1) becomes simply $|s_Q| \leqslant \lambda\, |r_Q|^{1-\theta}\, |t_Q|^{\theta}$ for all Q.

THEOREM 8.2. *Suppose* $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, p_1 \leqslant +\infty$, $0 < q_0, q_1 \leqslant +\infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$, *and* $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$. *Then*

$$\dot{\mathbf{f}}_p^{\alpha q} \approx (\dot{\mathbf{f}}_{p_0}^{\alpha_0 q_0})^{1-\theta}\, (\dot{\mathbf{f}}_{p_1}^{\alpha_1 q_1})^{\theta}.$$

Proof. We shall first prove the theorem assuming that $p_0, p_1 < +\infty$. Let $X_0 = \dot{\mathbf{f}}_{p_0}^{\alpha_0 q_0}$ and $X_1 = \dot{\mathbf{f}}_{p_1}^{\alpha_1 q_1}$. Suppose $s \in X_0^{1-\theta} X_1^{\theta}$. Given $\varepsilon > 0$, let $B = (1+\varepsilon)\, \|s\|_{X_0^{1-\theta}X_1^{\theta}}$. Then there exist sequences r and t such that $\|r\|_{X_0} \leqslant 1$, $\|t\|_{X_1} \leqslant 1$, and $|s_Q| \leqslant B\, |r_Q|^{1-\theta}\, |t_Q|^{\theta}$ for all Q. Applying Hölder's inequality with conjugate exponents $q_0/(1-\theta)q$ and $q_1/q\theta$ yields

$$\begin{aligned}
&\left(\sum_Q (|Q|^{-\alpha/n}\, |s_Q|\, \tilde{\chi}_Q)^q\right)^{1/q} \\
&\quad\leqslant B\left(\sum_Q (|Q|^{-\alpha_0/n}\, |r_Q|\, \tilde{\chi}_Q)^{q(1-\theta)}\, (|Q|^{-\alpha_1/n}\, |t_Q|\, \tilde{\chi}_Q)^{q\theta}\right)^{1/q} \\
&\quad\leqslant B\left(\sum_Q (|Q|^{-\alpha_0/n}\, |r_Q|\, \tilde{\chi}_Q)^{q_0}\right)^{(1-\theta)/q_0} \left(\sum_Q (|Q|^{-\alpha_1/n}\, |t_Q|\, \tilde{\chi}_Q)^{q_1}\right)^{\theta/q_1}. \qquad (8.2)
\end{aligned}$$

Applying Hölder's inequality again with conjugate indices $p_0/(1-\theta)p$ and $p_1/p\theta$ and letting $\varepsilon \to 0$ gives

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant \|s\|_{X_0^{1-\theta}X_1^{\theta}}\, \|r\|_{X_0}^{1-\theta}\, \|t\|_{X_1}^{\theta} \leqslant \|s\|_{X_0^{1-\theta}X_1^{\theta}}. \qquad (8.3)$$

Now suppose $s \in \dot{\mathbf{f}}_p^{\alpha q}$, and, to begin with, $q_0, q_1 < +\infty$. To prove the estimate converse to (8.3), we may assume $p_0/q_0 \leqslant p_1/q_1$, since the contrary case follows from this one by interchanging X_0 with X_1 and θ with $1-\theta$.

For $k \in \mathbb{Z}$, let

$$A_k = \left\{ x : \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q(x))^q \right)^{1/q} > 2^k \right\}$$

and

$$C_k = \{Q \text{ dyadic: } |Q \cap A_k| \geqslant |Q|/2 \text{ and } |Q \cap A_{k+1}| < |Q|/2\}.$$

Note that if $Q \notin \bigcup_{k \in \mathbb{Z}} C_k$, then $s_Q = 0$. Define sequences r and t by setting

$$r_Q = (|s_Q|/A_Q)^{q/q_0} \qquad \text{and} \qquad t_Q = (|s_Q|/B_Q)^{q/q_1},$$

where

$$A_Q = 2^{k\gamma} |Q|^u, \qquad u = \frac{\alpha}{n} + \frac{1}{2} - \frac{q_0}{q}\left[\frac{\alpha_0}{n} + \frac{1}{2}\right]$$

and

$$B_Q = 2^{k\delta} |Q|^v, \qquad v = \frac{\alpha}{n} + \frac{1}{2} - \frac{q_1}{q}\left[\frac{\alpha_1}{n} + \frac{1}{2}\right],$$

if $Q \in C_k$, and $r_Q = t_Q = 0$ if $Q \notin \bigcup_{k \in \mathbb{Z}} C_k$, for

$$\gamma = 1 - pq_0/qp_0 \qquad \text{and} \qquad \delta = 1 - pq_1/qp_1.$$

A calculation shows that $|s_Q| = |r_Q|^{1-\theta} |t_Q|^{\theta}$. We would like to prove that

$$\|r\|_{X_0} \leqslant C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}^{p/p_0} \qquad \text{and} \qquad \|t\|_{X_1} \leqslant C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}^{p/p_1}. \tag{8.4}$$

Assuming (8.4) for the moment, we have

$$|s_Q| = C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}} (r_Q / C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}^{p/p_0})^{1-\theta} (t_Q / C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}^{p/p_1})^{\theta}.$$

Thus (8.4) yields $\|s\|_{X_0^{1-\theta} X_1^{\theta}} \leqslant C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}$.

To prove (8.4), we notice that by Proposition 2.7 we have

$$\begin{aligned} \|r\|_{X_0}^{p_0} &\leqslant C \int \left(\sum_{k \in \mathbb{Z}} \sum_{Q \in C_k} |Q|^{-(\alpha_0/n + 1/2) q_0} A_Q^{-q} |s_Q|^q \chi_{Q \cap A_k} \right)^{p_0/q_0} \\ &\leqslant C \int \left(\sum_{k \in \mathbb{Z}} \chi_{A_k} \sum_{Q \in C_k} 2^{-k\gamma q} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q)^q \right)^{p_0/q_0}. \end{aligned}$$

On the set A_k, $2^k < (\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q(x))^q)^{1/q}$, and since $\gamma \leqslant 0$, we may replace 2^k by this larger quantity. The right-hand side can then be estimated by $C \|s\|_{\dot{\mathbf{f}}_p^{\alpha q}}^p$.

The second estimate in (8.4) is similar. Replace α_0, q_0, p_0, and γ with α_1, q_1, p_1, and δ, respectively, and $\chi_{Q \cap A_k}$ with $\chi_{Q \cap A_{k+1}^c}$. Since $\delta \geqslant 0$ we can estimate $2^{-(k+1)}$ by $(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q(x))^q)^{-1/q}$ on A_{k+1}^c, and this leads to the desired conclusion.

If $q_1 = +\infty$ and $q_0 < +\infty$, the same arguments with the usual interpretation work; take A_Q as above and $t_Q = A_Q^{1/\theta}$, to get (8.4). Similarly if $q_1 < +\infty$ and $q_0 = +\infty$. If $q_0 = q_1 = +\infty$, selecting r_Q and t_Q so that

$$(|Q|^{-\alpha_0/n} |r_Q| \tilde{\chi}_Q)^{p_0} = (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q)^p = (|Q|^{-\alpha_1/n} |t_Q| \tilde{\chi}_Q)^{p_1}$$

yields equality in (8.4) and, hence, the result.

This proves the theorem when p_0 and p_1 are finite. Let us now indicate the (minor) changes necessary to remove this restriction.

The idea is simply to use Corollary 5.6. We first consider the case when only one is infinite, say $p_0 < +\infty$ and $p_1 = +\infty$. Assume for now, in addition, $q_0, q_1 < +\infty$. Corollary 5.6 then provides us with a set E_Q for each dyadic cube Q. In (8.2) we replace $\tilde{\chi}_Q$ by $\tilde{\chi}_{E_Q}$ and estimate the second factor by its L^∞-norm. By using Proposition 2.7 and Corollary 5.6 (in the other direction), we get $\|s\|_{\dot{f}_p^{\alpha q}} \leqslant C \|s\|_{X_0^{1-\theta} X_1^\theta}$ in place of (8.3). To show the converse we let $\delta = 1$ and define r and t as before. We then need to replace the estimate involving t in (8.4) by

$$\|t\|_{X_1} \leqslant C.$$

This inequality follows as before once we notice that by Corollary 5.6 with $E_Q = Q \cap A_{k+1}^c$,

$$\|t\|_{X_1} \leqslant C \left\| \left(\sum_{k \in \mathbb{Z}} \sum_{Q \in C_k} |Q|^{-(\alpha_1/n + 1/2) q_1} B_Q^{-q} |s_Q|^q \chi_{Q \cap A_{k+1}^c} \right)^{1/q_1} \right\|_{L^\infty}.$$

The case when $p_0 < +\infty$, $q_0 \leqslant +\infty$, and $p_1 = q_1 = +\infty$ is elementary. The first part of the proof is direct; for the second part, replace the definitions of r_Q and t_Q given there by

$$r_Q = (|s_Q| / |Q|^{\theta(\alpha_1/n + 1/2)})^{1/(1-\theta)} \qquad \text{and} \qquad t_Q = |Q|^{\alpha_1/n + 1/2}. \tag{8.5}$$

The case when both p_0 and p_1 are infinite (and q_0 and q_1 finite) is, in fact, the most complicated. However, it still only requires changes along similar lines. Suppose $r \in X_0 = \dot{f}_\infty^{\alpha_0 q_0}$ and $t \in X_1 = \dot{f}_\infty^{\alpha_1 q_1}$. According to Corollary 5.6, applied with $\frac{1}{2}$ replaced by $\varepsilon = \frac{9}{10}$, there exist for each dyadic cube Q subsets $E_Q^0 = E_Q^0(r)$ with $|E_Q^0| / |Q| > \frac{9}{10}$ and

$\|r\|_{X_0} \approx \|(\sum_Q (|Q|^{-\alpha_0/n} |r_Q| \tilde{\chi}_{E_Q^0})^{q_0})^{1/q_0}\|_{L^\infty}$. There are also sets E_Q^1 corresponding to t with a similar property. Notice that $E_Q \equiv E_Q^0 \cap E_Q^1$ satisfies $|E_Q|/|Q| > \frac{8}{10}$. In (8.2) we now replace $\tilde{\chi}_Q$ by $\tilde{\chi}_{E_Q}$ and argue as before to obtain $\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha q}} \leqslant C \|s\|_{X_0^{1-\theta} X_1^\theta}$. To prove the converse we must modify the definition of the sets A_k and C_k. Suppose $s \in \dot{\mathbf{f}}_\infty^{\alpha q}$. Corollary 5.6, again with $\varepsilon = \frac{9}{10}$, then gives us sets E_Q satisfying, in particular, $|E_Q|/|Q| > \frac{9}{10}$. For $k \in \mathbb{Z}$ we let

$$\mathscr{A}_k = \left\{x : \left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_{E_Q}(x))^q\right)^{1/q} > 2^k\right\},$$

and define $\mathscr{C}_k$ analogous to the sets C_k above with A_k replaced by $\mathscr{A}_k$. Note that, since $|E_Q|/|Q| > \frac{9}{10}$, we must still have $s_Q = 0$ if $Q \notin \bigcup_{k \in \mathbb{Z}} \mathscr{C}_k$. Regard $p/p_0 = p/p_1 = 1$ in the definition of δ and γ and in (8.4). In the proof of (8.4) we now replace $Q \cap A_k$ by $E_Q \cap \mathscr{A}_k$ if $Q \in \mathscr{C}_k$; note that $|E_Q \cap \mathscr{A}_k|/|Q| > \frac{9}{10} - \frac{1}{2} = \frac{4}{10}$. Similarly, we replace $Q \cap A_{k+1}^c$ by $E_Q \cap \mathscr{A}_{k+1}^c$. The estimates of $\|r\|_{X_0}$ and $\|t\|_{X_1}$ can now be carried out as before.

The final case is $p_0 = p_1 = +\infty$ and $q_0 \leqslant +\infty$, $q_1 = +\infty$. Now one direction is immediate from the definitions. For the other we again define r and t by (8.5). Instead of (8.4) we then have $\|r\|_{X_0} \leqslant C \|s\|_{\dot{\mathbf{f}}_\infty^{\alpha q}}^{1/(1-\theta)}$ and $\|t\|_{X_1} = 1$. ∎

We can obtain results regarding complex interpolation from Theorem 8.2. Let $[X_0, X_1]_\theta$ denote the space obtained from X_0 and X_1 by the complex interpolation method (cf. [Cal1; or Be-L, Chap. 4]). Suppose X_0 and X_1 are Banach lattices on a measure space (M, μ), and let $X = X_0^{1-\theta} X_1^\theta$ for some $\theta \in (0, 1)$. Suppose X has the property that the conditions $f \in X$, $|f_n(x)| \leqslant |f(x)|$, μ-a.e. for each $n \in \mathbb{Z}^+$, and $\lim_{n\to\infty} f_n = 0$, μ-a.e., imply $\lim_{n\to\infty} \|f_n\|_X = 0$. Calderón [Cal1, p. 125] then shows that $X_0^{1-\theta} X_1^\theta = [X_0, X_1]_\theta$. Hence, we obtain the following.

Corollary 8.3. *Suppose* $\alpha_0, \alpha_1 \in \mathbb{R}$, $1 \leqslant p_0, q_0 < +\infty$, $1 \leqslant p_1, q_1 \leqslant +\infty$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then*

$$[\dot{\mathbf{f}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{f}}_{p_1}^{\alpha_1 q_1}]_\theta \approx \dot{\mathbf{f}}_p^{\alpha q} \tag{8.6}$$

and

$$[\dot{\mathbf{F}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1 q_1}]_\theta \approx \dot{\mathbf{F}}_p^{\alpha q}. \tag{8.7}$$

Proof. Since $p, q < +\infty$, the property needed to apply Calderón's result to $X = \dot{\mathbf{f}}_p^{\alpha q}$ follows easily from the dominated convergence theorem. Hence Theorem 8.2 yields (8.6). Now (8.7) follows from (8.6) and the retraction diagram (Theorems 2.2 and 5.2), as usual. ∎

Special cases of (8.7) are $[H^1, \mathrm{BMO}]_\theta = L^p$, $1/p = 1-\theta$, and $[H^1, L^{p_1}]_\theta = L^p$, $1/p = 1-\theta+\theta/p_1$, $p_1 < +\infty$, which are well known (see, e.g., the survey in [Jon]). The result for general α's and q's, at least when $1 < p_i, q_i < +\infty$, $i=0, 1$, is due to Triebel (cf. [Tr2]). Some of the results in the extreme cases $p_i = 1$ and/or $p_i = +\infty$, $i=0, 1$, may be new.

In the case where we let $q_0 = q_1 = +\infty$ and/or $p_0 = p_1 = +\infty$ in the setting of Corollary 8.3, it follows from Theorem 8.2 and a general result of Shestakov ([Sh], or see [N, p. 140]) that $[\dot{\mathbf{f}}_{p_0}^{\alpha_0\infty}, \dot{\mathbf{f}}_{p_1}^{\alpha_1\infty}]_\theta$ is the closure of $\dot{\mathbf{f}}_{p_0}^{\alpha_0\infty} \cap \dot{\mathbf{f}}_{p_1}^{\alpha_1\infty}$ in $\dot{\mathbf{f}}_p^{\alpha\infty}$ and that $[\dot{\mathbf{f}}_\infty^{\alpha_0 q_0}, \dot{\mathbf{f}}_\infty^{\alpha_1 q_1}]_\theta$ is the closure of $\dot{\mathbf{f}}_\infty^{\alpha_0 q_0} \cap \dot{\mathbf{f}}_\infty^{\alpha_1 q_1}$ in $\dot{\mathbf{f}}_\infty^{\alpha q}$.

We could also consider (8.6)–(8.7) for $p, q < 1$; it turns out that there are many ways to extend the complex method to the quasi-Banach space case (e.g., [Riv; Cal-T; J-J; Tr2]). However, we will not pursue this. Instead we consider two alternate methods of interpolation whose extension to the quasi-Banach case is straightforward and for which the interpolation property is immediate.

The first of these methods is due to Gagliardo and is denoted $\langle A_0, A_1 \rangle_\theta$ in [N] (cf. also [O; P6]). In [N] it is proved that for quasi-Banach lattices satisfying certain conditions (easily checked for the $\dot{\mathbf{f}}$-spaces),

$$\langle X_0, X_1 \rangle_\theta = (X_0^{1-\theta} X_1^\theta)^\circ. \tag{8.8}$$

Here, in general, X° denotes the closure of $X_0 \cap X_1$ in X. This now readily yields the following result for the $\dot{\mathbf{f}}$- and $\dot{\mathbf{F}}$-spaces.

COROLLARY 8.4. *Suppose* $\alpha_0, \alpha_1 \in \mathbb{R}$, $0 < p_0, q_0 < +\infty$, $0 < p_1, q_1 \leqslant +\infty$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, *and* $1/q = (1-\theta)/q_0 + \theta/q_1$. *Then*

$$\langle \dot{\mathbf{f}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{f}}_{p_1}^{\alpha_1 q_1} \rangle_\theta \approx \dot{\mathbf{f}}_p^{\alpha q} \tag{8.9}$$

and

$$\langle \dot{\mathbf{F}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1 q_1} \rangle_\theta \approx \dot{\mathbf{F}}_p^{\alpha q}. \tag{8.10}$$

Proof. By the trivial fact that finite sequences are dense in $\dot{\mathbf{f}}_p^{\alpha q}$, $p, q < +\infty$, we have that $\dot{\mathbf{f}}_p^{\alpha q} = (\dot{\mathbf{f}}_p^{\alpha q})^\circ$. Hence, (8.9) follows by combining Theorem 8.2 and (8.8). By the standard retraction argument we then also get (8.10). ∎

Corollaries 8.3–8.4 yield the interpolation property for the $\dot{\mathbf{f}}$- and $\dot{\mathbf{F}}$-spaces for nearly all possible values of the indices α, p, and q. However, there is another method of interpolation which very easily yields the interpolation property for all possible values of the indices. This method, due to

Gagliardo, Peetre [P6], and Gustavsson and Peetre [Gu-P], is known as the *$\pm$-method of interpolation* and is denoted $\langle A_0, A_1, \theta\rangle$ in [N].

Again under appropriate mild conditions, which are satisfied in the case we are about to discuss, we have

$$X_0^{1-\theta}X_1^{\theta} \subset \langle X_0, X_1, \theta\rangle \subset (X_0^{1-\theta}X_1^{\theta})^{\sim} \tag{8.11}$$

(see [N]). Here, in general, $X^{\sim}$ is the Gagliardo closure of X in X_0+X_1; recall that $x \in X^{\sim}$ if and only if there exists $\{x_i\}_{i=0}^{\infty}$ with $x_i \in X$ such that $x_i \to x$ in X_0+X_1 as $i \to +\infty$ and $\|x_i\|_X \leqslant \lambda$. The norm on $X^{\sim}$ is then the infimum of all such λ's.

THEOREM 8.5. *Suppose $\alpha_0, \alpha_1 \in \mathbb{R}$, $0<p_0, q_0 \leqslant +\infty$, $0<p_1, q_1 \leqslant +\infty$, $\alpha=(1-\theta)\alpha_0+\theta\alpha_1$, $1/p=(1-\theta)/p_0+\theta/p_1$, and $1/q=(1-\theta)/q_0+\theta/q_1$. Then*

$$\langle \dot{\mathbf{f}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{f}}_{p_1}^{\alpha_1 q_1}, \theta\rangle \approx \dot{\mathbf{f}}_p^{\alpha q} \tag{8.12}$$

and

$$\langle \dot{\mathbf{F}}_{p_0}^{\alpha_0 q_0}, \dot{\mathbf{F}}_{p_1}^{\alpha_1 q_1}, \theta\rangle \approx \dot{\mathbf{F}}_p^{\alpha q}. \tag{8.13}$$

Proof. Theorem 8.2 and (8.11) will give (8.12) as soon as we have verified that

$$\dot{\mathbf{f}}_p^{\alpha q} = (\dot{\mathbf{f}}_p^{\alpha q})^{\sim}. \tag{8.14}$$

Trivially $\dot{\mathbf{f}}_p^{\alpha q} \subset (\dot{\mathbf{f}}_p^{\alpha q})^{\sim}$, so we need to prove the converse. Suppose $s \in (\dot{\mathbf{f}}_p^{\alpha q})^{\sim}$. By definition this means that there exists a sequence $\{s_i\}_{i=0}^{\infty}$ such that $\|s_i\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant \lambda$ and $s_i \to s$ in $\dot{\mathbf{f}}_{p_0}^{\alpha_0 q_0} + \dot{\mathbf{f}}_{p_1}^{\alpha_1 q_1}$ as $i \to +\infty$. In other words, there are sequences $\{s_i^0\}_{i=0}^{\infty}$ and $\{s_i^1\}_{i=0}^{\infty}$ such that

$$s - s_i = s_i^0 + s_i^1$$

and

$$\|s_i^j\|_{\dot{\mathbf{f}}_{p_j}^{\alpha_j q_j}} \to 0, \qquad j=0, 1,$$

as $i \to +\infty$. Since $|(s_i^j)_Q| \leqslant |Q|^{\alpha_j/n+1/2-1/p_j} \|s_i^j\|_{\dot{\mathbf{f}}_{p_j}^{\alpha_j q_j}}$, we have $(s_i^j)_Q \to 0$, $j=0, 1$, and, using the identity above, $(s_i)_Q \to s_Q$ as $i \to +\infty$ for each cube Q. This implies that

$$\left(\sum_Q (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q)^q\right)^{1/q} \leqslant \liminf_{i\to\infty} \left(\sum_Q (|Q|^{-\alpha/n} |(s_i)_Q| \tilde{\chi}_Q)^q\right)^{1/q}$$

(when $p_0=p_1=+\infty$ we should modify this slightly and only sum over cubes $Q \subset P$ for an arbitrary fixed cube P). This yields

$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant \liminf_{i\to\infty} \|s_i\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant \lambda$, by Fatou's lemma, and completes the proof of (8.14). Using the retract diagrams again we also obtain (8.13). ∎

Notice, in particular, that BMO $= \dot{\mathbf{F}}_\infty^{02}$ can be obtained by interpolating between $\dot{\mathbf{F}}_\infty^{01}$ and $\dot{\mathbf{F}}_\infty^{0\infty}$, for instance. This answers a question recorded in [P8]. Similarly, the theorem tells us how to interpolate between $\dot{\mathbf{F}}_\infty^{02}$ and $\dot{\mathbf{F}}_\infty^{0\infty}$, and this yields the analogous results for BMOA and the Bloch-space, giving at least a partial solution to a problem by Peetre [P9, p. 237].

9. The Algebra of Almost Diagonal Operators

In this section we further discuss the class of almost diagonal matrices (introduced in Section 3). We prove (Theorem 9.1) that this class is closed under composition and often under taking inverses. The class of all operators on the distribution space level, which correspond to almost diagonal matrices, is then also an algebra under composition. We consider various characterizations of this algebra and of the families of distributions naturally associated with it. These distribution families generalize families of smooth molecules; however, in the case $\alpha = 0$ the two notions coincide (Theorem 9.15). We will see that our algebra contains fairly general families of Calderón–Zygmund and Fourier multiplier operators (Examples 9.18 and 9.19). We have again put some of the proofs in an appendix, Appendix D.

For fixed α, p, and q, the class of almost diagonal operators of $\dot{\mathbf{f}}_p^{\alpha q}$ can be made into a normed space. Given an almost diagonal operator A with matrix $\{a_{QP}\}_{Q,P}$, we define

$$\|A\|_\varepsilon = \sup_{Q,P} |a_{QP}|/\omega_{QP}(\varepsilon)$$

and

$$\|A\|_{\mathbf{ad}} = \inf_{\varepsilon>0} \|A\|_\varepsilon.$$

Since $\|A\|_\varepsilon$ is a nondecreasing function of ε, it easily follows that $\|A\|_{\mathbf{ad}}$ is a norm. We denote the class of almost diagonal operators on $\dot{\mathbf{f}}_p^{\alpha q}$ equipped with this norm by $\mathbf{ad}$ or $\mathbf{ad}_p^{\alpha q}$.

Theorem 9.1. *Let $\alpha \in \mathbb{R}$ and $0 < p, q \leqslant +\infty$.*

(i) *If $A, B \in \mathbf{ad}_p^{\alpha q}$, then $A \circ B \in \mathbf{ad}_p^{\alpha q}$.*

(ii) *There exists $\delta = \delta(\varepsilon) > 0$ such that if $A \in \mathbf{ad}_p^{\alpha q}$ and $\|I - A\|_\varepsilon < \delta$, then A is invertible and $A^{-1} \in \mathbf{ad}_p^{\alpha q}$.*

Proof. See Appendix D. ∎

In Theorems 3.5 and 3.7 we obtained certain basic estimates under relatively restrictive conditions. We now consider the most general families of functions for which these estimates hold (see Theorem 9.9). The fact that $\mathbf{ad}_p^{\alpha q}$ is an algebra (Theorem 9.1(i)) is the crucial property used in analyzing these families.

For each dyadic cube P, let e^P be the sequence defined by $(e^P)_Q = 1$ if $P = Q$ and $= 0$ otherwise. Each operator $A \in \mathbf{ad}_p^{\alpha q}$ corresponds to a family of sequences $\{Ae^P\}_P$ obtained by perturbing the standard unit basis $\{e^P\}_P$ by A. That is, if A is represented by the matrix $\{a_{QP}\}_{Q,P}$, then Ae^P is the sequence with $(Ae^P)_Q = a_{QP}$. Theorem 9.1 implies that the collection of all such families,

$$U = \{\{Ae^P\}_P : Ae^P = \{a_{QP}\}_Q \text{ for some } A \in \mathbf{ad}_p^{\alpha q}\},$$

is stable under $\mathbf{ad}_p^{\alpha q}$. Notice also that since e^P is an element of $\dot{\mathbf{f}}_p^{\alpha q}$ and an almost diagonal operator is bounded, each sequence Ae^P also belongs to $\dot{\mathbf{f}}_p^{\alpha q}$.

Let us now fix φ and ψ satisfying (2.1)–(2.4). The image of the collection U under the inverse φ-transform T_ψ consists of families $\{m_P\}_P$ of elements $m_P = T_\psi Ae^P = \sum_Q a_{QP}\psi_Q \in \dot{\mathbf{F}}_p^{\alpha q}$. By Theorems 9.1, 2.2, and 5.2 this collection is precisely

$$M = M(\varphi) = \{\{m_P\}_P : \{\langle m_P, \varphi_Q\rangle\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}\}.$$

We shall say that $\{m_P\}_P \in M$ is an **Ad**-*family* (or $\mathbf{Ad}_p^{\alpha q}$-*family*). By Lemma 3.6, any family of smooth molecules is an **Ad**-*family*.

The definition of **Ad**-family is independent of the choice of φ.

PROPOSITION 9.2. *Suppose $\varphi^{(1)}$, $\psi^{(1)}$ and $\varphi^{(2)}$, $\psi^{(2)}$ each satisfy (2.1)–(2.4). Then $M(\varphi^{(1)}) = M(\varphi^{(2)})$.*

Proof. Suppose $\{m_P\}_P \in M(\varphi^{(2)})$. By Theorems 2.2 and 5.2,

$$m_P = \Sigma_R \langle m_P, \varphi_R^{(2)}\rangle\, \psi_R^{(2)}.$$

Hence,

$$\langle m_P, \varphi_Q^{(1)}\rangle = \sum_R \langle m_P, \varphi_R^{(2)}\rangle\langle \psi_R^{(2)}, \varphi_Q^{(1)}\rangle.$$

This corresponds to the composition of two operators with matrices $\{\langle m_P, \varphi_Q^{(2)}\rangle\}_{Q,P}$ and $\{\langle \psi_P^{(2)}, \varphi_Q^{(1)}\rangle\}_{Q,P}$, respectively. By assumption, $\{\langle m_P, \varphi_Q^{(2)}\rangle\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$, and $\{\langle \psi_P^{(2)}, \varphi_Q^{(1)}\rangle\}_{Q,P}$ belongs to all $\mathbf{ad}_p^{\alpha q}$-spaces. Theorem 9.1(i) implies that the composition is also in $\mathbf{ad}_p^{\alpha q}$ which means that $\{m_P\}_P \in M(\varphi^{(1)})$. By symmetry, this completes the proof. ∎

In fact, there are many other equivalent characterizations as we shall see next. Some of the ideas in the discussion below can be traced to Peetre's book [P3, Chap. 8] and the references given there.

Given families $\{\sigma_Q\}_Q$ and $\{\eta_Q\}_Q$, let $\{\langle\sigma_P|\eta_Q\rangle\}_{Q,P}$ be the matrix with entries

$$\langle\sigma_P|\eta_Q\rangle \equiv \sup_{y\in P_0} |\langle\sigma_P^y, \eta_Q\rangle|,$$

where $\sigma_P^y(\cdot)=\sigma_P(\cdot\,-y)$ and P_0 is the cube centered at the origin with twice the sidelength of P. We say that $\{\sigma_P\}_P$ is a *uniform* $\mathbf{Ad}_p^{\alpha q}$-family if the matrix $\{\langle\sigma_P|\varphi_Q\rangle\}_{Q,P}\in\mathbf{ad}_p^{\alpha q}$. Lemma 3.6 implies that a family of smooth molecules is a uniform **Ad**-family, and it is easy to see that this new notion is less restrictive in general.

We are interested in finding conditions on families $\{\sigma_P\}_P$ which enable us to characterize **Ad**-families $\{m_P\}_P$ in terms of the matrix $\{\langle\sigma_Q|m_P\rangle\}_{Q,P}$. Recall that $J=n/\min(1,p,q)$.

THEOREM 9.3. *Let $\alpha\in\mathbb{R}$ and $0<p, q\leqslant+\infty$, and let $\{\sigma_P\}_P$ be a uniform $\mathbf{Ad}_p^{\beta q}$-family with $\beta=-\alpha+J-n$. If $\{m_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family, then*

$$\{\langle\sigma_Q|m_P\rangle\}_{Q,P}=\{\sup_{y\in Q_0}|\langle m_P,\sigma_Q^y\rangle|\}_{Q,P}\in\mathbf{ad}_p^{\alpha q}. \tag{9.1}$$

Proof. According to Proposition 9.2 we may assume that the function φ in the definition of $M(\varphi)$ is such that $\varphi=\psi$. Now, by Theorems 2.2 and 5.2, we have the representation $\sigma_Q^y=\sum_R\langle\sigma_Q^y,\varphi_R\rangle\,\varphi_R$ and, consequently,

$$\langle m_P,\sigma_Q^y\rangle=\sum_R\overline{\langle\sigma_Q^y,\varphi_R\rangle}\langle m_P,\varphi_R\rangle.$$

This implies, with the notation above, that

$$\langle\sigma_Q|m_P\rangle\leqslant\sum_R\langle\sigma_Q|\varphi_R\rangle|\langle m_P,\varphi_R\rangle|.$$

Notice that $\{\langle\sigma_Q|\varphi_P\rangle\}_{Q,P}\in\mathbf{ad}_p^{\alpha q}$; using the definition of $\omega_{QP}(\varepsilon)$, it is readily seen that this is equivalent to our assumption that the transpose $\{\langle\sigma_P|\varphi_Q\rangle\}_{Q,P}\in\mathbf{ad}_p^{\beta q}$, with $\beta=-\alpha+J-n$. Thus $\{\langle\sigma_Q|m_P\rangle\}_{Q,P}$ corresponds to and operator dominated by the composition of two operators in $\mathbf{ad}_p^{\alpha q}$, and, by Theorem 9.1(i), this operator is in $\mathbf{ad}_p^{\alpha q}$ as well.

A result in the opposite direction, sufficient for our purposes, is the following.

THEOREM 9.4. *Suppose* $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$, *and let* σ *be a function satisfying*

$$\hat{\sigma}(\xi) \in C^\infty, \qquad |\hat{\sigma}(\xi)| \geqslant c > 0 \qquad \text{if } \tfrac{1}{2} \leqslant |\xi| \leqslant 2. \tag{9.2}$$

Set $\sigma_P(\cdot) = |P|^{-1/2} \sigma(2^\nu \cdot - k)$ *if* $P = P_{\nu k}$. *Suppose* $\{\sigma_P\}_P$ *is a uniform* $\mathbf{Ad}_p^{\beta q}$*-family with* $\beta = -\alpha + J - n$. *If* (9.1) *holds, then* $\{m_P\}_P$ *is an* $\mathbf{Ad}_p^{\alpha q}$*-family.*

Proof. Our assumptions on σ guarantee that there is function $\tilde{\varphi}$ satisfying (2.1)–(2.3) such that $\varphi = \tilde{\varphi} * \sigma$. Hence, $\varphi_Q = \tilde{\varphi}_\nu * \sigma_Q$ if $l(Q) = 2^{-\nu}$ and

$$\langle m_P, \varphi_Q \rangle = \int \langle m_P, \sigma_Q^y \rangle \bar{\tilde{\varphi}}_\nu(y)\, dy = \sum_l \int_{Q_{\nu l}} \langle m_P, \sigma_Q^y \rangle \bar{\tilde{\varphi}}_\nu(y)\, dy.$$

We define $\{a_{QP}\}_{Q,P}$ to be the matrix with entries $a_{QP} = 0$ if $l(Q) \neq l(P)$ and $a_{QP} = \sup_{y \in Q_{\nu,l-k}} 2^{-\nu n} |\tilde{\varphi}_\nu(y)| = \sup_{y \in Q_{0,l-k}} |\tilde{\varphi}(y)|$ if $P = P_{\nu l}$ and $Q = Q_{\nu k}$. Clearly this matrix corresponds to an operator in $\mathbf{ad}_p^{\alpha q}$, since $\tilde{\varphi}$ is rapidly decreasing. The identity above shows that

$$|\langle m_P, \varphi_Q \rangle| \leqslant \sum_R \langle \sigma_R | m_P \rangle\, a_{QR}.$$

Once again we thus obtain an estimate involving the composition of two operators in $\mathbf{ad}_p^{\alpha q}$ and, applying Theorem 9.1(i), we get the desired conclusion. ∎

Assume for a moment that (9.2) holds and that $\sigma_P(\cdot) = |P|^{-1/2} \sigma(2^\nu \cdot - k)$ if $P = P_{\nu k}$. Then under a certain condition, namely that $\{\sigma_P\}_P$ is a uniform $\mathbf{Ad}_p^{\beta q}$-family, we have that (9.1) is equivalent to $\{m_P\}_P$ being an $\mathbf{Ad}_p^{\alpha q}$-family. In fact, this condition on $\{\sigma_P\}_P$ is sharp. To see this, note that $\{\varphi_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family. Hence, given the equivalence, (9.1) holds with $\{m_P\}_P$ replaced by $\{\varphi_P\}_P$. This gives $\{\langle \sigma_Q | \varphi_P \rangle\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$, or, equivalently, $\{\langle \sigma_P | \varphi_Q \rangle\}_{Q,P} \in \mathbf{ad}_p^{\beta q}$.

Remark 9.5. There are also analogues of Theorems 9.3 and 9.4 with the single function σ replaced by a finite family $\{\sigma^i\}_{i=1}^K$. Instead of (9.2) we then require that

$$\hat{\sigma}^i \in C^\infty, \bigcup_i \{\hat{\sigma}^i \neq 0\} \supset \{\tfrac{1}{2} \leqslant |\xi| \leqslant 2\}.$$

Suppose each $\{\sigma_P^i\}_P$ is a uniform $\mathbf{Ad}_p^{\alpha q}$-family with $\beta = -\alpha + J - n$. Then $\{m_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family if and only if

$$\{\langle \sigma_Q^i | m_P \rangle\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}, \qquad i = 1, \ldots, K.$$

The only modification of the proof required in this case is to note that we can find $\{\tilde{\varphi}^i\}_{i=1}^K$, with each $\tilde{\varphi}^i$ satisfying (2.1)–(2.3) such that $\varphi = \sum_i \tilde{\varphi}^i * \sigma^i$.

Remark 9.6. Similarly, $\{m_P\}_P$ is a uniform $\mathbf{Ad}_p^{\alpha q}$-family if and only if

$$\{\sup_{y \in P_0} \sup_{z \in Q_0} |\langle m_P^y, \sigma_Q^z \rangle|\} \in \mathbf{ad}_p^{\alpha q}$$

for a (or, equivalently, any) fixed uniform $\mathbf{Ad}_p^{\beta q}$-family $\{\sigma_P\}_P$, $\beta = -\alpha + J - n$, derived from a single function σ satisfying (9.1). This can be proved in virtually the same way as Theorems 9.3 and 9.4.

EXAMPLE 9.7. Consider for a moment the one-dimensional case. Let $\sigma(x)$ be the sawtooth

$$\sigma(x) = \begin{cases} x, & 0 \leqslant x < 1, \\ 2 - x, & 1 \leqslant x < 2, \\ 0, & 2 \leqslant x, \end{cases} \qquad \sigma(-x) = -\sigma(x),$$

and define σ_P in the usual way. Then $\{\sigma_P\}_P$ is a family of smooth molecules for $\dot{\mathbf{F}}_p^{\alpha q}$ for $-1 < \alpha < 1$, $1 \leqslant p, q \leqslant +\infty$, and thus also a uniform $\mathbf{Ad}_p^{\alpha q}$-family for these values of the parameters. Furthermore, $|\hat{\sigma}(\xi)| > 0$ on $\frac{1}{2} \leqslant |\xi| \leqslant 2$. Combining Theorems 9.3 and 9.4 we see $\{m_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family if and only if (9.1) holds.

EXAMPLE 9.8. Also in $\mathbb{R}^1$, let $\sigma(x)$ be the step function

$$\sigma(x) = \begin{cases} -1, & 0 \leqslant x < 1, \\ 1, & 1 \leqslant x < 2, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\{\sigma_P\}_P$ is a family of smooth molecules for $\dot{\mathbf{F}}_p^{\beta q}$ for $-1 < \beta < 0$, $1 \leqslant p, q \leqslant +\infty$, and $|\hat{\sigma}(\xi)| > 0$ on $\frac{1}{2} \leqslant |\xi| \leqslant 2$. By Theorems 9.3 and 9.4, $\{m_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family, $0 < \alpha < 1$, $1 \leqslant p, q \leqslant +\infty$, if and only if $\{\sup_{y \in Q_0} |\langle m_P, \sigma_Q^y \rangle|\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$. Written out, this becomes a fairly explicit condition on $\{m_P\}_P$. Given a dyadic interval Q, we let Q^+ and Q^- be its right and left halves, respectively. The condition is that

$$|Q|^{-1/2} \left| \int_{Q^+} m_P(x+y)\, dx - \int_{Q^-} m_P(x+y)\, dx \right| \leqslant C\omega_{QP}(\varepsilon), \qquad y \in Q_0,$$

for some $\varepsilon > 0$ and with C independent of y (and Q, P, of course).

Examples 9.7 and 9.8 also have analogues in higher dimensions. For instance, let $\sigma^i(x)$, $i=1, ..., n$, be defined by

$$\sigma^i(x)=\begin{cases} -1, & 0\leqslant x_i<1, \\ 1, & 1\leqslant x_i\leqslant 2, \end{cases} \quad 0\leqslant x_1, ..., x_n\leqslant 2,$$

$\sigma^i(x) = 0$ otherwise. By using Remark 9.5 we easily see that these functions can be used to characterize $\mathbf{Ad}_p^{\alpha q}$-families when $0<\alpha<1$ and $1\leqslant p, q\leqslant +\infty$.

Several of the results we have previously stated for families of smooth molecules have obvious generalizations to $\mathbf{Ad}_p^{\alpha q}$-families. For example, Theorems 3.5 and 3.7 can be generalized as follows.

THEOREM 9.9. *Suppose* $\alpha\in\mathbb{R}$, *and* $0<p, q\leqslant +\infty$.

(i) *If* $f=\sum_Q s_Q m_Q$, *where* $\{m_Q\}_Q$ *is an* $\mathbf{Ad}_p^{\alpha q}$-*family, then*

$$\| f \|_{\dot{\mathbf{F}}_p^{\alpha q}}\leqslant c\, \| \{s_Q\}_Q \|_{\dot{\mathbf{f}}_p^{\alpha q}}.$$

(ii) *If* $f\in\dot{\mathbf{F}}_p^{\alpha q}$ *and* $\{b_Q\}_Q$ *is an* $\mathbf{Ad}_p^{\beta q}$-*family, where* $\beta=-\alpha+J-n$, *then*

$$\| \{\langle f, b_Q\rangle\}_Q \|_{\dot{\mathbf{f}}_p^{\alpha q}}\leqslant c\, \| f \|_{\dot{\mathbf{F}}_p^{\alpha q}}.$$

Proof. For (i), see the proof of Theorem 3.5. For (ii), note that the condition $\{\langle\psi_P, b_Q\rangle\}_{Q,P}\in \mathbf{ad}_p^{\alpha q}$ is the same as our assumption on $\{b_Q\}_Q$ and follow the proof of Theorem 3.7. ∎

Let us now consider operators between $\dot{\mathbf{F}}_p^{\alpha q}$-spaces. Once again, we fix φ and ψ satisfying (2.1)–(2.4). If T is a continuous linear operator from $\mathscr{S}$ to $\mathscr{S}'$, we say that T is *almost diagonal* on $\dot{\mathbf{F}}_p^{\alpha q}$ if the corresponding matrix $\{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P}$ is in $\mathbf{ad}_p^{\alpha q}$. We denote the class of such almost diagonal operators by $\mathbf{Ad}_p^{\alpha q}$. Note that if S and T have associated matrices $A=\{\langle S\psi_P, \varphi_Q\rangle\}_{Q,P}$ and $B=\{\langle T\psi_P, \varphi_Q\rangle\}_{Q,P}$, then by Lemma 2.1, $S\psi_P=\sum_R \langle S\psi_P, \varphi_R\rangle\, \psi_R$. Hence,

$$\langle TS\psi_P, \varphi_Q\rangle=\sum_R \langle S\psi_P, \varphi_R\rangle\langle T\psi_R, \varphi_Q\rangle=(BA)_{QP};$$

i.e., TS has associated matrix BA. Theorem 9.1 thus tells us that $\mathbf{Ad}_p^{\alpha q}$ is an algebra as well.

It is immediate by the definitions that an operator T is in $\mathbf{Ad}_p^{\alpha q}$ if and only if T maps the family $\{\psi_P\}_P$ into an $\mathbf{Ad}_p^{\alpha q}$-family. In fact, such operators map general $\mathbf{Ad}_p^{\alpha q}$-families into $\mathbf{Ad}_p^{\alpha q}$-families.

PROPOSITION 9.10. *Suppose* $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$, *and let* $T \in \mathbf{Ad}_p^{\alpha q}$. *If* $\{m_P\}_P$ *is an* $\mathbf{Ad}_p^{\alpha q}$*-family, then* $\{Tm_P\}_P$ *is also an* $\mathbf{Ad}_p^{\alpha q}$*-family. Furthermore, if* $\{m_P\}_P$ *is a uniform* $\mathbf{Ad}_p^{\alpha q}$*-family, then* $\{\sup_{y \in P_0} |\langle T(m_P^y), \varphi_Q \rangle|\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$.

Proof. Since $m_P = \sum_R \langle m_P, \varphi_R \rangle \psi_R$, we have

$$\langle Tm_P, \varphi_Q \rangle = \sum_R \langle T\psi_R, \varphi_Q \rangle \langle m_P, \varphi_R \rangle$$

and this again corresponds to the composition of two operators in $\mathbf{ad}_p^{\alpha q}$. Hence, $\{Tm_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family. Similarly, to show the last part of the theorem, we use that

$$\sup_{y \in P_0} |\langle T(m_P^y), \varphi_Q \rangle| \leqslant \sum_R |\langle T\psi_R, \varphi_Q \rangle| \langle m_P | \varphi_R \rangle. \quad \blacksquare$$

In the other direction we have the next result.

PROPOSITION 9.11. *Suppose* $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$, *and let* τ *be a function satisfying*

$$\hat{\tau}(\xi) \in C^\infty, \qquad |\hat{\tau}(\xi)| \leqslant c > 0 \qquad \text{if } \tfrac{1}{2} \leqslant |\xi| \leqslant 2.$$

Set $\tau_P(\cdot) = |P|^{-1/2} \tau(2^\nu \cdot - k)$ *if* $P = P_{\nu k}$. *Suppose* $\{\tau_P\}_P$ *is a uniform* $\mathbf{Ad}_p^{\alpha q}$ *– family. If*

$$\{\sup_{y \in P_0} |\langle T(\tau_P^y), \varphi_Q \rangle|\}_{Q,P} \in \mathbf{ad}_p^{\alpha q},$$

then $T \in \mathbf{Ad}_p^{\alpha q}$.

Proof. As in the proof of Theorem 9.4 we have $\psi = \tilde{\psi} * \tau$ with $\tilde{\psi}$ satisfying (2.1)–(2.3). Also as in that proof, this leads to the identity

$$\langle T\psi_P, \varphi_Q \rangle = \int \langle T(\tau_P^y), \varphi_Q \rangle \tilde{\psi}_\mu(y)\, dy = \sum_l \int_{P_{\mu l}} \langle T(\tau_P^y), \varphi_Q \rangle \tilde{\psi}_\mu(y)\, dy,$$

$l(P) = 2^{-\mu}$, and the estimate

$$|\langle T\psi_P, \varphi_Q \rangle| \leqslant \sum_R \sup_{y \in R_0} |\langle T(\tau_R^y), \varphi_Q \rangle|\, b_{RP}$$

with $b_{QP} = 0$ if $l(Q) \neq l(P)$ and $b_{QP} = \sup_{y \in Q_{0,k-l}} |\tilde{\psi}(y)|$ if $P = P_{\mu l}$ and $Q = Q_{\mu k}$. Applying Thereom 9.1(i) concludes the proof. $\blacksquare$

Suppose that a family $\{\tau_P\}_P$ of functions has the property that $\psi_P = \sum_R a_{RP} \tau_R$ with $\{a_{QP}\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$. Then a linear operator T belongs to

$\mathbf{Ad}_p^{\alpha q}$ if $\{T\tau_P\}_P$ is an $\mathbf{Ad}_p^{\alpha q}$-family. This fact is a simple consequence of Theorem 9.1(i)), since

$$\langle T\psi_P, \varphi_Q\rangle = \sum_R a_{RP}\langle T\tau_R, \varphi_Q\rangle.$$

Similarly, suppose we consider the collection of all smooth atoms for $\dot{\mathbf{F}}_p^{\alpha q}$. We will say that a linear operator T *maps families of smooth atoms* (for $\dot{\mathbf{F}}_p^{\alpha q}$) *to* $\mathbf{Ad}_p^{\alpha q}$*-families* if, whenever $\{a_P\}_P$ is a family of smooth atoms, there exist constants C and $\varepsilon > 0$, independent of $\{a_P\}_P$, such that $|\langle Ta_P, \varphi_Q\rangle| \leqslant C\omega_{QP}(\varepsilon)$. Similarly, we say that T *maps smooth atoms to smooth molecules* (for $\dot{\mathbf{F}}_p^{\alpha q}$) if there exists $\delta > \alpha^* = \alpha - [\alpha]$, $M > J = n/\min(1, p, q)$, and a constant C such that $\{Ta_Q/C\}_Q$ is a family of smooth (δ, M)-molecules whenever $\{a_Q\}_Q$ is a family of smooth atoms for $\dot{\mathbf{F}}_p^{\alpha q}$ (with δ, M, and C independent of $\{a_Q\}_Q$).

We now have the analogue of the characterization of $\mathbf{Ad}_p^{\alpha q}$ in Propositions 9.10–9.11) with the family $\{\tau_P\}_P$ replaced by the collection of all smooth atoms.

Proposition 9.12. *Suppose $\alpha \in \mathbb{R}$, and $0 < p, q \leqslant +\infty$. Then T maps families of smooth atoms to $\mathbf{Ad}_p^{\alpha q}$-families if and only if $T \in \mathbf{Ad}_p^{\alpha q}$. In particular, if T maps smooth atoms to smooth molecules, then $T \in \mathbf{Ad}_p^{\alpha q}$.*

Proof. We first claim that for each dyadic cube P there exists a family $\{a_R\}_R$ of smooth atoms and a sequence $\{t_{RP}\}_R$ such that $\psi_P = \sum_R t_{RP} a_R$, where $|t_{RP}| \leqslant C\omega_{RP}(\varepsilon)$ for some C and $\varepsilon > 0$ independent of P and R. To prove this, we see from the representation of ψ_Q in the proof of Theorem 4.1, that for any $L > 0$, we can write

$$\psi_{Q_{\nu l}} = \sum_{k \in \mathbb{Z}^n} C(1 + |k|)^{-L} a_{Q_{\nu, k+l}},$$

where $a_{Q_{\nu,k+l}} \equiv 2^{\nu n/2}(1 + |k|)^L g_k(2^\nu x - l)/C$ is a smooth atom if C is sufficiently large, depending on L. Changing notation yields the claim.

The claim and the argument in the paragraph following the proof of Proposition 9.11 yield the "only if" part of the first statement we are proving. The second statement follows immediately from the first by Lemma 3.6. Similarly, since smooth atoms are smooth molecules, Proposition 9.10 (or, rather, its proof, to obtain the required uniformity in δ and J) yields the remaining "if" statement. ∎

We can characterize $\mathbf{Ad}_p^{\alpha q}$-families by representations like those for the ψ_P's in the proof of the last proposition.

PROPOSITION 9.13. *Suppose* $\alpha \in \mathbb{R}$, *and* $0 < p, q \leqslant +\infty$. *Then* $\{m_P\}_P$ *is an* $\mathbf{Ad}_p^{\alpha q}$*-family if and only if there exists a family of smooth atoms* $\{a_R^P\}_R$ *for each* P, *and a matrix* $\{t_{QP}\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$, *such that* $m_P = \sum_R t_{RP} a_R^P$. *In particular, any family of smooth molecules can be represented this way.*

Proof. We apply Lemma 2.1, and the result for the family $\{\psi_P\}_P$ in the proof of Proposition 9.12, to write

$$\begin{aligned} m_P &= \sum_Q \langle m_P, \varphi_Q \rangle \psi_Q \\ &= \sum_R \sum_Q \langle m_P, \varphi_Q \rangle \tilde{t}_{RQ} \tilde{a}_R^Q, \end{aligned}$$

where $\tilde{a}_R^Q$ is a smooth atom corresponding to the cube R, for each Q, and $|\tilde{t}_{RQ}| \leqslant C\omega_{RQ}(\varepsilon)$, for some C and $\varepsilon > 0$. Let

$$t_{RP} = \sum_Q |\langle m_P, \varphi_Q \rangle| \, |\tilde{t}_{RQ}|$$

and

$$a_R^P = \sum_Q \langle m_P, \varphi_Q \rangle \tilde{t}_{RQ} \tilde{a}_R^Q / t_{RP}.$$

Then each a_R^P is a smooth atom (corresponding to R), since each $\tilde{a}_R^Q$ is and since the sum is dominated by a convex combination. Now $\{m_P\}_P \in \mathbf{Ad}_p^{\alpha q}$, so Proposition 9.1(i) shows that $\{t_{QP}\}_{Q,P} \in \mathbf{ad}_p^{\alpha q}$, completing the proof of the "only if" statement.

Noting that $\{a_R^P\}_R$ is an $\mathbf{Ad}_p^{\alpha q}$-family for each P (with uniform C and ε), Proposition 9.1(i) yields the "if" part, since $\langle m_P, \varphi_Q \rangle = \sum_R t_{RP} \langle a_R^P, \varphi_Q \rangle$. ∎

The identity $m_P = \sum_R t_{RP} a_R^P$ in the last proposition must be properly interpreted to avoid problems with polynomials. This comes from the fact that the representation $m_P = \sum_Q \langle m_P, \varphi_Q \rangle \psi_Q$ converges in $\mathcal{S}'/\mathcal{P}$. However, if, say, $g \in \mathcal{S}_0$, the expression $\sum_R t_{RP} \langle a_R^P, g \rangle$ is absolutely convergent and coincides with $\langle m_P, g \rangle$.

In the special case $\alpha = 0$, we can characterize $\mathbf{Ad}_p^{\alpha q}$-families and the algebra $\mathbf{Ad}_p^{\alpha q}$ in terms of smooth atoms and molecules. This is a consequence of the following technical lemma, whose analogue for $\alpha \neq 0$ is false.

LEMMA 9.14. *Suppose* $0 < p, q \leqslant +\infty$, $\varepsilon > 0$, *and* $0 < \delta$, $\tilde{\varepsilon} < \varepsilon/2$. *Let* $\{g_P\}_P$ *be a family of smooth* $(\delta, J + \tilde{\varepsilon})$*-molecules* (*for* $\dot{\mathbf{F}}_p^{0q}$), *let* Q *be a fixed dyadic cube, and let* $\{a_{QP}\}_P$ *be a sequence satisfying* $|a_{QP}| \leqslant C\omega_{PQ}(\varepsilon)$ *for all*

P. Define $m_Q = \sum_P a_{QP}\, g_P$. *Then there exists* $\tilde{C}$ *(depending only on the fixed parameters) such that* $m_Q/\tilde{C}$ *is a smooth* $(\delta, J+\tilde{\varepsilon})$*-molecule.*

Proof. See Appendix D. ∎

We can now characterize $\mathbf{Ad}_p^{0q}$-families as follows.

THEOREM 9.15. *Suppose* $0 < p, q \leqslant +\infty$. *Then* $\{m_P\}_P$ *is an* $\mathbf{Ad}_p^{0q}$*-family if and only if* $\{m_P\}_P$ *is a family of smooth molecules (for* $\dot{\mathbf{F}}_p^{0q}$*).*

Proof. One direction follows from Lemma 3.6. The other is a consequence of Proposition 9.13, Lemma 9.14, and the fact that smooth atoms are smooth molecules. ∎

Similarly, we can characterize the algebra $\mathbf{Ad}_p^{0q}$. We say that T maps *smooth molecules to smooth molecules* (for $\dot{\mathbf{F}}_p^{0q}$) if, whenever $\{m_Q\}_Q$ is a family of smoth (δ, M)-molecules $(\delta > 0, M > J)$, there exist $\tilde{\delta} > 0$, $\tilde{M} > J$, and a constant C, depending on δ and N, such that $\{Tm_Q/C\}_Q$ is a family of smooth $(\tilde{\delta}, \tilde{M})$-molecules (for $\dot{\mathbf{F}}_p^{0q}$).

THEOREM 9.16. *Suppose* $0 < p, q \leqslant +\infty$. *The following are equivalent*:

(i) $T \in \mathbf{Ad}_p^{0q}$;

(ii) *T maps smooth atoms to smooth molecules*;

(iii) *T maps smooth molecules to smooth molecules.*

Proof. The equivalence of (i) and (ii) follows from Proposition 9.12 and Theorem 9.15. Also, that (i) implies (iii) is a consequence of Theorem 9.15 and the fact that an operator in $\mathbf{Ad}_p^{\alpha q}$ preserves $\mathbf{Ad}_p^{\alpha q}$-families, i.e., Proposition 9.10. Finally, (iii) trivially gives (ii).

Remark 9.17. In Section 4 we considered decompositions of the form $f = \sum_Q \langle f, \tau^Q \rangle \sigma^Q$, in which one of the families $\{\tau^Q\}_Q$ or $\{\sigma^Q\}_Q$ could be chosen in a certain convenient, explicit way. We can now discuss the extent to which we can also control the other family.

Suppose $\alpha \in \mathbb{R}$, and $0 < p, q \leqslant +\infty$. Then we have the following:

(i) In Theorem 4.2, there exists $\mu_0 \leqslant 0$ such that if $\mu \leqslant \mu_0$, then $\{\tau^Q\}_Q$ is an $\mathbf{Ad}_p^{\beta q}$-family, where $\beta = -\alpha + J - n$.

(ii) In Theorem 4.4, there exists $\mu_0 \leqslant 0$ such that if $\mu \leqslant \mu_0$, then $\{\sigma^Q\}_Q$ is an $\mathbf{Ad}_p^{\alpha q}$-family.

We will show (ii) first. From the proof of Theorem 4.4, we see that $(I - T_\mu)\,\psi_P = C \sum_R s_R m_R$, where $\{m_R\}_R$ is a family of smooth molecules, and

$$|s_R| = |s_R(\psi_P)| = |a_{RP}| \leqslant c 2^{\mu\rho} \omega_{RP}(\varepsilon)$$

for some $\varepsilon, \rho > 0$ (see (4.26)). Hence, by Lemma 3.6 and Theorem 9.1(i),

$$|\langle (I - T_\mu)\, \psi_P, \varphi_Q \rangle| \leqslant c 2^{\mu\rho} \omega_{QP}(\tilde{\varepsilon}),$$

for some $\tilde{\varepsilon} > 0$. By Theorem 9.1(ii), there exists μ_0 such that $T_\mu^{-1} \in \mathbf{Ad}_p^{\alpha q}$ for $\mu \leqslant \mu_0$. By (4.15)–(4.17), $\{\eta_Q\}_Q$ is a family of smooth molecules and hence an $\mathbf{Ad}_p^{\alpha q}$-family (Lemma 3.6). Since $\sigma^Q = T_\mu^{-1} \eta_Q$, Proposition 9.10 implies that $\{\sigma^Q\}_Q$ is an $\mathbf{Ad}_p^{\alpha q}$-family.

We now show (i), which is similar. By the proof of Theorem 4.2, we have $(I - T_\mu)\, \psi_P = C 2^{\mu\delta} \sum_R s_R m_R$, where $\{m_R\}_R$ is a family of smooth molecules, and

$$s_R = S_R(\psi_P) = |R|^{-1/2} \int_R |\tilde{\varphi}_\nu * \psi_P(y)|\, dy \qquad \text{if} \quad l(R) = 2^{-\nu}.$$

It is easy to check that $|s_R| \leqslant C\omega_{RP}(\varepsilon)$ for some $\varepsilon > 0$. So by the argument above, $T_\mu^{-1} \in \mathbf{Ad}_p^{\alpha q}$. Since the adjoint of an operator corresponds to the conjugate transpose of the corresponding matrix, we have $(T_\mu^{-1})^* \in \mathbf{Ad}_p^{\beta q}$. Since $\tau^Q = (T_\mu^{-1})^* \eta_Q$, the result follows as in the previous case. This completes the proofs of (i) and (ii).

By Theorem 9.9, (i) and (ii) above imply (4.5) and (4.6), respectively. Thus we have obtained stronger conclusions regarding the families $\{\sigma^Q\}_Q$ and $\{\tau^Q\}_Q$. Using our characterizations of $\mathbf{Ad}_p^{\alpha q}$-families, e.g., Proposition 9.13, we obtain more explicit information on these families. In particular, by Theorem 9.15, $\{\tau^Q\}_Q$ in (i) is a family of smooth molecules when $\beta = 0$, and similarly for $\{\sigma^Q\}_Q$ in (ii) when $\alpha = 0$.

The algebra $\mathbf{Ad}_p^{\alpha q}$ contains many classes of operators usually encountered in harmonic analysis. An example is the class of Calderón–Zygmund kernel operators; we recall the standard notation and discuss some known results about this class next.

EXAMPLE 9.18. Suppose $T: \mathscr{S} \to \mathscr{S}'$ is a continuous linear operator. The Schwartz kernel theorem guarantees the existence of a kernel $K \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that $\langle T\Phi, \Psi \rangle = K(\Psi \otimes \Phi)$ for all $\Phi, \Psi \in \mathscr{S}$. K is a *Calderón–Zygmund kernel* if it is given by a continuous function K on $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ satisfying

(i) $|K(x, y)| \leqslant C\, |x - y|^{-n}$,

(ii) $|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leqslant C\, |x - x'|^\varepsilon\, |x - y|^{-(n+\varepsilon)}$, provided $2\, |x - x'| \leqslant |x - y|$,

for some $0 < \varepsilon \leqslant 1$, and

$$\langle T\Phi, \Psi \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)\, \Phi(y)\, \Psi(x)\, dx\, dy$$

whenever Φ, $\Psi \in \mathscr{D}$ and $\operatorname{supp} \Phi \cap \operatorname{supp} \Psi = \varnothing$. We write $T \in CZK(\varepsilon)$ if T's kernel satisfies these conditions for a fixed ε, or just $T \in CZK$ if the particular value of ε is not important.

A continuous linear operator $T : \mathscr{D} \to \mathscr{D}'$ satisfies the weak boundedness property (WBP) if

$$|\langle T\Phi, \Psi \rangle| \leqslant Ct^n(\|\Phi\|_{L^\infty} + t\,\|\nabla\Phi\|_{L^\infty})(\|\Psi\|_{L^\infty} + t\,\|\nabla\Psi\|_{L^\infty})$$

for all Φ, $\Psi \in \mathscr{D}$ with supports having diameter at most $t > 0$.

The fundamental result about operators $T \in CZK$ is the result by David and Journé ([DJ]). This reslt states that an operator $T \in CZK$ is bounded on L^2 if and only $T1 \in \mathrm{BMO}$, $T^*1 \in \mathrm{BMO}$, and $T \in WBP$. The proof of this can be reduced to the case when T satisfies $T1 = T^*1 = 0$ (cf. [DJ]).

In [Fr-H-J-W] we showed that if $T \in CZK(\varepsilon) \cap \mathrm{WBP}$ and $T1 = 0$, then T maps smooth atoms to smooth molecules for $\dot{\mathbf{F}}_p^{\alpha q}$ whenever $0 < \alpha < \varepsilon \leqslant 1$, $1 \leqslant p, q \leqslant +\infty$. In paticular, by Proposition 9.12, we have $T \in \mathbf{Ad}_p^{\alpha q}$. Under the additional assumption $T^*1 = 0$, we also proved that T maps smooth atoms into smooth molecules for $\dot{\mathbf{F}}_p^{0q}$ $(1 \leqslant p, q \leqslant +\infty)$, and, hence, $T \in \mathbf{Ad}_p^{0q}$. In particular, we have that T is bounded on $\dot{\mathbf{F}}_2^{02} \approx L^2$, i.e., the (reduced) David–Journé result. The approach in [Fr-H-J-W] has been extended to cover the full range $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$ in [FTW; Torr]; the main difference is that stronger regularity and cancellation conditions are assumed in the case of more general indices.

For another example, let us consider Fourier multiplier operators. For $m \in L^\infty$, let T_m be the associated Fourier multiplier operator, i.e., $(T_m\varphi)^\wedge\,(\xi) = m(\xi)\,\hat{\varphi}(\xi)$ for, say, $\varphi \in \mathscr{S}_0$. We will say that T_m is bounded on $\dot{\mathbf{F}}_p^{\alpha q}$ if $\|T_m f\|_{\dot{\mathbf{F}}_p^{\alpha q}} \leqslant c\, \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}$ for $f \in \mathscr{S}_0$. Of course, if $0 < p$, $q < +\infty$, $\mathscr{S}_0$ is dense in $\dot{\mathbf{F}}_p^{\alpha q}$, so T_m can be extended to a bounded operator on (all of) $\dot{\mathbf{F}}_p^{\alpha q}$. In the cases $p = +\infty$ or $q = +\infty$, so that $\mathscr{S}_0$ is not dense, this convention is a convenient abuse of terminology. We also introduce the notation

$$R_\nu = \{x \in \mathbb{R}^n : 2^{\nu-1} \leqslant x \leqslant 2^{\nu+1}\}$$

for $\nu \in \mathbb{Z}$.

EXAMPLE 9.19. Let $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$, and $J = n/\min(1, p, q)$. Suppose that the function m satisfies

$$\sup_\nu \sum_{|\gamma| \leqslant [J+1]} 2^{\nu|\gamma|} 2^{-\nu n} \int_{R_\nu} |\partial^\gamma m(\xi)|\, d\xi < +\infty. \tag{9.3}$$

Then T_m is almost diagonal and hence extends to a bounded operator on $\dot{\mathbf{F}}_p^{\alpha q}$. More precisely, if $\varepsilon = [J+1] - J$, then

$$|\langle T_m \psi_P, \varphi_Q \rangle| \leqslant c_m \omega_{QP}(\varepsilon). \tag{9.4}$$

To show this we fix Q and P with $l(Q)=2^{-\nu}$ and $l(P)=2^{-\mu}$ and put $j=\nu-\mu$ and $a_{QP}=\langle T_m\psi_P, \varphi_Q\rangle$. We have

$$a_{QP}=(2\pi)^{-n}\langle m\hat{\psi}_P, \hat{\varphi}_Q\rangle,$$

so by (2.2), $a_{QP}=0$ if $|j|>1$. If $|j|\leqslant 1$, then $l(P)\approx l(Q)$, so the required estimate is merely

$$A=\sup_{P,Q}\,(1+l(Q)^{-1}|x_P-x_Q|)^{[J+1]}\,|a_{QP}|<+\infty.$$

Now $\langle m\hat{\psi}_P, \hat{\varphi}_Q\rangle=2^{-(\mu+\nu)n/2}h(x_Q-x_P)$, where

$$h(x)=h_{\mu\nu}(x)=(2\pi)^n\,(m\hat{\psi}_\mu\,\bar{\hat{\varphi}}_\nu)^\vee\,(x)=(2\pi)^n\,2^{\nu n}(m(2^\nu\cdot)\,\hat{\psi}(2^j\cdot)\,\bar{\hat{\varphi}}(\cdot))^\vee\,(2^\nu x).$$

Letting $\chi_j(x)=\hat{\psi}(2^jx)\,\bar{\hat{\varphi}}(x)$ and replacing $2^\nu x$ by x, we have

$$A\leqslant c\sup_{\nu,x}\,(1+|x|)^{[J+1]}\,|(m(2^\nu\cdot)\,\chi_j(\cdot))^\vee\,(x)|.$$

Since $(1+|x|)^{[J+1]}\approx\sum_{|\gamma|\leqslant[J+1]}|x|^{|\gamma|}$, we obtain

$$\begin{aligned}A&\leqslant c\sup_{\nu,x}\sum_{|\gamma|\leqslant[J+1]}|(\partial^\gamma(m(2^\nu\cdot)\,\chi_j(\cdot)))^\vee\,(x)|\\&\leqslant c\sup_\nu\int_{R_0}\sum_{|\gamma|\leqslant[J+1]}2^{\nu|\gamma|}\,|(\partial^\gamma m)(2^\nu\xi)|\,d\xi,\end{aligned}$$

by the Riemann–Lebesgue lemma and the chain rule, since $\chi_j\in\mathscr{D}$ and $\operatorname{supp}\chi_j\subset R_0$. Changing variables shows that A is dominated by the expression in (9.3), so (9.4) holds.

Of course, there are more general classes of bounded multiplier operators than those satisfying the L^1-Mihlin condition (9.3). For example, the familiar Hörmander (Fourier) multiplier theorem states that L^p-boundedness $(1<p<+\infty)$ holds under a weaker assumption, only involving an L^2 condition on $[n/2+1]$ derivatives. We can obtain this result, and generalizations of it, by a variation of our argument above. We do this in the next section (Corollary 10.7 and Remark 10.9) by studying more precise criteria for boundedness of matrices on $\dot{\mathbf{f}}_p^{\alpha q}$.

10. Schur's Lemma and Further Results on Operators

We turn now to a more careful study of conditions for the boundedness of matrices on the $\dot{\mathbf{f}}_p^{\alpha q}$ spaces. We saw in Section 9 that the class of almost diagonal operators contains many familiar examples and has an interesting structure. However, our methods can be modified to give much sharper boundedness criteria than almost diagonality. It fact, we will find necessary and sufficient conditions for a positive matrix to be bounded on all the spaces $\dot{\mathbf{f}}_p^{\alpha q}$, $1 \leqslant p, q \leqslant +\infty$ (for each fixed α). As an application we obtain a Fourier multiplier theorem for $\dot{\mathbf{F}}_p^{\alpha q}$ which generalizes the well-known Hörmander theorem for L^p and its Hardy space analogue (compare Example 9.19 and Remark 10.9 and note that (10.21) is weaker than (9.3), e.g., by the Sobolev imbedding theorem). Returning to our discussion of bounded positive matrix operators, we note several characterizations, closely related to the classical Schur's lemma and to some work of Rubio de Francia. We conclude the section with some associated results about extrapolation of operators on the $\dot{\mathbf{F}}_p^{\alpha q}$ spaces, which also follow along the lines of Rubio de Francia's work.

Let us start with the simplest situation, namely with $q = p \leqslant 1$. Since $\dot{\mathbf{f}}_p^{\alpha p}$ is a weighted l^p-space, we have that an operator A is bounded from $\dot{\mathbf{f}}_p^{\alpha p}$ to another p-normed space X, $0 < p \leqslant 1$, if and only if A is bounded on the standard basis vectors $\{e^P\}_P$, i.e.,

$$\| Ae^P \|_X \leqslant C \, \| e^P \|_{\dot{\mathbf{f}}_p^{\alpha p}}.$$

(Recall that a space X is *p-normed* if $\| \cdot \|_X^p$ satisfies the triangle inequality.) If X is a space of sequences $\{s_Q\}_Q$, then A is represented by the matrix $\{a_{QP}\}_{Q,P} = \{(Ae^P)_Q\}_{Q,P}$. In terms of this matrix the above characterization becomes

$$\sup_P |P|^{\alpha/n + 1/2 - 1/p} \, \| \{a_{QP}\}_Q \|_X < +\infty. \tag{10.1}$$

In the particular case $X = \dot{\mathbf{f}}_p^{\alpha p}$ this says that

$$\sup_P \sum_Q (|a_{QP}| (|Q|/|P|)^{-\alpha/n - 1/2 + 1/p})^p < +\infty. \tag{10.2}$$

Dually, suppose $p = q = +\infty$. A linear operator A with matrix $\{a_{QP}\}_{Q,P}$ is bounded on $\dot{\mathbf{f}}_\infty^{\alpha\infty}$ if and only if

$$\sup_Q \sum_P |a_{QP}| (|Q|/|P|)^{-\alpha/n - 1/2} < +\infty. \tag{10.3}$$

When $1 < p = q < +\infty$ there is no complete characterization. Schur's lemma (e.g., see [Ga]) is a substitute for matrices with nonnegative entries. With our notations this lemma states (cf. [Ja5, p. 396]) that if A is a positive operator with matrix $\{a_{QP}\}_{Q,P}$ (i.e., $a_{QP} \geqslant 0$), then A is bounded on $\dot{\mathbf{f}}_p^{\alpha p}$, $\alpha \in \mathbb{R}$, $1 < p < +\infty$, if and only if there exists a positive sequence $\{u_Q\}_Q$ such that

$$\sum_P a_{QP}(|Q|/|P|)^{-\alpha/n - 1/2 + 1/p} u_P^{p'} \leqslant C u_Q^{p'}$$

and

$$\sum_Q a_{QP}(|Q|/|P|)^{-\alpha/n - 1/2 + 1/p} u_Q^{p} \leqslant C u_P^{p}.$$

(The general statement of Schur's lemma is that $Tf = \int K(x, y) f(y)\, d\mu(y)$, $K(x, y) \geqslant 0$, is bounded on $L^p(d\mu)$, $1 < p < +\infty$, if and only if there exists a positive function u such that $\int K(x, y)\, u(y)^{p'}\, d\mu(y) \leqslant c u^{p'}(x)$ and $\int K(x, y)\, u(x)^p\, d\mu(x) \leqslant c u^p(y)$.)

The cases $q \neq p$ are more complicated. We will obtain certain general boundedness criteria in Theorems 10.5 and 10.11–10.13 for positive operators; of course, more generally, an operator with matrix $\{a_{QP}\}_{Q,P}$ is bounded if the operator with matrix $\{|a_{QP}|\}_{Q,P}$ is bounded.

When $p \leqslant 1$ and $q = +\infty$ and, dually, when $p = +\infty$, $q = 1$, we have complete, rather explicit characterizations. Given a matrix $A = \{a_{QP}\}_{Q,P}$, we let $a^*_{QP} \equiv \bar{a}_{PQ}$, $A^* = \{a^*_{QP}\}_{Q,P}$, and $|A| = \{|a_{QP}|\}_{Q,P}$.

PROPOSITION 10.1. *Let A be a positive operator with matrix $\{a_{QP}\}_{Q,P}$.*

(i) *If $p \leqslant 1$, then A is bounded on $\dot{\mathbf{f}}_p^{\alpha\infty}$ if and only if*

$$\sup_{P_0} \frac{1}{|P_0|^{1/p}} \left\| \left\{ \sum_{P \subset P_0} a_{QP} |P|^{\alpha/n + 1/2} \right\}_Q \right\|_{\dot{\mathbf{f}}_p^{\alpha\infty}} < +\infty. \tag{10.4}$$

(ii) *A is bounded on $\dot{\mathbf{f}}_\infty^{\alpha 1}$ if and only if*

$$\sup_{Q_0} \frac{1}{|Q_0|} \left\| \left\{ \sum_{Q \subset Q_0} a_{QP} |Q|^{-\alpha/n + 1/2} \right\}_P \right\|_{\dot{\mathbf{f}}_1^{-\alpha\infty}} < +\infty. \tag{10.5}$$

Proof. To prove (i) we note that A is bounded on $\dot{\mathbf{f}}_p^{\alpha\infty}$ if and only if it is bounded for an arbitrary atom (cf. Theorem 7.2). In fact, since A is positive, we only need to consider atoms of the particular form $r = \{|P|^{\alpha/n - 1/2}/|P_0|^{1/p}\}_{P \subset P_0}$ for an arbitrary dyadic cube P_0. Written out, this is exactly (i).

Now (ii) follows from (i) by a duality argument. Suppose first (10.5) holds. By Remark 5.10 and Proposition 5.5, we have that $|\langle s, t\rangle| \leqslant c\, \|s\|_{\dot{\mathbf{f}}_1^{-\alpha\infty}} \|t\|_{\dot{\mathbf{f}}_\infty^{\alpha 1}}$. Then, by a constructive argument, similar to (but even easier than) the second part of the proof of Theorem 5.9, we see that

$$\|t\|_{\dot{\mathbf{f}}_\infty^{\alpha 1}} \approx \sup\{|\langle s, t\rangle| : s \text{ is finite and } \|s\|_{\dot{\mathbf{f}}_1^{-\alpha\infty}} \leqslant 1\}. \tag{10.6}$$

By (10.6), $\{a^*_{QP}\}_{Q,P}$ is bounded on $\dot{\mathbf{f}}_1^{-\alpha\infty}$, and the results follows.

For the other direction of (ii), let $\mathring{\mathbf{f}}_1^{\alpha\infty}$ denote the closure of finite sequences in $\dot{\mathbf{f}}_1^{\alpha\infty}$. Then $(\mathring{\mathbf{f}}_1^{-\alpha\infty})^* \approx \dot{\mathbf{f}}_\infty^{\alpha 1}$, by (10.6) and the fact that $(\mathring{\mathbf{f}}_1^{-\alpha\infty})^*$ must be a sequence space. By the Hahn–Banach theorem, we obtain the analogue of (10.6) with $\dot{\mathbf{f}}_\infty^{\alpha 1}$ and $\mathring{\mathbf{f}}_1^{-\alpha\infty}$ interchanged. Now the conclusion follows as before. ∎

Suppose $q < +\infty$. As in the case $q = +\infty$, an operator A is bounded on $\dot{\mathbf{f}}_p^{\alpha q}$, $0 < p \leqslant 1$, $p < q$, if and only if it is bounded on atoms (for $\dot{\mathbf{f}}_p^{\alpha q}$). However, when q is finite, the resulting condition is, unfortunately, not very explicit.

Let $\mathbf{b}$ denote the class of matrices A such that $|A|$ is bounded on $\dot{\mathbf{f}}_p^{0q}$ for all $1 \leqslant p, q \leqslant +\infty$. Proposition 10.1 gives the following explicit characterization of the class $\mathbf{b}$.

COROLLARY 10.2. *A matrix $\{a_{QP}\}_{Q,P}$ belongs to $\mathbf{b}$ if and only if $\{|a_{QP}|\}_{Q,P}$ satisfies the conditions* (10.2)–(10.5) *with $\alpha = 0$ and $p = 1$.*

Proof. This follows from Proposition 10.1 by interpolation (in fact, only Proposition 8.1 and Theorem 8.2 are used). ∎

Clearly, $\mathbf{b}$ is closed under composition and, by definition, under taking "adjoints," i.e., $A \in \mathbf{b}$ implies $A^* \in \mathbf{b}$. We equip the algebra $\mathbf{b}$ with the obvious norm, namely the maximum of the quantities, with $\alpha = 0$ and $p = 1$, that are assumed to be finite in (10.2)–(10.5). Similarly to Theorem 9.1(ii), there exists $\varepsilon > 0$ with the property that if $\|A - I\|_{\mathbf{b}} < \varepsilon$, then A^{-1} exists and belongs to $\mathbf{b}$. In particular, the class $\mathbf{b}$ has all the essential properties we used about $\mathbf{ad}_p^{\alpha q}$ in Section 9. Hence, there are analogues for $\mathbf{b}$ of the results in the previous section (up through Proposition 9.11) describing the basic features of the algebra $\mathbf{ad}_p^{\alpha q}$.

The class $\mathbf{b}$ is clearly larger than the class of almost diagonal operators on $\dot{\mathbf{f}}_p^{0q}$, $1 \leqslant p, q \leqslant +\infty$, since almost diagonal operators are bounded (Theorem 3.3). Our next theorem makes the comparison between these two classes easier and shows that the class $\mathbf{b}$ is considerably larger.

THEOREM 10.3. *If the matrix $A = \{a_{QP}\}_{Q,P}$ satisfies*

$$B_s \equiv \sup_Q \left(\sum_P (|a_{QP}|/(\omega_{QP}(\varepsilon)))^s \, \omega_{QP}(\varepsilon)(|P|/|Q|)^{1/2} \right)^{1/s} < +\infty \tag{10.7}$$

and

$$B_s^* \equiv \sup_P \left(\sum_Q (|a_{QP}|/(\omega_{QP}(\varepsilon)))^s \, \omega_{QP}(\varepsilon)(|Q|/|P|)^{1/2} \right)^{1/2} < +\infty, \tag{10.8}$$

for some $\varepsilon > 0$ and $s > 1$, then $\{a_{QP}\}_{Q,P} \in \mathbf{b}$. In particular, A is a bounded operator on $\dot{\mathbf{f}}_p^{0q}$ for all $1 \leqslant p, q \leqslant +\infty$.

Proof. It is easy to see that

$$\sup_{Q} \sum_{P} \omega_{QP}(\varepsilon)(|P|/|Q|)^{1/2} < +\infty.$$

By Hölder's inequality, B_s is an (essentially) increasing function of s (i.e., $B_r \leqslant c_{rs} B_s$ if $r \leqslant s$). Hence, the analogous condition with $s=1$ is also satisfied, and this is exactly the condition (10.3). Similarly, (10.8) implies the condition (10.2) with $p=1$. By the symmetry of our assumptions, it is thus enough to show that

$$\frac{1}{|P_0|} \left\| \left\{ \sum_{P \subset P_0} |a_{QP}| \, |P|^{1/2} \right\}_Q \right\|_{\dot{f}_1^{0\infty}} \leqslant c. \tag{10.9}$$

To prove this, we put the cubes Q in the two disjoint families $C = \{Q : Q \subset 20P_0\}$ and $D = \{Q : Q \not\subset 20P_0\}$. For the cubes in C the required estimate is immediate. We have just seen that (10.7) implies (10.3). Hence,

$$\begin{aligned} \mathrm{I} &\equiv \frac{1}{|P_0|} \left\| \left\{ \sum_{P \subset P_0} |a_{QP}| \, |P|^{1/2} \right\}_{Q \in C} \right\|_{\dot{f}_1^{0\infty}} \\ &= \frac{1}{|P_0|} \int_{20P_0} \sup_Q \left(\sum_{P \subset P_0} |a_{QP}| \, |P|^{1/2} \, \tilde{\chi}_Q(x) \right) dx \\ &\leqslant c \sup_Q \left(\sum_{P \subset P_0} |a_{QP}| (|P|/|Q|)^{1/2} \right) \leqslant c B_s. \end{aligned}$$

For the family D we get, by using the imbedding $\dot{f}_1^{01} \to \dot{f}_1^{0\infty}$,

$$\begin{aligned} \mathrm{II} &\equiv \frac{1}{|P_0|} \left\| \left\{ \sum_{P \subset P_0} |a_{QP}| \, |P|^{1/2} \right\}_{Q \in D} \right\|_{\dot{f}_1^{0\infty}} \\ &\leqslant \frac{1}{|P_0|} \left\| \left\{ \sum_{P \subset P_0} |a_{QP}| \, |P|^{1/2} \right\}_{Q \in D} \right\|_{\dot{f}_1^{01}} \\ &= \frac{1}{|P_0|} \sum_{P \subset P_0} \sum_{Q \in D} |a_{QP}| \, |P|^{1/2} \, |Q|^{1/2}. \end{aligned}$$

By Hölder's inequality this is dominated by

$$B_s^* \frac{1}{|P_0|} \sum_{P C P_0} \left(\sum_{Q \in D} |P|^{s'} \omega_{QP}(\varepsilon)(|Q|/|P|)^{1/2} \right)^{1/s'}.$$

By an elementary calculation, summarized in the lemma below, and using that there are $|P_0|/|P|$ cubes P of the same sidelength $l(P)$ contained in P_0, we have

$$\mathrm{II} \leqslant cB_s^* \frac{1}{|P_0|} \sum_{P \subset P_0} |P| (l(P)/l(P_0))^{\varepsilon/2s'} \leqslant c_{\varepsilon,s} B_s^*.$$

This shows that I and II are finite and gives the desired conclusion (10.9). ∎

LEMMA 10.4. *For $P \subset P_0$ fixed and $D = \{Q : Q \nsubseteq 20P_0\}$ we have*

$$\sum_{Q \in D} \omega_{QP}(\varepsilon)(|Q|/|P|)^{1/2} \leqslant c_\varepsilon (l(P)/l(P_0))^{\varepsilon/2}.$$

Proof. Let $2^{-p} = l(P)$ and $2^{-p_0} = l(P_0)$, and define $D_\nu = \{Q \in D : l(Q) = 2^{-\nu}\}$, $\nu \in \mathbb{Z}$. By some elementary calculations we obtain that $\sum_{Q \in D_\nu} \omega_{QP}(\varepsilon)(|Q|/|P|)^{1/2}$ is bounded by $c_\varepsilon 2^{\varepsilon(\nu - p)/2}$ if $\nu < p_0$, and by $c_\varepsilon 2^{\varepsilon(p_0 - (\nu + p)/2)}$ if $\nu \geqslant p_0$. Here, to obtain the second estimate, we consider $p < \nu$ and $p_0 \leqslant \nu \leqslant p$, separately. Using these estimates and summing the obvious geometric series yield the desired conclusion. ∎

As we noticed in the proof of the theorem, the quantity B_s in (10.7) is an (essentially) increasing function of s. If we let $s \to +\infty$ in (10.7), then in the limit we get our almost diagonality condition (3.1).

Suppose that $\Omega = \{\Omega_{QP}\}_{Q,P}$ is a positive matrix satisfying

$$\sup_{P_0} \frac{1}{|P_0|} \sum_{P \subset P_0} |P| \left(\sum_{Q \nsubseteq 20P_0} \Omega_{QP} (|Q|/|P|)^{1/2} \right)^{1/s'} < +\infty, \tag{10.10}$$

for some fixed $s > 1$. We note that the proof of the theorem shows that the matrix $A = \{a_{QP}\}_{Q,P}$ corresponds to a bounded operator on $\dot{f}_1^{0\infty}$ if, for each cube P_0,

$$\sup_{P \subseteq P_0} \left(\sum_{Q \nsubseteq 20P_0} (|a_{QP}|/\Omega_{QP})^s \, \Omega_{QP} (|Q|/|P|)^{1/2} \right)^{1/s} \leqslant C, \tag{10.11}$$

for some constant C (independent of P_0), and

$$A \text{ is bounded on } \dot{f}_\infty^{0\infty}. \tag{10.12}$$

(Note by (10.3) that (10.12) is equivalent to $\sup_Q \sum_P |a_{QP}| (|P|/|Q|)^{1/2} \leqslant C$.)

Similar results can also be proved for operators on the spaces $\dot{f}_1^{0q}$, $1 \leqslant q < +\infty$. We shall give one example, and to keep notations fairly simple, we restrict ourselves to the case of $\dot{f}_1^{02}$.

THEOREM 10.5. *Suppose $\Omega = \{\Omega_{QP}\}_{Q,P}$ is a positive matrix satisfying*

$$\sup_{P_0} \frac{1}{|P_0|} \sum_{P \subseteq P_0} |P| \sum_{Q \nsubseteq 20P_0} \Omega_{QP}(|Q|/|P|)^{1/2} < +\infty. \tag{10.13}$$

If the matrix $A = \{a_{QP}\}_{Q,P}$ is bounded on $\dot{\mathbf{f}}_2^{02}$ and satisfies

$$C^* \equiv \sup_P \left(\sum_Q (|a_{QP}|/\Omega_{QP})^2 \, \Omega_{QP}(|Q|/|P|)^{1/2} \right)^{1/2} < +\infty, \tag{10.14}$$

then A is bounded on $\dot{\mathbf{f}}_1^{02}$.

Proof. The proof parallels that of Theorem 10.3. Using the decomposition of $\dot{\mathbf{f}}_1^{02}$ into 2-atoms (cf. Remark 7.3), it is enough to show that

$$\| \{(Ar)_Q\}_Q \|_{\dot{\mathbf{f}}_1^{02}} \leqslant c \tag{10.15}$$

for an arbitrary 2-atom $r = \{r_P\}_{P \subset P_0}$. Recall that $r = \{r_P\}_{P \subset P_0}$ is a 2-atom if

$$\| \{r_P\}_{P \subset P_0} \|_{\dot{\mathbf{f}}_2^{02}} \leqslant |P_0|^{-1/2}.$$

To prove (10.15), we again divide the cubes Q into the two disjoint families $C = \{Q : Q \subset 20P_0\}$ and $D = \{Q : Q \nsubseteq 20P_0\}$. Using first Hölder's inequality and then that A is bounded on $\dot{\mathbf{f}}_2^{02}$, we have

$$\begin{aligned}
&\| \{(Ar)_Q\}_{Q \in C} \|_{\dot{\mathbf{f}}_1^{02}} \\
&\quad \leqslant \int_{20P_0} \left(\sum_Q \left(\sum_{P \subset P_0} |a_{QP} r_P| \, \tilde{\chi}_Q(x) \right)^2 \right)^{1/2} dx \\
&\quad \leqslant c\, |P_0|^{1/2} \, \| \{(Ar)_Q\}_Q \|_{\dot{\mathbf{f}}_2^{02}} \leqslant c\, |P_0|^{1/2} \, \| \{r_P\}_P \|_{\dot{\mathbf{f}}_2^{02}} \leqslant c.
\end{aligned}$$

By the imbedding $\dot{\mathbf{f}}_1^{01} \to \dot{\mathbf{f}}_1^{02}$,

$$\begin{aligned}
\| \{(Ar)_Q\}_{Q \in D} \|_{\dot{\mathbf{f}}_1^{02}} &\leqslant \| \{(Ar)_Q\}_{Q \in D} \|_{\dot{\mathbf{f}}_1^{01}} \\
&\leqslant \sum_{P \subset P_0} \sum_{Q \in D} |a_{QP}| \, |r_P| \, |Q|^{1/2}.
\end{aligned}$$

Applying Cauchy–Schwarz twice, this can be estimated by

$$\begin{aligned}
&C^* \sum_{P \subset P_0} |r_P| \, |P|^{1/2} \left(\sum_{Q \in D} \Omega_{QP}(|Q|/|P|)^{1/2} \right)^{1/2} \\
&\quad \leqslant C^* \, \| \{r_P\}_P \|_{\dot{\mathbf{f}}_2^{02}} \left(\sum_{P \subset P_0} |P| \sum_{Q \in D} \Omega_{QP}(|Q|/|P|)^{1/2} \right)^{1/2}.
\end{aligned}$$

Since $\|\{r_P\}_P\|_{\dot{\mathbf{f}}_2^{02}} \leqslant |P_0|^{-1/2}$, our assumption (10.13) implies that the last expression is finite. The proof is complete. ∎

We may also use the atomic decomposition to prove results for $p, q \leqslant 1$. However, as we have already seen in the proof of Theorem 3.3, a simpler argument can often be used to reduce to the case $p, q > 1$. Consider, for example, the following general version of Theorem 10.3. We temporarily set

$$\tilde{\omega}_{QP}(\varepsilon) = \left(1 + \frac{|x_Q - x_P|}{\max(l(P), l(Q))}\right)^{-n-\varepsilon} \min\left[\left(\frac{l(Q)}{l(P)}\right), \left(\frac{l(P)}{l(Q)}\right)\right]^{(n+\varepsilon)/2};$$

i.e., $\tilde{\omega}_{QP}(\varepsilon)$ is indeendent of α, q, and p; when $\alpha = 0$ and $p, q \geqslant 1$, $\omega_{QP}(\varepsilon)$ coincides with $\tilde{\omega}_{QP}(\varepsilon)$ (cf. (3.1)).

COROLLARY 10.6. *Suppose* $0 < p, q \leqslant +\infty$ *and* $r = \min(1, p, q)$. *If the matrix* $A = \{a_{QP}\}_{Q,P}$ *satisfies*

$$\sup_Q \sum_P \left(|a_{QP}|\,(|P|/|Q|)^{1/2}\right)^r \left(\tilde{\omega}_{QP}(\varepsilon)(|P|/|Q|)^{1/2}\right)^{-\delta} < +\infty \tag{10.16}$$

and

$$\sup_P \sum_Q \left(|a_{QP}|\,(|Q|/|P|)^{1/r-1/2}\right)^r \left(\tilde{\omega}_{QP}(\varepsilon)(|P|/|Q|)^{1/2}\right)^{-\delta} < +\infty, \tag{10.17}$$

for some $\varepsilon, \delta > 0$, *then* A *is a bounded operator on* $\dot{\mathbf{f}}_p^{0q}$.

Proof. We pick t so that $\delta = r/t - 1$, in particular, $0 < t < r \leqslant 1$. We then let $\tilde{A} = \{\tilde{a}_{QP}\}_{Q,P}$ be defined by

$$\tilde{a}_{QP} = |a_{QP}|^t\,(|Q|/|P|)^{1/2 - t/2}.$$

We have p/t, $q/t > 1$, and the boundedness of $\tilde{A}$ on $\dot{\mathbf{f}}_{p/t}^{0,q/t}$ will imply the boundedness of A on $\dot{\mathbf{f}}_p^{0q}$, as in the proof of Theorem 3.3. But (10.7) and (10.8) for $\tilde{A} = \{\tilde{a}_{QP}\}_{Q,P}$ are exactly our assumptions (10.16) and (10.17), respectively, with $s = r/t$. So Therorem 10.3 yields the required boundedness of $\tilde{A}$. ∎

We remark that when studying the boundedness of matrices on $\dot{\mathbf{f}}_p^{\alpha q}$, we usually pick $\alpha = 0$ to simplify notation. This causes no real loss of generality, since a matrix $A = \{a_{QP}\}_{Q,P}$ is bounded on $\dot{\mathbf{f}}_p^{\alpha q}$ if and only if

$$\tilde{A} = \{(|Q|/|P|)^{-\alpha/n}\, a_{QP}\}_{Q,P} \tag{10.18}$$

is bounded on $\dot{\mathbf{f}}_p^{0q}$.

Let us now consider Fourier multipliers again.

COROLLARY 10.7. *Let $\alpha \in \mathbb{R}$, $0<p, q \leqslant +\infty$, and $r=\min(1, p, q)$. Suppose that T_m is a Fourier multiplier operator with multiplier m, which satisfies*

$$\sup_{\nu} \|(1+|x|)^{\delta} (m(2^{\nu}\cdot)\,\bar{\varphi}(\cdot))^{\vee}(x)\|_{L^r} < +\infty \tag{10.19}$$

for some $\delta>0$. Then T_m is a bounded operator on $\dot{\mathbf{F}}_p^{\alpha q}$.

Proof. It is sufficient to show that the operator on $\dot{\mathbf{f}}_p^{\alpha q}$ with matrix $\{a_{QP}\}_{Q,P} = \{\langle T_m \psi_P, \varphi_Q\rangle\}_{Q,P}$ is bounded (cf. Proposition 3.1). Since T_m is a Fourier multiplier operator, $a_{QP}=0$ unless $\frac{1}{2} \leqslant l(Q)/l(P) \leqslant 2$. By making the reduction (10.18) we may take $\alpha=0$. Hence, we only need to prove that the conditions (10.16) and (10.17) are satisfied. In fact, for a Fourier multiplier operator these are equivalent, and, furthermore, they are both equivalent to (10.19). This is a consequence of the general fact (cf. Proposition 10.14) that the retract diagram in Theorem 2.2, and its analogue for Besov spaces, also holds with certain more general measures $w(x)\,dx$ replacing dx. In Appendix E we sketch a direct proof in this particular case. Modulo the proof of the following lemma, the proof is thus complete. ∎

LEMMA 10.8. *Let $\delta, \varepsilon>0$, and let $\{a_{QP}\}_{Q,P} = \{\langle T_m \psi_P, \varphi_Q\rangle\}_{Q,P}$. Then*

$$\sup_{Q} \sum_{P} (|a_{QP}|\,(|P|/|Q|)^{1/2})^r\,(\tilde{\omega}_{QP}(\varepsilon)(|P|/|Q|)^{1/2})^{-\delta}$$
$$\leqslant c \sup_{\nu} \|(1+|x|)^{\tilde{\delta}} (m(2^{\nu}\cdot)\,\bar{\varphi}(\cdot))^{\vee}(x)\|_{L^r}^r,$$

where $\tilde{\delta} = \delta(n+\varepsilon)/r$.

Proof. See Appendix E. ∎

Notice that if we formally set $\delta=0$ in (10.19), then we obtain the condition

$$\sup_{\nu} \|(m(2^{\nu}\cdot)\,\bar{\varphi})^{\vee}(x)\|_{L^r} < +\infty. \tag{10.20}$$

In fact, when $p=q \leqslant 1$, this condition is equivalent (see Corollary 10.7 and Appendix E) to (10.2) with $\alpha=0$ for the matrix with entries $a_{QP} = \langle T_m \psi_P, \varphi_Q\rangle$. Thus (10.20) is necessary and sufficient for a multiplier operator to be bounded on $\dot{\mathbf{F}}_p^{0p}$, $0<p=r\leqslant 1$. When $p=q=1$ this is a result of Taibleson (cf. [Tai] and also [Tay, p. 264]), and appears to be folklore for $p=q<1$.

Remark 10.9. By Hölder's inequality, Corollary 10.7 easily implies the following general Fourier multiplier result of the Hörmander–Mihlin type, which, in the case of general α, p, and q, is due to Triebel (cf. [Tr2]). Recall that $L^2_\beta \approx \mathbf{F}_2^{\beta 2}$ is the usual Bessel potential space, with norm $\|f\|_{L^2_\beta} = \|\hat{f}(\xi)(1+|\xi|^2)^{\beta/2}\|_{L^2}$.

Suppose $\alpha \in \mathbb{R}$, $0 < p, q \leqslant +\infty$, $J = n/\min(1, p, q)$, and T_m is a Fourier multiplier operator with multiplier m, which satisfies

$$\sup_\nu \|m(2^\nu \xi)\, \bar{\varphi}(\xi)\|_{L^2_{J-n/2+\varepsilon}} < +\infty, \tag{10.21}$$

for some $\varepsilon > 0$. Then T_m is a bounded operator on $\dot{\mathbf{F}}_p^{\alpha q}$.

As remarked in the Introduction, special cases of this are the results of Hörmander [Hör1] for L^p, $1 < p < +\infty$, Fefferman–Stein [Fef-S2] for H^1, and Calderón–Torchinsky [Cal-T] for H^p, $0 < p \leqslant 1$.

It is well known (see, e.g., [Ba-S, p. 21]) that Hörmander's theorem fails if we only assume (10.21) with $\varepsilon = 0$, when $p = 1$ and $q = 2$, i.e., $n/2$ derivatives. To get an idea how close we are to a sharp result with our rather simple approach, we record the following consequences of Theorem 10.5 and our remarks concerning (10.10)–(10.12).

COROLLARY 10.10. *Suppose $\Phi(t)$, $t \geqslant 0$, is a nondecreasing function satisfying $\Phi(0) = 1$, $\Phi(2t) \leqslant c\Phi(t)$, $t \geqslant 0$, and that T_m is a Fourier multiplier operator (on $\mathbb{R}^n$) with multiplier m such that*

$$\sup_\nu \|\Phi(|x|)^{1/2}\, (m(2^\nu \cdot)\, \bar{\varphi}(\cdot))^\vee\, (x)\|_{L^2} < +\infty. \tag{10.22}$$

(i) *If*

$$\int_2^\infty \left[\int_{\{x \in \mathbb{R}^n : |x| > t\}} \frac{1}{\Phi(|x|)}\, dx\right]^{1/2} \frac{dt}{t} < +\infty, \tag{10.23}$$

then T_m is a bounded operator on $\dot{\mathbf{F}}_p^{\alpha q}$, $\alpha \in \mathbb{R}$, $1 \leqslant p, q \leqslant +\infty$.

(ii) *If*

$$\int_2^\infty \int_{\{x \in \mathbb{R}^n : |x| > t\}} \frac{1}{\Phi(|x|)}\, dx\, \frac{dt}{t} < +\infty, \tag{10.24}$$

then T_m is a bounded operator on $\dot{\mathbf{F}}_1^{\alpha 2}$, $\alpha \in \mathbb{R}$.

Proof. We define the matrix $\Omega = \{\Omega_{QP}\}$ by

$$\Omega_{QP} = 1/\Phi(|x_P - x_Q|/\max(l(P), l(Q))) \qquad \text{if} \quad \tfrac{1}{2} \leqslant l(P)/l(Q) \leqslant 2,$$

and 0, otherwise. By discretizing, it follows that (10.23) and (10.24) are equivalent to (10.10) with $s'=2$, and (10.13), respectively. The analogue of Lemma 10.8 is also true (cf. Remark E.1):

$$\sup_P \Big(\sum_Q (|a_{QP}|/\Omega_{QP})^2\, \Omega_{QP}(|Q|/|P|)^{1/2}\Big)^{1/2}$$
$$\leqslant C \sup_\nu \|\Phi(|x|)^{1/2}\,(m(2^\nu\cdot)\,\bar{\hat{\varphi}}(\cdot))^\vee\,(x)\|_{L^2},$$

with $\{a_{QP}\}_{Q,P}=\{\langle T_m\psi_P, \varphi_Q\rangle\}_{Q,P}$. To prove (i) it remains only to verify (10.12). For this, first note $\int 1/\Phi(|x|)\,dx<+\infty$ by our assumptions. Hence, by applying Cauchy–Schwarz and using (10.22), (10.20) follows. By Taibleson's theorem noted above, T_m is bounded on $\dot{\mathbf{F}}_1^{01}$, and thus by duality on $\dot{\mathbf{F}}_\infty^{0\infty}$. Passing back to the matrix A (cf. Proposition 3.1), we obtain (10.12). This completes the proof of (i). To complete (ii), our definition of a multiplier operator includes the assumption $m\in L^\infty$. So T_m is bounded on L^2 and $\{a_{QP}\}_{Q,P}$ is thus bounded on $\dot{\mathbf{f}}_2^{02}$. ∎

For example, in (i) we may take

$$\Phi(x)=(1+|x|)^n\,(1+\log^+|x|)^{3+\varepsilon}$$

for any fixed $\varepsilon>0$, and in (ii)

$$\Phi(x)=(1+|x|)^n\,(1+\log^+|x|)^{2+\varepsilon}.$$

Roughly speaking, to obtain a version of Hörmander's theorem on $H^1\approx\dot{\mathbf{F}}_1^{02}$ with our approach, we thus need $n/2$ "full" derivatives and a little more than one "logarithmic derivative." It would be interesting to see if our methods could be applied to obtain the results of Baernstein and Sawyer [Ba-S].

We return to our study of positive operators bounded on $\dot{\mathbf{f}}_p^{\alpha q}$ when $q\neq p$. The next two theorems are (essentially) special cases of results due to Maurey [Mau] and, especially, Rubio de Francia [RdF2] (cf. also [GC-RdF]).

Let w be a nonnegative function. We define the weighted sequence space $\dot{\mathbf{f}}_p^{\alpha q}(w)$, $0<p<+\infty$, by requiring

$$\|s\|_{\dot{\mathbf{f}}_p^{\alpha q}(w)}=\left\|\left(\sum_Q (|Q|^{-\alpha/n}\,|s_Q|\,\tilde{\chi}_Q)^q\right)^{1/q}\right\|_{L^p(w\,dx)}<+\infty.$$

THEOREM 10.11. *Let* $\alpha\in\mathbb{R}$, *and* $0<q<p<+\infty$, *and set* $r=(p/q)'$, $\beta=n/2$. *For a positive operator A the following are equivalent*:

(i) *A is bounded on* $\dot{\mathbf{f}}_p^{\alpha q}$.

(ii) *For each positive sequence* $t=\{t_Q\}_Q \in \dot{\mathfrak{f}}_r^{\beta\infty}$ *there exists a positive sequence* $\tau=\{\tau_Q\}_Q \in \dot{\mathfrak{f}}_r^{\beta\infty}$ *such that*

$$t_Q \leqslant \tau_Q,$$
$$\|\tau\|_{\dot{\mathfrak{f}}_r^{\beta\infty}} \leqslant C\, \|t\|_{\dot{\mathfrak{f}}_r^{\beta\infty}}$$

and

$$\sum_Q (|Q|^{-\alpha/n-1/2}(As)_Q)^q\, \tau_Q \leqslant C \sum_Q (|Q|^{-\alpha/n-1/2} s_Q)^q\, \tau_Q$$

for all nonnegative sequences $s=\{s_Q\}_Q$.

(iii) *For each positive* $v \in L^r$ *there exists a positive function* $w \in L^r$ *such that*

$$v(x) \leqslant w(x) \qquad a.e.,$$
$$\|w\|_{L^r} \leqslant C\, \|v\|_{L^r},$$

and

$$\|As\|_{\dot{\mathfrak{f}}_q^{\alpha q}(w)} \leqslant C\, \|s\|_{\dot{\mathfrak{f}}_q^{\alpha q}(w)}$$

for all nonnegative sequences $s=\{s_Q\}_Q$.

The constants C are independent of t and v, respectively.

THEOREM 10.12. *Let* $\alpha \in \mathbb{R}$, *and* $0<p<q<+\infty$, *and set* $r=(q/p)'/(q/p)$, $\beta=n(q/q'-\frac{1}{2})$. *For a positive operator A the following are equivalent*:

(i) *A is bounded on* $\dot{\mathfrak{f}}_p^{\alpha q}$.

(ii) *For each positive sequence* $t=\{t_Q\}_Q \in \dot{\mathfrak{f}}_r^{\beta\infty}$ *there exists a positive sequence* $\tau=\{\tau_Q\}_Q \in \dot{\mathfrak{f}}_r^{\beta\infty}$ *such that*

$$t_Q \leqslant \tau_Q,$$
$$\|\tau\|_{\dot{\mathfrak{f}}_r^{\beta\infty}} \leqslant C\, \|t\|_{\dot{\mathfrak{f}}_r^{\beta\infty}},$$

and

$$\sum_Q (|Q|^{-\alpha/n+1/2}(As)_Q)^q\, \tau_Q^{-1} \leqslant C \sum_Q (|Q|^{-\alpha/n+1/2} s_Q)^q\, \tau_Q^{-1}$$

for all nonnegative sequences $s=\{s_Q\}_Q$.

(iii) *For each positive function* $v \in L^r$ *there exists a positive function* $w \in L^r$ *such that*

$$v(x) \leqslant w(x) \qquad a.e.,$$
$$\|w\|_{L^r} \leqslant C\, \|v\|_{L^r},$$

and

$$\|As\|_{\dot{\mathfrak{f}}_q^{\alpha q}(w^{-1})} \leqslant C\,\|s\|_{\dot{\mathfrak{f}}_q^{\alpha q}(w^{-1})}$$

for all nonnegative sequences $s=\{s_Q\}_Q$.

The constants C are independent of t and v, respectively.

The proofs of Theorems 10.11 and 10.12 are in Appendix E; ultimately they depend on the Hahn–Banach theorem.

There are also versions of these theorems with the boundedness of A on $\dot{\mathfrak{f}}_p^{\alpha q}$ being equivalent to boundedness on some other, weighted, $\dot{\mathfrak{f}}_p^{\alpha q}$-space. The "diagonal" case we have stated here is of particular interest since we can use Schur's lemma to get one step further.

THEOREM 10.13. *Let* $\alpha \in \mathbb{R}$ *and* $1 \leqslant q < p < +\infty$, *and set* $r=(p/q)'$, $\beta = n/2$. For *a positive operator* A *with matrix* $\{a_{QP}\}_{Q,P}$ *the following are equivalent*:

(i) *A is bounded on* $\dot{\mathfrak{f}}_p^{\alpha q}$.

(ii) *For each positive sequence* $t=\{t_Q\}_Q \in \dot{\mathfrak{f}}_r^{\beta\infty}$ *there exist positive sequences* $\tau=\{\tau_Q\}_Q \in \dot{\mathfrak{f}}_r^{\beta\infty}$ *and* $u=\{u_Q\}_Q$ *such that*

$$t_Q \leqslant \tau_Q,$$

$$\|\tau\|_{\dot{\mathfrak{f}}_r^{\beta\infty}} \leqslant C\,\|t\|_{\dot{\mathfrak{f}}_r^{\beta\infty}},$$

$$\sum_P a_{QP}(|Q|/|P|)^{-\alpha/n-1/2}\,(\tau_Q/\tau_P)^{1/q}\,u_P^{q'} \leqslant C u_Q^{q'},$$

and

$$\sum_Q a_{QP}(|Q|/|P|)^{-\alpha/n-1/2}\,(\tau_Q/\tau_P)^{1/q}\,u_Q^{q} \leqslant C u_P^{q}.$$

Proof. This is just Schur's lemma combined with Theorem 10.11. We use counting measure in Schur's lemma and replace s_P in Theorem 10.11(ii) by $\tilde{s}_P = |P|^{-\alpha/n-1/2}\,\tau_P^{1/q} s_P$. ∎

There is a similar theorem when $p<q$; we then just replace (τ_Q/τ_P) by (τ_P/τ_Q), use the value of β in Theorem 10.12, and replace the exponent $-\alpha/n-\frac{1}{2}$ by $-\alpha/n+\frac{1}{2}$.

One of the applications of characterizations like those in Theorems 10.11–10.12 is to prove extrapolation theorems, see [RdF2]. We shall only consider one theorem of this kind, not directly relying on the previous theorems. Let w be a nonnegative function satisfying the "doubling condition" $w(2Q) \leqslant cw(Q)$, Q dyadic, where, for a measurable set E, $w(E)=\int_E w(x)\,dx$. We define the weighted Triebel–Lizorkin space $\dot{\mathbf{F}}_p^{\alpha q}(w)$, $\alpha\in\mathbb{R}$, $0<p<+\infty$, and $0<q\leqslant+\infty$, by requiring

$$\|f\|_{\dot{\mathbf{F}}_p^{\alpha q}(w)}=\left\|\left(\sum_Q (2^{\alpha\nu}\,|\varphi_\nu * f|)^q\right)^{1/q}\right\|_{L^p(w\,dx)}<+\infty.$$

Proposition 10.14. *Suppose* $\alpha\in\mathbb{R}, 0<p<+\infty, 0<q\leqslant+\infty$, φ *and* ψ *satisfy* (2.1)–(2.4) *and that* w *is doubling. The operators* $S_\varphi\colon \dot{\mathbf{F}}_p^{\alpha q}(w)\to\dot{\mathbf{f}}_p^{\alpha q}(w)$ *and* $T_\psi:\dot{\mathbf{f}}_p^{\alpha q}(w)\to\dot{\mathbf{F}}_p^{\alpha q}(w)$ *are bounded. Furthermore,* $T_\psi\circ S_\varphi$ *is the identity on* $\dot{\mathbf{F}}_p^{\alpha q}(w)$.

Proof. The proof follows the same lines as that of Theorem 2.2 and is outlined in Appendix E. ∎

Let us recall the definition of Muckenhoupt's A_p-classes (cf. [*GC*-RdF]). Let $1\leqslant p<+\infty$. We say that a nonnegative function w belongs to A_p, $w\in A_p$, if

$$\frac{1}{|Q|}\int_Q w(x)\,dx\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1}\leqslant c$$

uniformly for all cubes Q (not necessarily dyadic). When $p=1$, this should be interpreted as

$$\frac{1}{|Q|}\int_Q w(x)\,dx\leqslant c\ \operatorname*{ess.\,inf}_{x\in Q} w(x).$$

The classes A_p are increasing in p, i.e.,

$$A_{p_0}\subset A_{p_1}\qquad\text{if}\quad p_0\leqslant p_1.$$

We also set

$$A_\infty=\bigcup_{p>1} A_p.$$

We recall the following fact about A_p-weights.

Lemma 10.15. *Let* $1<p<+\infty$. *The following are equivalent*:

(i) $w\in A_p$;

(ii) *there is a constant* c *such that given any function* $h\in L^1(dx)$, $\|h\|_{L^1(dx)}=1$, *there are weights* $w_1, w_2\in A_1$ *with* $w=w_1^{1-p}w_2$, $h\leqslant w_1w_2$, *and* $\|w_1w_2\|_{L^1(dx)}\leqslant c$.

Proof. This is a simple consequence of Rubio de Francia's [RdF1] proof of Peter Jones' factorization of A_p-weights; see also Corollary 6.1 and Lemma 6.5 in [Ja5]. ▮

It is well known that $\dot{\mathbf{F}}_p^{02}(w) \approx L^p(w)$ as long as $w \in A_p$ (this is essentially just the fact that the square function is bounded on $L^p(w)$ when $w \in A_p$). The next theorem thus generalizes Rubio de Francia's theorem about extrapolation of weighted L^p-spaces (cf. [GC-RdF]).

THEOREM 10.16. *Suppose that $T: \dot{\mathbf{F}}_{p_0}^{0q_0}(w) \to \dot{\mathbf{F}}_{p_0}^{0q_0}(w)$ is a bounded linear operator whenever $w \in A_{p_0/\lambda}$, for some fixed $0 < p_0 < +\infty$, $0 < q_0 \leqslant +\infty$, and $0 < \lambda \leqslant p_0$. Then $T: \dot{\mathbf{F}}_p^{0q_0}(w) \to \dot{\mathbf{F}}_p^{0q_0}(w)$ is bounded whenever $w \in A_{p/\lambda}$, for each $\lambda < p < +\infty$.*

Proof. The proof follows a standard procedure (see [RdF1; GC-RdF; Ja5]). For a distribution h, we set

$$S(h)(x) = \left(\sum_\nu |\varphi_\nu * h(x)|^{q_0}\right)^{1/q_0}$$

(with a φ satisfying (2.1)–(2.3) as usual). Let us first assume $p > p_0$. For $w \in A_{p/\lambda}$ and f fixed, there is a function $g \in L^{(p/p_0)'}(w\,dx)$, $\|g\|_{L^{(p/p_0)'}(w\,dx)} = 1$, such that

$$\int |S(Tf)|^p\, w(x)\, dx = \left(\int |S(Tf)(x)|^{p_0}\, g(x)\, w(x)\, dx\right)^{p/p_0}.$$

Since $\|g^{(p/p_0)'} w\|_{L^1(dx)} = 1$, Lemma 10.15 yields A_1-weights w_1 and w_2 such that $w = w_1^{1-p/\lambda} w_2$, $g^{(p/p_0)'} w \leqslant w_1 w_2$ and $\|w_1 w_2\|_{L^1(dx)} \leqslant c$. As a consequence, $gw \leqslant w_1^{1-p_0/\lambda} w_2$, and, since $w_1, w_2 \in A_1$, we have that $w_1^{1-p_0/\lambda} w_2 \in A_{p_0/\lambda}$. By hypothesis, then,

$$\int |S(Tf)|^{p_0} gw\, dx \leqslant \int |S(Tf)|^{p_0}\, w_1^{1-p_0/\lambda} w_2\, dx$$

$$\leqslant c \int |S(f)|^{p_0}\, w_1^{1-p_0/\lambda} w_2\, dx.$$

By Hölder's inequality this is dominated by

$$c\left(\int |S(f)|^p\, w_1^{1-p/\lambda} w_2\, dx\right)^{p_0/p} (\|w_1 w_2\|_{L^1(dx)})^{1/(p/p_0)'}$$

$$\leqslant c\left(\int |S(f)|^p\, w\, dx\right)^{p_0/p},$$

which is what we had to prove.

If we instead assume that $p_0 > p$, then, given $S(f) \in L^p(w\,dx)$ with $w \in A_{p/\lambda}$, there exists $g \in L^{p/(p_0-p)}(w\,dx)$, $\|g\|_{L^{p/(p_0-p)}(w\,dx)} = 1$, such that

$$\int |S(f)|^p\, w(x)\,dx = \left(\int |S(f)(x)|^{p_0}\,(g(x))^{-1}\,w(x)\,dx\right)^{p/p_0}.$$

(Take $g = (Sf/\|Sf\|_{L^p(w)})^{p_0-p}$.) Lemma 10.15 implies that there are weights $w_1, w_2 \in A_1$ such that $w = w_1^{1-p/\lambda} w_2$, $g^{p/(p_0-p)} w \leqslant w_1 w_2$, and $\|w_1 w_2\|_{L^1(dx)} \leqslant c$. Now $w_1^{1-p_0/\lambda} w_2 \leqslant g^{-1} w$ and $w_1^{1-p_0/\lambda} w_2 \in A_{p_0/\lambda}$, so we can use the boundedness for $A_{p_0/\lambda}$-weights and Hölder's inequality as before. ∎

THEOREM 10.17. *Suppose* T_m *is a Fourier multiplier operator which is bounded on* $\dot{\mathbf{F}}_{p_0}^{0q_0}(w)$ *whenever* $w \in A_{p_0/\lambda}$, *for some fixed* $0 < p_0, q_0 < +\infty$ *and* $0 < \lambda \leqslant p_0$. *Then* $T_m : \dot{\mathbf{F}}_p^{0q}(w) \to \dot{\mathbf{F}}_p^{0q}(w)$ *is bounded whenever* $w \in A_{p/\lambda}$, *for all* $\lambda < p, q < +\infty$.

Proof. The simple observation we need about multiplier operators is that if T_m is bounded on $\dot{\mathbf{F}}_{p_0}^{0q_0}(w)$, then it is also bounded on the "diagonal" space $\dot{\mathbf{F}}_{p_0}^{0p_0}(w)$. (This follows, for the weighted case, by the usual argument (see [Tr2]) combined with the fact that convolution with φ_ν is uniformly bounded on $\dot{\mathbf{F}}_p^{0q}(w)$, cf. [Ja6].) Suppose now that we want to verify that T_m is bounded on $\dot{\mathbf{F}}_p^{0q}(w)$. Since T is bounded on $\dot{\mathbf{F}}_{p_0}^{0q_0}(w)$, $w \in A_{p_0/\lambda}$, Theorem 10.16 implies the boundedness on $\dot{\mathbf{F}}_q^{0q_0}(w)$, $w \in A_{q/\lambda}$, and the observation then shows that T_m is bounded on $\dot{\mathbf{F}}_q^{0q}(w)$, $w \in A_{q/\lambda}$. Applying Theorem 10.16 again yields that T_m is bounded on $\dot{\mathbf{F}}_p^{0q}(w)$, $w \in A_{p/\lambda}$. This completes the proof. ∎

Notice that the proofs of Theorems 10.16 and 10.17 use virtually nothing about our particular Littelewood–Paley function $(\sum_\nu |\varphi_\nu * f(x)|^q)^{1/q}$, and similar theorems can be proved for more general Littlewood–Paley functions, such as those related to the Nagel–Stein–Wainger results [NSW], (cf. [Ja5, Theorem 7.3, p. 407]).

COROLLARY 10.18. *Suppose* T_m *is a Fourier multiplier operator which is bounded on* $\dot{\mathbf{F}}_{p_0}^{0q_0}(w)$ *whenever* $w \in A_\infty$, *for some fixed* $0 < p_0, q_0 < +\infty$. *Then* $T_m : \dot{\mathbf{F}}_p^{0q}(w) \to \dot{\mathbf{F}}_p^{0q}(w)$ *is bounded whenever* $w \in A_\infty$ *for all* $0 < p, q < +\infty$.

Proof. Let $w \in A_\infty$. Then, by definition, there exists λ such that $w \in A_{p/\lambda}$. By assumption, T_m is bounded on $\dot{\mathbf{F}}_{p_0}^{0q_0}(v)$, $v \in A_{p_0/\lambda} \subset A_\infty$. By Theorem 10.17, T_m is bounded on $\dot{\mathbf{F}}_p^{0q}(w)$. ∎

Theorem 10.17 and Corollary 10.18 should be compared to the classical example by Stein–Zygmund [St-Z] of a multiplier operator bounded on $\dot{\mathbf{F}}_p^{0p}$ which is not bounded on $\dot{\mathbf{F}}_p^{02}$ if $p \neq 2$.

11. Trace Results

The pointwise restriction, or trace, operator $\operatorname{Tr} f(x')=f(x',0)$, $x' \in R^{n-1}$, is originally defined for $f \in \mathscr{S}(\mathbb{R}^n)$ or $f \in \mathscr{S}_0(\mathbb{R}^n)$ (see [Tr 2, p. 237]). If Tr extends to a continuous map from X into Y for some function or distribution spaces X and Y, we say the trace of X exists in Y. If this extension is onto we write $\operatorname{Tr} X = Y$. We have the following results for the $\dot{\mathbf{F}}_p^{\alpha q}$-spaces.

THEOREM 11.1. *Let $\alpha \in \mathbb{R}$, $0<p<+\infty$, and $0<q \leqslant +\infty$.*

(a) *If $\alpha > (n-1)(1/p-1)_+ + 1/p$, then* $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n) = \dot{\mathbf{F}}_p^{\alpha-1/p,p}(\mathbb{R}^{n-1})$.

(b) *If $0<p\leqslant 1$,* $\operatorname{Tr} \dot{\mathbf{F}}_p^{1/p,q}(\mathbb{R}^n) = L^p(\mathbb{R}^{n-1})$, *while if $1<p+\infty$,* $\operatorname{Tr} \dot{\mathbf{F}}_p^{1/p,q}(\mathbb{R}^n)$ *does not exist in $\mathscr{S}'/\mathscr{P}(\mathbb{R}^{n-1})$.*

(c) *If $\alpha < 1/p$,* $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$ *does not exist in $\mathscr{S}'/\mathscr{P}(\mathbb{R}^{n-1})$ or in $L^p + L^\infty(\mathbb{R}^{n-1})$.*

(d) *If $0<p<1$ and $1/p<\alpha<(n-1)(1/p-1)+1/p$,* $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$ *exists in $L^p + L^\infty(\mathbb{R}^{n-1})$, but not in $\mathscr{S}'/\mathscr{P}(\mathbb{R}^{n-1})$.*

(e) *If $0<p<1$ and $\alpha=(n-1)(1/p-1)+1/p$,* $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$ *exists in $L^p + L^\infty(\mathbb{R}^{n-1})$ and in $\mathscr{S}'/\mathscr{P}(\mathbb{R}^{n-1})$.*

Here (a) is known [Ja3] and indicates the essential peculiarity that $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}$ does not depend on q. Also, (d) is known [Ja4; P5] and (c) is essentially known [Tr1] as well. Perhaps (b) and (e) are new.

Proof. This is easy from the standpoint of the smooth atomic decomposition in Theorem 4.1 if we apply Proposition 2.7. We will show directly that $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}$ is independent of q. Given this, we have $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q} = \operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha p}$, and all conclusions follow from the results in [Fr-J1, Section 5], since $\dot{\mathbf{B}}_p^{\alpha p} = \dot{\mathbf{F}}_p^{\alpha p}$. (We point out here that the statement in the introduction of [Fr-J1] has $\max(1,p)$ incorrectly in place of $\min(1,p)$; the statement in Section 5 is correct, however.)

Suppose $0<q<r\leqslant +\infty$. Then trivially $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q} \subset \operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha r}$ and so we need to prove the converse inclusion. We denote a general point x in $\mathbb{R}^n$ by $x=(x',x_n)$, $x' \in \mathbb{R}^{n-1}$, and $x_n \in \mathbb{R}$. Let $A=\{Q \subset \mathbb{R}^n : Q \text{ is a dyadic cube and } \bar{Q} \cap \{x \in R^n : x_n = 0\} \neq \varnothing\}$.

Now, if $f \in \dot{\mathbf{F}}_p^{\alpha r}(\mathbb{R}^n)$, then by Theorem 4.1, $f=\sum_Q s_Q a_Q$, where each a_Q is a smooth atom (for $\dot{\mathbf{F}}_p^{\alpha r}$) associated with Q and the sequence $s=\{s_Q\}_{Q \text{ dyadic}}$ satisfies $\|s\|_{\dot{\mathbf{f}}_p^{\alpha r}} \approx \|f\|_{\dot{\mathbf{F}}_p^{\alpha r}}$. We claim that there is an $\tilde{f} \in \dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$ with $\operatorname{Tr} f = \operatorname{Tr} \tilde{f}$. To see this, we define the sequence $\tilde{s}$ by setting $\tilde{s}_Q = s_Q$ if $Q \in A$ and $\tilde{s}_Q = 0$ otherwise. Note that for each $Q \in A$ there exists a function $\tilde{a}_Q(x)$ satisfying $\tilde{a}_Q(x',0)=a_Q(x',0)$ such that $\tilde{a}_Q$ is, up to a constant, a smooth atom for Q for $\dot{\mathbf{F}}_p^{\alpha q}$. For $q \geq \min(1, p)$, we can just take $\tilde{a}_Q = a_Q$; in general, we can take $\tilde{a}_Q(x', x_n) = g(x_n)\, a_Q(x', 0)$, where $g \in \mathscr{S}(\mathbb{R})$, $g(0) = 1$, $\int t^k g(t)\, dt = 0$ for all $k \leq N$ (defined above), $\operatorname{supp} g \subset (-l(Q), l(Q))$, and $|d^j g(t)/dt^j| \leq c_j l(Q)^{-j}$ for all $t \in \mathbb{R}$ and $j \geq 0$. We now let $\tilde{f} = \sum_Q \tilde{s}_Q \tilde{a}_Q$. We clearly have $\operatorname{Tr} f = \operatorname{Tr} \tilde{f}$. For $Q \in A$, we let $E_Q = \{x \in Q : l(Q)/2 < |x_n| \leq l(Q)\}$. Obviously, $|E_Q|/|Q| = \frac{1}{2}$, so by Proposition 2.7,

$$\|\tilde{s}\|_{\dot{\mathbf{f}}_p^{\alpha q}} \approx \left\| \left(\sum_{Q \in A} (|Q|^{-\alpha/n} |s_Q|\, \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p}.$$

However, for $Q \in A$, the E_Q's are disjoint so the q and $1/q$ in this last expression cancel and can be replaced by r and $1/r$. In particular, $\tilde{s} \in \dot{\mathbf{f}}_p^{\alpha q}$ and $\tilde{f} \in \dot{\mathbf{F}}_p^{\alpha q}(\mathbb{R}^n)$. This verifies our claim and shows that $\operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha r} \subset \operatorname{Tr} \dot{\mathbf{F}}_p^{\alpha q}$ and, consequently, the trace is independent of q. ∎

The case $p = +\infty$ is essentially the same, except that we need to be more careful about the meaning of the trace operator, since $\mathscr{S}_0$ is not dense in $\dot{\mathbf{F}}_\infty^{\alpha q}$.

We first note that the trivial imbeddings of $\dot{\mathbf{f}}_p^{\alpha q}$ and $\dot{\mathbf{F}}_p^{\alpha q}$ into $\dot{\mathbf{f}}_p^{\alpha r}$ and $\dot{\mathbf{F}}_p^{\alpha r}$, respectively, for $0 < q \leq r \leq +\infty$, also hold for $p = +\infty$. For $r < +\infty$ this follows from the definitions (5.1) and (5.4) combined with the imbedding of l^q into l^r and Hölder's inequality. Recall the definition $\|s\|_{\dot{\mathbf{f}}_\infty^{\alpha\infty}} = \sup_Q |Q|^{-\alpha/n - 1/2} |s_Q|$. If $r = +\infty$, then, the result for the $\dot{\mathbf{f}}$-spaces is trivial by (5.5) and implies the result for the $\dot{\mathbf{F}}$-spaces.

For $\alpha > 0$, $\dot{\mathbf{F}}_\infty^{\alpha\infty}$ is a homogeneous version of C^α, and it is well known that each equivalence class in $\dot{\mathbf{F}}_\infty^{\alpha\infty}$ has a continuous representative. Hence the trace operator is defined on $\dot{\mathbf{F}}_\infty^{\alpha q} \subset \dot{\mathbf{F}}_\infty^{\alpha\infty}$ by pointwise restriction for $0 < q \leq +\infty$ and $a > 0$. For $\alpha \leq 0$, we shall say that $\operatorname{Tr} \dot{\mathbf{F}}_\infty^{\alpha q}$ does not exist if the restriction operator does not have a continuous extension from $\mathscr{S}_0$ to the closure of $\mathscr{S}_0$ in $\dot{\mathbf{F}}_\infty^{\alpha q}$. With this understanding, we have the $p = +\infty$ analogue of Theorem 11.1.

THEOREM 11.2. *Let* $\alpha \in \mathbb{R}$, *and* $0 < q \leq +\infty$.

(a) *If* $\alpha > 0$, *then* $\operatorname{Tr} \dot{\mathbf{F}}_\infty^{\alpha q}(\mathbb{R}^n) = \dot{\mathbf{F}}_\infty^{\alpha\infty}(\mathbb{R}^{n-1})$.

(b) *If* $\alpha \leq 0$, $\operatorname{Tr} \dot{\mathbf{F}}_\infty^{\alpha q}(\mathbb{R}^n)$ *does not exist in* $\mathscr{S}'/\mathscr{P}(\mathbb{R}^{n-1})$.

Proof. As in Theorem 11.1, we only need to show that $\operatorname{Tr} \dot{\mathbf{F}}_\infty^{\alpha q}$ is independent of q, since then the results follow from the corresponding results for $\dot{\mathbf{F}}_\infty^{\alpha\infty} = \dot{\mathbf{B}}_\infty^{\alpha\infty}$ in [Fr-J1]. Let $f \in \dot{\mathbf{F}}_\infty^{\alpha\infty}$ and write $f = \sum_Q s_Q a_Q$, where each a_Q is a smooth atom and $s \in \dot{\mathbf{f}}_\infty^{\alpha\infty}$. Define A, $\tilde{s}$, $\tilde{f}$, and E_Q exactly as in the proof of Theorem 11.1. By Proposition 5.4,

$$\|\tilde{s}\|_{\dot{\mathbf{f}}_\infty^{\alpha q}} \approx \sup_{P \text{ dyadic}} \left(\frac{1}{|P|} \int_P \sum_{Q \in A,\, Q \subset P} (|Q|^{-\alpha/n} |s_Q|\, \tilde{\chi}_{E_Q}(x))^q \, dx \right)^{1/q}.$$

The term inside the supremum is 0 unless $P \in A$. If $P \in A$, then $\sum_{Q \subset P, Q \in A} \chi_{E_Q} = \chi_P$. Hence,

$$\|\tilde{s}\|_{\dot{\mathbf{f}}_\infty^{\alpha q}} \leqslant c \sup_Q (|Q|^{-\alpha/n - 1/2} |\tilde{s}_Q|) = c \, \|\tilde{s}\|_{\dot{\mathbf{f}}_\infty^{\alpha\infty}}.$$

Thus by the $p = +\infty$ version of Theorem 3.5, $\tilde{f} \in \dot{\mathbf{F}}_\infty^{\alpha q}$. Since $\operatorname{Tr} \tilde{f} = \operatorname{Tr} f$, we have $\operatorname{Tr} \dot{\mathbf{F}}_\infty^{\alpha\infty} \subset \operatorname{Tr} \dot{\mathbf{F}}_\infty^{\alpha q}$. The converse inclusion is trivial by the aforementioned imbedding, and the result follows. ∎

Part (a) of Theorem 11.2 for $q > 1$ has been proved by Marshall [Mar].

12. Inhomogeneous Triebel–Lizorkin Spaces

Until now we have considered only the homogeneous spaces $\dot{\mathbf{F}}_p^{\alpha q}$. The usual Sobolev speces, however, are included within the scale of the inhomogeneous spaces $\mathbf{F}_p^{\alpha q}(\mathbb{R}^n)$. More generally, we have $L_\alpha^p \approx \mathbf{F}_p^{\alpha 2}$ for $1 < p < +\infty$ and $\alpha > 0$ [Tr2, p. 87], where L_α^p is the usual Bessel potential space. (We have $\mathbf{F}_p^{02} \approx L^p$ for $1 < p < +\infty$, as for $\dot{\mathbf{F}}_p^{02}$.) The inhomogeneous spaces have the advantage of being invariant under diffeomorphisms [Tr2, p. 174] and hence are sometimes more appropriate for local problems. All of our methods and results so far easily adapt to the inhomogeneous case, except for a few notational inconveniences. Hence, the purpose of this section is only to set notation and summarize the corresponding results for the inhomogeneous case, pointing out occasionally where differences arise.

Select a function $\Phi \in \mathscr{S}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp} \hat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2\}, \tag{12.1}$$

and

$$|\hat{\Phi}(\xi)| \geqslant c > 0 \qquad \text{if} \quad |\xi| \leqslant \tfrac{5}{3}. \tag{12.2}$$

Let φ satisfy (2.1)–(2.3) and define $\{\varphi_\nu\}_{\nu \in \mathbb{Z}}$ as usual. For $\alpha \in \mathbb{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$, $\mathbf{F}_p^{\alpha q}$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathbf{F}_p^{\alpha q}} = \|\Phi * f\|_{L^p} + \left\| \left(\sum_{\nu=1}^{\infty} (2^{\nu\alpha} |\varphi_\nu * f|)^q \right)^{1/q} \right\|_{L^p} < +\infty. \tag{12.3}$$

Note that $\|\cdot\|_{\mathbf{F}_p^{\alpha q}}$ is a (quasi-)norm on $\mathscr{S}'$ (rather than $\mathscr{S}'/\mathscr{P}$), since $\hat{\Phi}(0) \neq 0$. In (12.3) the lower frequencies of f have been combined in one term; naturally there are many equivalent ways to do this. For example, if we set $\Phi_k(x) = 2^{kn} \Phi(2^k x)$ for $k \in \mathbb{Z}$, we have the following easy fact.

LEMMA 12.1. *For each $k \in \mathbb{Z}$, we have*

$$\|\Phi_k * f\|_{L^p} + \left\| \left(\sum_{\nu=k+1}^{\infty} (2^{\nu\alpha} |\varphi_\nu * f|)^q \right)^{1/q} \right\|_{L^p} \approx \|f\|_{\mathbf{F}_p^{\alpha q}}$$

for $f \in \mathscr{S}'$, with constants depending on k, α, p, and q.

Proof. Suppose for example that $k < 0$. Suppose we wish to prove that $\|\Phi * f\|_{L^p}$ is dominated by the left side (LS) of the asserted equivalence. We can find $\eta_\nu \in \mathscr{S}$, $\nu = k, ..., 1$, such that

$$\Phi * f = \left(\Phi_k * \eta_k + \sum_{\nu=k+1}^{1} \varphi_\nu * \eta_\nu \right) * f,$$

by (2.3) and (12.2). Then

$$\|\Phi_k * f\|_{L^p} \leqslant c_p \left(\|\Phi_k * \eta_k * f\|_{L^p} + \sum_{\nu=k+1}^{1} \|\varphi_\nu * \eta_\nu * f\|_{L^p} \right)$$

$$\leqslant c \left(\|\Phi_k * f\|_{L^p} + \sum_{\nu=k+1}^{1} \|\varphi_\nu * f\|_{L^p} \right) \leqslant c(\text{LS}).$$

Here, if $1 \leqslant p \leqslant +\infty$, we have used Minkowski's inequality, and, if $0 < p < 1$, a standard result for functions of exponential type, which follows from the Plancherel–Pólya theorem [Pl-P], or see, e.g., [Fr-J1, (2.11)]. The converse estimates hold for similar reasons, e.g., writing $\Phi_k * f = \Phi * f * \eta$, and so on. The case $k > 0$ follows in the same way. ∎

Our inhomogeneous sequence spaces $\mathbf{f}_p^{\alpha q}$ will be indexed by the set of dyadic cubes Q with $l(Q) \leqslant 1$. For $s = \{s_Q\}_{l(Q) \leqslant 1}$, $\alpha \in \mathbb{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$, we set

$$\|s\|_{\mathbf{f}_p^{\alpha q}} = \left\| \left(\sum_{l(Q) \leqslant 1} (|Q|^{-\alpha/n} |s_Q| \tilde{\chi}_Q)^q \right)^{1/q} \right\|_{L^p} < +\infty.$$

The relation between $\mathbf{f}_p^{\alpha q}$ and $\dot{\mathbf{f}}_p^{\alpha q}$ is trivial. To be explicit, define $V: \mathbf{f}_p^{\alpha q} \to \dot{\mathbf{f}}_p^{\alpha q}$ by setting $(Vs)_Q = s_Q$ if $l(Q) \leqslant 1$, and $(Vs)_Q = 0$ otherwise. Obviously V is an isometric imbedding of $\mathbf{f}_p^{\alpha q}$ in $\dot{\mathbf{f}}_p^{\alpha q}$. Virtually any result for $\dot{\mathbf{f}}_p^{\alpha q}$ has an immediate analogue for $\mathbf{f}_p^{\alpha q}$ obtained by applying the homogeneous results to Vs. If $W: \dot{\mathbf{f}}_p^{\alpha q} \to \mathbf{f}_p^{\alpha q}$ is defined by setting $(Ws)_Q = s_Q$ for $l(Q) \leqslant 1$, then W is continuous and $W \circ V$ is the identity on $\mathbf{f}_p^{\alpha q}$. So $\mathbf{f}_p^{\alpha q}$ is a retract of $\dot{\mathbf{f}}_p^{\alpha q}$.

Next we want to show that the relation between $\mathbf{f}_p^{\alpha q}$ and $\mathbf{F}_p^{\alpha q}$ is as one expects. Given φ and Φ as above, we can select ψ satisfying (2.1)–(2.3) and $\Psi \in \mathscr{S}$ satisfying the same conditions as Φ in (12.1)–(12.2), such that

$$\hat{\Phi}(\xi)\, \hat{\Psi}(\xi) + \sum_{\nu=1}^{\infty} \hat{\varphi}(2^{-\nu}\xi)\, \hat{\psi}(2^{-\nu}\xi) = 1 \qquad \text{for all } \xi,$$

where, as usual, $\tilde{\Phi}(x)=\overline{\Phi(-x)}$, and similarly for $\tilde{\varphi}_\nu$. For $Q=Q_{0k}$ (so that $l(Q)=1$), set $\Phi_Q(x)=\Phi(x-k)$, and similarly for Ψ_Q. As in Lemma 2.1, we then have the identity

$$f=\sum_{l(Q)=1}\langle f,\Phi_Q\rangle\Psi_Q+\sum_{\nu=1}^{\infty}\sum_{l(Q)=2^{-\nu}}\langle f,\varphi_Q\rangle\psi_Q, \tag{12.4}$$

for $f\in\mathscr{S}'$ (with convergence in $\mathscr{S}'$). We define the φ-transform $S_\varphi:\mathbf{F}_p^{\alpha q}\to\mathbf{f}_p^{\alpha q}$ by setting $(S_\varphi f)_Q=\langle f,\varphi_Q\rangle$ if $l(Q)<1$ and $(S_\varphi f)_Q=\langle f,\Phi_Q\rangle$ if $l(Q)=1$. We define the *inverse φ-transform* $T_\psi:\mathbf{f}_p^{\alpha q}\to\mathbf{F}_p^{\alpha q}$ by

$$T_\psi s=\sum_{l(Q)=1}s_Q\Psi_Q+\sum_{l(Q)<1}s_Q\psi_Q.$$

We now have the following analogue of Theorem 2.2.

THEOREM 12.2. *Suppose $\alpha\in\mathbb{R}$, $0<p<+\infty$, and $0<q\leqslant+\infty$. The operators $S_\varphi:\mathbf{F}_p^{\alpha q}\to\mathbf{f}_p^{\alpha q}$ and $T_\psi:\mathbf{f}_p^{\alpha q}\to\mathbf{F}_p^{\alpha q}$ are bounded. Furthermore, $T_\psi\circ S_\varphi$ is the identity on $\mathbf{F}_p^{\alpha q}$. In particular, $\|f\|_{\mathbf{F}_p^{\alpha q}}\approx\|S_\varphi f\|_{\mathbf{f}_p^{\alpha q}}$, and $\mathbf{F}_p^{\alpha q}$ can be identified with a complemented subspace of $\mathbf{f}_p^{\alpha q}$.*

Proof. Lemma 2.3 with $\mathbf{f}_p^{\alpha q}$ in place of $\dot{\mathbf{f}}_p^{\alpha q}$ follows immediately from the homogeneous result. Define $\sup_Q(f)$ and $\inf_{Q,\gamma}(f)$ as in Section 2 if $l(Q)<1$, and similarly for $l(Q)=1$, except with $\tilde{\Phi}$ in place of $\tilde{\varphi}_0$. Lemma A.4 only requires exponential type and hence applies to $\tilde{\Phi}*f$. Using Lemma 12.1, the proof of Lemma A.5 goes through, and hence Lemma 2.5 follows. This yields the boundedness of S_φ. Dropping terms like φ_{-1} and ψ_J for $l(J)>1$, and replacing φ_0 and ψ_J for $l(J)=1$ with Φ and Ψ_J, respectively, the proof of the boundedness of T_ψ in Section 2 goes through as well. By (12.4), $T_\psi\circ S_\varphi$ is the identity on $\mathbf{F}_p^{\alpha q}$. ∎

The analogues of Remark 2.6 and Proposition 2.7 are now clear.

We can formally define $S_\varphi^*:\mathscr{L}(\mathbf{F}_p^{\alpha q})\to\mathscr{L}(\mathbf{f}_p^{\alpha q})$ and $T_\psi^*:\mathscr{L}(\mathbf{f}_p^{\alpha q})\to\mathscr{L}(\mathbf{F}_p^{\alpha q})$ as in Section 3, obtaining the retract diagram at the operator level and the equivalence

$$\|S_\varphi^*B\|_{\mathscr{L}(\mathbf{f}_p^{\alpha q})}\approx\|B\|_{\mathscr{L}(\mathbf{F}_p^{\alpha q})}.$$

Here a linear operator B on $\mathbf{F}_p^{\alpha q}$, $0<q<+\infty$, is associated with the operator on $\mathbf{f}_p^{\alpha q}$ with matrix $\{a_{QP}\}_{l(Q)\leqslant1,\,l(P)\leqslant1}$ defined by $a_{QP}=\langle B\psi_P,\varphi_Q\rangle$ for $l(Q)<1$ and $l(P)<1$, and similarly in the other cases, except with ψ_P replaced by Ψ_P if $l(P)=1$ and φ_Q replaced by Φ_Q if $l(Q)=1$. We define almost diagonality by the condition

$$\sup_{l(Q)\leqslant1,\,l(P)\leqslant1}|a_{QP}|/\omega_{QP}(\varepsilon)<+\infty,$$

for $\omega_{QP}(\varepsilon)$ as before. The analogue of Theorem 3.3 for $\mathbf{f}_p^{\alpha q}$ now holds. We define a family $\{m_Q\}$ of smooth molecules for $\mathbf{F}_p^{\alpha q}$ as above if $l(Q) < 1$. If $l(Q) = 1$, we assume (3.5) (3.6) and

$$|m_Q(x)| \leqslant |Q|^{-1/2} (1 + l(Q)^{-1} |x - x_Q|)^{-M},$$

which differs from (3.4) only if $\alpha < 0$. We do not assume that m_Q has any vanishing moments if $l(Q) = 1$. With a similar modification of the conditions (3.7)–(3.10), we obtain that $\{\langle m_P, b_Q \rangle\}$ is almost diagonal as in Lemma 3.8. This follows simply because the vanishing moment conditions for m_Q (and the additional decay in the case $\alpha < 0$) are only used in Lemma 3.8 when we take $h = m_P$ and apply Lemma B.1, i.e., when $l(P) < l(Q)$. In the inhomogeneous case, this never happens if $l(P) = 1$ (similarly for b_Q if $l(Q) = 1$). Hence, we obtain the inhomogeneous analogue of Theorem 3.5. Similar modifications work in Theorem 3.7. For Remark 3.10 (and similarly for Proposition 3.11) we start with b as above and a function B satisfying (3.13)–(3.14) and $|B(x)| \leqslant (1 + |x|)^{-M}$. We define $B_Q(x) = B(x - x_Q)$ for $l(Q) = 1$, and b_Q as before for $l(Q) < 1$, to obtain

$$\|\tilde{B} * f\|_{L^p} + \left\| \left(\sum_{\nu=1}^{\infty} (2^{\nu\alpha} |\tilde{b}_\nu * f|)^q \right)^{1/q} \right\|_{L^p} \leqslant c \, \|f\|_{\mathbf{F}_p^{\alpha q}}.$$

For the analogue of the smooth atomic decomposition in Theorem 4.1 we obtain

$$f = \sum_{l(Q)=1} s_Q A_Q + \sum_{l(Q)<1} s_Q a_Q \tag{12.5}$$

with the a_Q's as before and each A_Q satisfying

$$\operatorname{supp} A_Q \subset 3Q \tag{12.6}$$

and

$$|\partial^\gamma A_Q(x)| \leqslant 1 \qquad \text{if} \quad |\gamma| \leqslant \tilde{K}. \tag{12.7}$$

Similarly, we obtain inhomogeneous versions of Theorem 4.2 and Corollary 4.3 (resp. Theorem 4.4 and Corollary 4.5) if we begin with functions u as above and U satisfying (4.9) (resp. (4.21)), except only with decay of order M, (4.10)–(4.11) (resp., (4.22)–(4.23)), and

$$|\hat{U}(\xi)| \geqslant c > 0 \qquad \text{if} \quad |\xi| \leqslant \tfrac{5}{3},$$

in place of (4.7). We define σ^Q (resp., τ^Q) as before for $l(Q) < 1$, while for $l(Q) = 1$ we use U in place of u.

For $\alpha \in \mathbb{R}$, and $0 < q \leqslant +\infty$, we let $\mathbf{F}_\infty^{\alpha q}$ be the set of all $f \in \mathscr{S}'$ such that

$$\|f\|_{\mathbf{F}_\infty^{\alpha q}} = \|\Phi * f\|_{L^\infty} + \sup_{l(P)<1} \left(\frac{1}{|P|} \int_P \sum_{\nu = -\log_2 l(P)}^{\infty} (2^{\nu\alpha} |\varphi_\nu * f(x)|)^q \, dx \right)^{1/q}$$
$$< +\infty. \tag{12.8}$$

We get an equivalent norm if we delete the $\|\Phi * f\|_{L^\infty}$ term, but take the supremum over $l(P) \leqslant 1$ and replace φ_0 by Φ. This follows from the fact that

$$\|\Phi * f\|_{L^\infty} \approx \sup_{l(P)=1} \left(\int_P |\Phi * f|^q \right)^{1/q},$$

which is a consequence of a local Plancherel–Pólya estimate, as in (2.11) of [Fr-J1].

We define $\mathbf{f}_\infty^{\alpha q}$ simply by restricting the supremum in (5.4) to $l(P) \leqslant 1$. With $V\colon \mathbf{f}_\infty^{\alpha q} \to \dot{\mathbf{f}}_\infty^{\alpha q}$ and $W\colon \dot{\mathbf{f}}_\infty^{\alpha q} \to \mathbf{f}_\infty^{\alpha q}$ formally defined as before, it follows easily again that V is an isometry and $W \circ V$ is the identity on $\mathbf{f}_\infty^{\alpha q}$, so $\mathbf{f}_\infty^{\alpha q}$ is a retract of $\dot{\mathbf{f}}_\infty^{\alpha q}$. Now the results of Section 5 carry over, restricting definitions, sums, and sups to $l(P) \leqslant 1$ and $l(Q) \leqslant 1$ whenever appropriate. In particular, since $\mathbf{F}_p^{02} \approx h^p$, $0 < p \leqslant 1$ (cf. [Tr, p. 50]), we have $\mathbf{F}_\infty^{02} \approx (\mathbf{F}_1^{02})^* \approx bmo$, where h^p and bmo are the local Hardy and BMO spaces (cf. [Go]).

The interpolation results in Section 6 easily yield their inhomogeneous counterparts. For $s = \{s_Q\}_{l(Q) \leqslant 1}$ we set $\|s\|_{\mathbf{f}_0} = |\bigcup_{s_Q \neq 0} Q|$ as above. Defining $V\colon \mathbf{f}_0 \to \dot{\mathbf{f}}_0$ and $W\colon \dot{\mathbf{f}}_0 \to \mathbf{f}_0$ as before, the fact that $\mathbf{f}_p^{\alpha q}$ is a retract of $\dot{\mathbf{f}}_p^{\alpha q}$ extends to $\mathbf{f}_0$ and $\dot{\mathbf{f}}_0$ also. This allows us to derive real interpolation results, including characterizations of K-functionals, for $\mathbf{f}_p^{\alpha q}$, from the results for $\dot{\mathbf{f}}_p^{\alpha q}$. Then the fact that $\mathbf{F}_p^{\alpha q}$ is a retract of $\mathbf{f}_p^{\alpha q}$ (by Theorem 12.2) gives corresponding results for the $\mathbf{F}_p^{\alpha q}$ spaces.

In analogy with our definition in Section 7, we say that $r = \{r_Q\}_{l(Q) \leqslant 1}$ is an atom for $\mathbf{f}_p^{\alpha q}$, $0 < p \leqslant 1$, $p \leqslant q \leqslant +\infty$, if r satisfies the definition of an atom for $\dot{\mathbf{f}}_p^{\alpha q}$ for some $\bar{Q}$ with $l(\bar{Q}) \leqslant 1$. Then Theorem 7.2, with $\dot{\mathbf{f}}_p^{\alpha q}$ replaced by $\mathbf{f}_p^{\alpha q}$, holds. However, the statement of Theorem 7.4 requires further modification in the inhomogeneous case. The reason for this is that in (12.5) the a_Q's may have vanishing moments, but the A_Q's may not. Hence, the two types of terms should not be combined. Instead we write $f = \sum_j \lambda_j A_j + \sum_i \gamma_i h_i$ with A_j's satisfying (12.6)–(12.7) and the h_j's being atoms for $\mathbf{F}_p^{\alpha q}$. We obtain

$$\|f\|_{\mathbf{F}_p^{\alpha q}} \approx \left(\sum |\lambda_j|^p \right)^{1/p} + \left(\sum |\gamma_i|^p \right)^{1/p},$$

in place of (ii) in Theorem 7.4, with a corresponding result for (i).

Theorem 8.2 yields its own inhomogeneous version by considering $Vs \in \dot{\mathbf{f}}_p^{\alpha q}$ for $s \in \mathbf{f}_p^{\alpha q}$. This yields all of the interpolation results in Section 8, with $\mathbf{f}_p^{\alpha q}$ in place of $\dot{\mathbf{f}}_p^{\alpha q}$, which alternatively can be obtained by retraction of the $\dot{\mathbf{f}}_p^{\alpha q}$ results. Retraction again gives the results for $\mathbf{F}_p^{\alpha q}$.

In Section 9, all of the results about sequences carry over immediately. With the same type of modifications regarding Φ and Ψ as above, the results on the distribution space level up through Proposition 9.13 have analogues. Lemma 9.14 and its consequences Theorems 9.15–9.16, which are restricted to $\alpha = 0$, do not have immediate counterparts. This is due to the fact that in Lemma 9.14, the vanishing moment condition is lost in the summation. Remark 9.17 and Examples 9.18–9.19 do have appropriate analogues.

The inhomogeneous versions of all of the results in Section 10 follow immediately, with the exception of the Fourier multiplier results. In fact, these only require minor modifications. E.g., in Corollary 10.7, the supremum is taken only over $\nu \geqslant 0$, and for $\nu = 0$, we replace $\hat{\varphi}$ by $\hat{\Phi}$ (cf. [Tr2, p. 72], for a similar formulation).

The trace results in Section 11 carry over to the $\mathbf{F}_p^{\alpha q}$ spaces, as well, by the same reduction to the diagonal case and then using the results for $\mathbf{F}_p^{\alpha p} = \mathbf{B}_p^{\alpha p}$ in [Fr-J1].

13. Pointwise Multipliers

In this section we shall consider pointwise multipliers on the inhomogeneous spaces $\mathbf{F}_p^{\alpha q}$. Our main result, Theorem 13.3, gives sufficient conditions for the characteristic function of a domain $\Omega \subseteq \mathbb{R}^n$ to be a pointwise multiplier on $\mathbf{F}_p^{\alpha q}$ for appropriate values of the indices. In Theorem 13.7 we also study a geometric condition characterizing the sets Ω for which a certain restriction phenomenon does not depend on q in $\mathbf{f}_p^{\alpha q}$.

For our purposes it suffices to consider locally bounded functions $b(x)$ as pointwise multipliers. In general, if X is a quasi-Banach space continuously imbedded in $\mathscr{S}'(\mathbb{R}^n)$, we define pointwise multipliers on X in the following way, e.g., as in [Tr2, p. 140]. Let $\gamma \in \mathscr{S}(\mathbb{R}^n)$ satisfy $\hat{\gamma}(\xi) = 1$ for $|\xi| \leqslant 1$ and let $\gamma_\nu(x) = 2^{\nu n}\gamma(2^\nu x)$ for $\nu = 0, 1, 2, \ldots$. Then for $b \in \mathscr{S}'$, the function $b_\nu = b * \gamma_\nu$ is smooth. We say that $b(x)$ is a *pointwise multiplier* for X, and write $b \in \mathbf{M}X$ if, for all $f \in X$, the sequence $b_\nu f$ converges in $\mathscr{S}'$ as $\nu \to \infty$ to an element $g \in X$, and there exists $c > 0$, independent of f, such that $\|g\|_X \leqslant c\, \|f\|_X$. If Y is a quasi-Banach space of locally bounded functions on $\mathbb{R}^n$, we write $Y \subseteq \mathbf{M}X$ if $b \in \mathbf{M}X$ for each $b \in Y$ and there exists $c > 0$ such that $\|\lim_{\nu \to \infty} b_\nu f\|_X \leqslant c\, \|b\|_Y\, \|f\|_X$ for all $b \in Y$ and $f \in X$. It is easy to see that if X is a Banach space, then $\mathbf{M}X = \mathbf{M}X^*$, but if X is quasi-Banach we only have $\mathbf{M}X \subseteq \mathbf{M}X^*$.

In case $\mathscr{S}$ is dense in X in the quasi-norm on X, e.g., $X=\mathbf{F}_p^{\alpha q}$ for $0<p,q<+\infty$, it is sufficient to show that $\|bf\|_X \leqslant c\,\|f\|_X$ for all $f\in\mathscr{S}$, and extend the action of b to all of X by linearity, to prove that $b\in\mathbf{M}X$. However, in the general case we require the interpretation above.

To start, we would like to make two observations about pointwise multipliers on the $\mathbf{F}_p^{\alpha q}$ spaces. The first is well known [Tr2, p. 143], but is particularly simple from our point of view.

Remark 13.1. For $\beta>0$, let $C^\beta=F_\infty^{\beta\infty}$ $(=\Lambda_\beta=B_\infty^{\beta\infty})$ denote the usual Hölder or Lipschitz spaces. We have $C^\beta\subseteq\mathbf{MF}_p^{\alpha q}$ if either

$$0<p,q\leqslant+\infty \qquad \text{and} \qquad \beta>\alpha>J-n, \tag{13.1}$$

or

$$1<p,q<+\infty \qquad \text{and} \qquad \beta>|\alpha|. \tag{13.2}$$

To see this, we consider (13.1) first. By (12.5), we can write each $f\in\mathbf{F}_p^{\alpha q}$ as $f=\sum_{l(Q)\leqslant 1} s_Q a_Q$, where each a_Q satisfies

$$\operatorname{supp} a_Q\subseteq 3Q, \tag{13.3}$$

and

$$|\partial^\gamma a_Q(x)|\leqslant|Q|^{-1/2-|\gamma|/n} \qquad \text{if} \quad |\gamma|\leqslant[\alpha+1]_+, \tag{13.4}$$

and the sequence $s=\{s_Q\}_{l(Q)\leqslant 1}$ satisfies

$$\|s\|_{\mathbf{f}_p^{\alpha q}}\leqslant c\,\|f\|_{\mathbf{F}_p^{\alpha q}}. \tag{13.5}$$

If we set $m_Q=ba_Q$, then $\operatorname{supp} m_Q\subseteq 3Q$, and since $l(Q)\leqslant 1$, it is easy to see that (3.4)–(3.6) hold for $\delta=\min(1,\beta-[\alpha])>\alpha^*$. Since (3.3) is void for $\alpha>J-n$, m_Q is a smooth molecule for Q. Thus by the inhomogeneous version of Theorem 3.5,

$$\|b\|_{\mathbf{F}_p^{\alpha q}}=\Big\|\sum_{l(Q)\leqslant 1} s_Q m_Q\Big\|_{\mathbf{F}_p^{\alpha q}}\leqslant c\,\|s\|_{\mathbf{f}_p^{\alpha q}}\leqslant c\,\|f\|_{\mathbf{F}_p^{\alpha q}}.$$

Then (13.2) follows from (13.1) by duality and interpolation (the inhomogeneous analogues of Remark 5.14 and Theorem 8.5).

Triebel states [Tr2, p. 143] that if $p\leqslant q$ and $\beta<\max(\alpha,J-n-\alpha)$, then there exists $g\in C^\beta$ which is not a pointwise multiplier for $\mathbf{F}_p^{\alpha q}$. This indicates that (3.4)–(3.6) are reasonably sharp conditions, since, for example if $J-n-\alpha<\alpha$ and $p\leqslant q$, we cannot take $0<\delta<\alpha^*$ in (3.6) and still obtain Theorem 3.5 in the homogeneous case, else our argument would apply for some $\beta<\alpha$, contradicting Triebel's remark.

Remark 13.2. For $q=2$ and $1<p<+\infty$, $\mathbf{F}_p^{02} \approx L^p$, so it is clear that $\mathbf{MF}_p^{02}=L^\infty$. This can be regarded as a limiting case of the result from Lemma 13.1 that for $1 \leqslant p, q \leqslant +\infty$, $C^\beta \subseteq \mathbf{MF}_p^{0q}$ for all $\beta>0$. For $1<p, q<+\infty$ and $\alpha \in \mathbb{R}$, duality and interpolation ($[\mathbf{F}_p^{\alpha q}, \mathbf{F}_{p'}^{-\alpha q'}]_{1/2}=\mathbf{F}_2^{02}=L^2$) show that $\mathbf{MF}_p^{\alpha q} \subseteq \mathbf{M}L^2=L^\infty$. From the trace problem in Section 11, one might expect that $\mathbf{MF}_p^{\alpha q}$ is "independent of q," i.e., that $\mathbf{MF}_p^{0q}=L^\infty$ for al q. However, it is easy to see that if the complex exponentials were pointwise multipliers on $\mathbf{f}_p^{0q}$ (or even if they just mapped $\mathbf{F}_p^{0q}$ into $\mathbf{F}_p^{0\infty}$), then this would imply that $\mathbf{F}_p^{0q} \subset L^p \approx \mathbf{F}_p^{02}$, $1<q \leqslant +\infty$. Of course, this is only possible if $q \leqslant 2$. Hence, by duality, if $1<p, q \leqslant +\infty$ and $q \neq 2$ we have $L^\infty \nsubseteq \mathbf{MF}_p^{0q}$. This answers a point that arose in [AF]. (To verify the statement about the complex exponentials, let φ satisfy (2.1)–(2.3) and $\hat{\varphi}(1, 0, ..., 0)=1$. Then $\|\sup_\nu |\varphi_\nu * f| \|_{L^p} \leqslant \|f\|_{\mathbf{F}_p^{0\infty}}$. If we let $\Theta(x)=e^{ix_1}\varphi(x)$, then $\hat{\Theta}(0)=1$ and $\|f\|_{L^p} \approx \lim_{\nu \to \infty} \|f * \Theta_\nu\|_{L^p} \leqslant \sup_\nu \|f * e^{i2^\nu x_1}\varphi_\nu\|_{L^p}=\sup_\nu \|(e^{-i2^\nu x_1}f) * \varphi_\nu\|_{L^p} \leqslant c\,\|f\|_{\mathbf{F}_p^{0q}}$.)

We now turn to the main problem we wish to consider. Let Ω be an open subset of $\mathbb{R}^n$ (not necessarily bounded or connected). We ask for conditions yielding $\chi=\chi_\Omega \in \mathbf{MF}_p^{\alpha q}$ for appropriate α, p, and q.

Fix Ω. We consider a quantitative condition on Ω. For $x \in \mathbb{R}^n$, let $\delta(x)$ be the distance from x to $\Omega^c=\mathbb{R}^n \backslash \Omega$. For $s>0$ and Q a cube in $\mathbb{R}^n$, not necessarily dyadic, let

$$\rho_s(Q, \Omega)=l(Q)\left(\frac{1}{|Q|}\int_{\Omega \cap Q} \frac{1}{\delta(x)^s}\,dx\right)^{1/s}.$$

We let D_s be the set of domains $\Omega \subseteq \mathbb{R}^n$ such that

$$\|\Omega\|_s \equiv \sup_{Q \text{ dyadic},\, l(Q) \leqslant 1} \rho_s(Q, \Omega)<+\infty. \tag{13.6}$$

We shall make several remarks about the condition $\Omega \in D_s$. First, the condition is monotonic; that is, by Hölder's inequality $D_s \subseteq D_t$ for $s \geqslant t$. To get a better understanding of the class D_s, we make an elementary computation in the special case $\Omega=\mathbb{R}_+^n=\{x \in \mathbb{R}^n : x_n>0\}$. Suppose Q is dyadic with $l(Q)=2^{-\mu}$, and $0<s<1$. Then

$$\begin{aligned}\rho_s(Q, \Omega)&=2^{-\mu}\left(2^{\mu n}\int_Q \frac{1}{|x_n|^s}\,dx\right)^{1/s}\\ &\leqslant 2^{-\mu+\mu/s}\left(\int_0^{2^{-\mu}} \frac{1}{t^s}\,dt\right)^{1/s}=(1-s)^{-1/s}.\end{aligned} \tag{13.7}$$

It follows that $\mathbb{R}_+^n \in D_s$ for $0<s<1$. This is typical for an Ω with a nice boundary.

Now suppose $\Omega = P^\circ$ for a dyadic cube P with $l(P) \leqslant 1$ (where E° denotes the interior of the set E). Let $\delta_i(x)$ be the distance from x to the ith face of P, $i = 1, ..., 2n$. Then obviously $\delta_i(x)^{-s} \leqslant \delta(x)^{-s} \leqslant 2\sum_{i=1}^n \delta_i(x)^{-s}$ for each $s > 0$. Hence, if $0 < s < 1$, the previous calculation and a dilation argument show that $\int_P \delta(x)^{-s}\, dx = c_n l(P)^{n-s}$. For an arbitrary Ω this implies that $\int_Q \delta(x)^{-s}\, dx \leqslant c_n l(Q)^{n-s}$ whenever Q is a cube with $Q \subseteq \Omega$ and $0 < s < 1$. As a consequence

$$\sup\{\rho_s(Q, \Omega) : Q^\circ \cap \partial\Omega = \varnothing\} \leqslant c_{n,s} < +\infty,$$

so we only need check (13.6) for Q such that $Q \cap \partial\Omega \neq \varnothing$, if $0 < s < 1$.

For that matter, it suffices to check (13.6) only for all sufficiently small dyadic cubes intersecting $\partial\Omega$. To see this, suppose $k \in \mathbb{Z}$, $k > 0$, and

$$A_k = \sup\{\rho_s(Q, \Omega) : Q \text{ is dyadic, } l(Q) \leqslant 2^{-k}\} < +\infty.$$

If Q is dyadic with $l(Q) = 2^{-k+1}$, and $\{Q_j\}_{j=1}^{2^n}$ are the dyadic subcubes of Q with $l(Q_j) = 2^{-k}$, then

$$\int_{Q \cap \Omega} \delta(x)^{-s}\, dx = \sum_{j=1}^{2^n} \int_{Q_j \cap \Omega} \delta(x)^{-s}\, dx \leqslant 2^s A_k^s l(Q)^{n-s}.$$

This shows that $A_{k-1} \leqslant 2A_k$ and $\|\Omega\|_s \leqslant 2^k A_k < +\infty$. Finally, it is clear that there exists $c_{n,s}$ such that for any cube, not necessarily dyadic, $\rho_s(Q, \Omega) \leqslant c_{n,s} \|\Omega\|_s$.

For our next theorem we also need a slight modification of the Whitney decomposition. For an open set $\Omega \subseteq \mathbb{R}^n$, let $\mathscr{F}_0$ be the set of dyadic cubes in the Whitney decomposition of Ω. The basic properties of this decomposition (see [St, pp. 167–169]) that we require are:

$$\bigcup_{Q \in \mathscr{F}_0} \bar{Q} = \Omega, \tag{13.8}$$

$$Q_1^\circ \cap Q_2^\circ = \varnothing \quad \text{if} \quad Q_1 \neq Q_2,\ Q_1, Q_2 \in \mathscr{F}_0, \tag{13.9}$$

$$\operatorname{diam} Q \leqslant \operatorname{dist}(Q, \Omega^c) \leqslant 4 \operatorname{diam} Q \quad \text{if} \quad Q \in \mathscr{F}_0, \tag{13.10}$$

$$\tfrac{1}{4} \leqslant l(Q_1)/l(Q_2) \leqslant 4 \quad \text{if} \quad Q_1, Q_2 \in \mathscr{F}_0 \quad \text{and} \quad \bar{Q}_1 \cap \bar{Q}_2 \neq \varnothing, \tag{13.11}$$

and

$$\text{if } x \in \Omega, \text{ then there exist at most } c_n \text{ cubes } Q \in \mathscr{F}_0 \text{ such that } x \in 1.1Q. \tag{13.12}$$

Since the dyadic cubes in our decomposition of $\mathbf{F}_p^{\alpha q}$ satisfy $l(Q) \leqslant 1$, we subdivide the cubes in $\mathscr{F}_0$ of sidelength greater than 1. Let

$$\mathscr{F}_1 = \{Q \in \mathscr{F}_0 : l(Q) > 1\}, \qquad \mathscr{F}_2 = \mathscr{F}_0 \backslash \mathscr{F}_1,$$

and

$$\mathscr{F}_3 = \{Q \text{ dyadic} : l(Q) = 1 \text{ and there exists } \tilde{Q} \in \mathscr{F}_1 \text{ such that } Q \subseteq \tilde{Q}\}.$$

The collection we shall use is $\mathscr{F} = \mathscr{F}_2 \cup \mathscr{F}_3$. It is easy to see then that (13.8)–(13.9) and (13.11)–(13.12), with $\mathscr{F}_0$ replaced by $\mathscr{F}$, still hold. In place of (13.10) we have in general only diam $Q \leqslant \operatorname{dist}(Q, \Omega^c)$ if $Q \in \mathscr{F}$. Let

$$\mathscr{C} = \{P \text{ dyadic} : l(P) \leqslant 1 \text{ and } (3P)^\circ \cap \partial\Omega \neq \varnothing\}.$$

Note that if $P \in \mathscr{C}$, then $\operatorname{dist}(P, \Omega^c) < \operatorname{diam} P$, and hence $P \notin \mathscr{F}$ and no dyadic cube containing P belongs to $\mathscr{F}$. For $P \in \mathscr{C}$, let

$$N(3P) = \left\{Q \in \mathscr{F} : \bar{Q} \bigcap \overline{3P} \neq \varnothing\right\}.$$

If $Q \in N(3P)$, then diam $Q \leqslant \operatorname{dist}(Q, \Omega^c) \leqslant 3 \operatorname{diam} P$. This implies that $l(Q) \leqslant 2l(P)$, since we must have $l(Q) = 2^k l(P)$ for some $k \in \mathbb{Z}$. In particular, it follows that

$$\bigcup \{Q : Q \in N(3P)\} \subseteq 7P \qquad \text{if} \quad P \in \mathscr{C}. \tag{13.13}$$

(The precise number 7 will not be important though.)

We also note that $\operatorname{dist}(Q, \Omega^c) \leqslant 4 \operatorname{diam} Q$ if $Q \in N(3P)$ for some $P \in \mathscr{C}$. (For $Q \in \mathscr{F}$ such that $l(Q) < 1$, this follows by (13.10). If $l(Q) = 1$, then $\operatorname{dist}(Q, \Omega^c) \leqslant 3 \operatorname{diam} P \leqslant 3\sqrt{n} = 3 \operatorname{diam} Q$, since $l(P) \leqslant 1$ for all $P \in \mathscr{C}$.). Hence, for all $P \in \mathscr{C}$, $Q \in N(3P)$, and $x \in Q$, we have

$$\sqrt{n}\, l(Q) \leqslant \operatorname{dist}(x, \Omega^c) \leqslant 5\sqrt{n}\, l(Q). \tag{13.14}$$

We construct a partition of unity of Ω corresponding to $\mathscr{F}$ in a standard way. Fix a function $\gamma \in \mathscr{D}$ satisfying $0 \leqslant \gamma(x) \leqslant 1$ for all x, and $\operatorname{supp} \gamma \subseteq [-0.1, 1.1]^n$. For P dyadic, let $\gamma^*(P)(x) = \gamma((x - x_P)/l(P))$. For $P \in \mathscr{F}$, let

$$\gamma(P)(x) = \gamma_P^*(x) \Big/ \sum_{Q \in \mathscr{F}} \gamma^*(Q)(x).$$

It is clear that $\sum_{P \in \mathscr{F}} \gamma(P)(x) = 1$ for $x \in \Omega$, $0 \leqslant \gamma(P)(x) \leqslant 1$ for all x and P, $\gamma(P) \in \mathscr{D}$ for all P, and $\operatorname{supp} \gamma(P) \subseteq 1.1P$. It also follows (as in [St, p. 174]) that

$$|\partial^\beta \gamma(P)(x)| \leqslant c_\beta l(P)^{-|\beta|} \qquad \text{for all } \beta \text{ and all } P \in \mathscr{F}. \tag{13.15}$$

Finally, by (13.11) for $\mathscr{F}$ (and since $\frac{1}{10} < \frac{1}{4}$), we have

$$\sum_{Q \in N(3P)} \gamma(Q)(x) = 1 \qquad \text{for} \quad x \in 3P \cap \Omega \quad \text{and} \quad P \in \mathscr{C}. \tag{13.16}$$

THEOREM 13.3. *Let Ω be a domain in $\mathbb{R}^n$, and let $\chi = \chi_\Omega$. Suppose $0 < p < +\infty$, $0 < q \leqslant +\infty$, and $\Omega \in D_s$ for some $s > 0$. Then $\chi \in \mathbf{MF}_p^{\alpha q}$ if either*

$$0 < p \leqslant 1 \qquad \textit{and} \qquad J - n < \alpha \leqslant s/p, \tag{13.17}$$

or

$$1 < p < +\infty \qquad \textit{and} \qquad J - n < \alpha < s/p. \tag{13.18}$$

Proof. First assume $q = +\infty$. By (12.5), we can write $f = \sum_P s_P a_P$, where

$$\| s \|_{\mathbf{f}_p^{\alpha\infty}} \leqslant c \, \| f \|_{\mathbf{F}_p^{\alpha\infty}}, \tag{13.19}$$

for $s = \{s_P\}_{P \text{ dyadic}}$, and each a_P satisfies $\operatorname{supp} a_P \subseteq 3P$, and

$$|\partial^\gamma a_P(x)| \leqslant |P|^{-1/2 - |\gamma|/n} \qquad \text{if} \quad |\gamma| \leqslant [\alpha + 1]_+. \tag{13.20}$$

(Since we are in the range $\alpha > J - n$, we have $N = -1$ and the moment condition (4.2) may be taken to be void.)

Let $\mathscr{B} = \{P \text{ dyadic: } l(P) \leqslant 1 \text{ and } (3P)^0 \subseteq \Omega\}$ and $\mathscr{E} = \{P \text{ dyadic: } l(P) \leqslant 1$ and $(3P)^0 \subseteq \Omega^c\}$. Let

$$f = \sum_{P \in \mathscr{C}} s_P a_P + \sum_{P \in \mathscr{B}} s_P a_P + \sum_{P \in \mathscr{E}} s_P a_P \equiv f_1 + f_2 + f_3.$$

Then $\chi f_3 = 0$ and $\chi f_2 = f_2$, hence $\| \chi f_2 \|_{\mathbf{F}_p^{\alpha p}} = \| f_2 \|_{\mathbf{F}_p^{\alpha p}} \leqslant c \, \| s \|_{\mathbf{f}_p^{\alpha p}} \leqslant c \, \| f \|_{\mathbf{F}_p^{\alpha p}}$. So we only need to consider f_1, and thus we assume $s_P = 0$ if $P \notin \mathscr{C}$.

Given $P \in \mathscr{C}$, by (13.16) we can write

$$\chi a_P(x) = \sum_{Q \in N(3P)} \gamma(Q)(x) \, a_P(x).$$

By Leibniz's rule, (13.15), (13.20), and the fact that $l(Q) \leqslant 2l(P)$ for $Q \in N(3P)$, we have

$$\begin{aligned} &|\partial^\beta(\gamma(Q)(x) \, a_P(x))| \\ &\quad \leqslant \sum_{\rho + \sigma = \beta} c_{\rho,\sigma} l(Q)^{-|\rho|} \, |P|^{-1/2} \, l(P)^{-|\sigma|} \leqslant C \, |Q|^{-1/2} \, l(Q)^{-|\beta|}, \end{aligned}$$

if $|\beta| \leqslant [\alpha + 1]_+$. Hence, if, for $P \in \mathscr{C}$ and $Q \in N(3P)$, we define

$$b_Q(P)(x) = \frac{1}{C} \left(\frac{|Q|}{|P|} \right)^{-1/2} \gamma(Q)(x) \, a_P(x),$$

then $b_Q(P)$ is a smooth atom corresponding to Q. We have

$$\chi f = \sum_{P \in \mathscr{C}} s_P \chi a_P = C \sum_{P \in \mathscr{C}} s_P \sum_{Q \in N(3P)} (|Q|/|P|)^{1/2}\, b_Q(P).$$

Define $t_Q = 0$ and $h_Q(x) = 0$ if $Q \notin \mathscr{F}$, and if $Q \in \mathscr{F}$ set

$$t_Q = \sum_{P \in \mathscr{C} : Q \in N(3P)} (|Q|/|P|)^{1/2}\, |s_P|$$

and

$$h_Q = \frac{1}{t_Q} \sum_{P \in \mathscr{C} : Q \in N(3P)} (|Q|/|P|)^{1/2}\, s_P b_Q(P)(x).$$

Then $\chi f = \sum_{Q \in \mathscr{F}} t_Q h_Q$. Each h_Q is a smooth atom corresponding to Q, since estimates for $\partial^\gamma h_Q$ are dominated by those for a convex combination of the smooth atoms $b_Q(P)$ for Q. By the inhomogeneous version of Theorem 3.5, we have $\|\chi f\|_{F_p^{\alpha\infty}} \leqslant c\, \|t\|_{\mathbf{f}_p^{\alpha\infty}}$. Hence, by (13.19), the conclusion follows if

$$\|t\|_{\mathbf{f}_p^{\alpha\infty}} \leqslant c\, \|s\|_{\mathbf{f}_p^{\alpha\infty}}. \tag{13.21}$$

If we define the matrix $A = \{a_{QP}\}_{Q,P}$ by

$$a_{QP} = (|Q|/|P|)^{1/2} \qquad \text{if} \quad P \in \mathscr{C} \quad \text{and} \quad Q \in N(3P),$$

and $a_{QP} = 0$ otherwise, then $t = A(s)$. We thus see that (13.21) is equivalent to the fact that A is bounded on $\mathbf{f}_p^{\alpha\infty}$. To prove this we consider the cases $0 < p \leqslant 1$ and $p > 1$, separately.

Suppose $0 < p \leqslant 1$. For $Q \in \mathscr{F}$ and P_0 fixed, we set $\mathscr{C}_0 = \mathscr{C}_{Q,P_0,\Omega} = \{P \in \mathscr{C} : P \subset P_0 \text{ and } Q \in N(3P)\}$. Using Proposition 10.1, we see that A is bounded on $\mathbf{f}_p^{\alpha\infty}$ if and only if

$$\|A\|_{\alpha,p,\infty} \equiv \sup_{P_0} \frac{1}{|P_0|^{1/p}} \left\| \left\{ \sum_{P \in \mathscr{C}_Q} |Q|^{1/2}\, |P|^{\alpha/n} \right\}_{Q \in \mathscr{F}} \right\|_{\mathbf{f}_p^{\alpha\infty}} < +\infty.$$

Notice now that there are at most a fixed number of dyadic cubes P of a given sidelength in $\mathscr{C}_Q$ (for Q fixed), since such a cube must satisfy $l(Q) \leqslant 2l(P)$ and, by (13.13), $Q \subset 7P$. Hence, as long as $\alpha > 0$, the sum in (13.22) is (essentially) geometric and can be estimated by $c\, |Q|^{1/2}\, |P_0|^{\alpha/n}$. Inserting this in (13.22), we find that

$$\|A\|_{\alpha,p,\infty} \leqslant c \sup_{P_0} \frac{1}{|P_0|^{1/p - \alpha/n}} \|\{|Q|^{1/2}\}_{Q : \mathscr{C}_Q \neq \varnothing}\|_{\mathbf{f}_p^{\alpha\infty}}.$$

If $\mathscr{C}_Q \neq \varnothing$, then, by (13.13), $Q \subset 7P_0 \cap \Omega$, and, by (13.14), $|Q|^{1/n} \approx \delta(x)$ for $x \in Q$. This and the fact that the cubes Q in $\mathscr{F}$ are pairwise disjoint shows that

$$\| \{|Q|^{1/2}\}_{Q:\mathscr{C}_Q \neq \varnothing} \|^p_{\mathbf{f}_p^{\alpha\infty}} \leqslant c \int_{7P_0 \cap \Omega} \frac{1}{\delta(x)^{\alpha p}}\, dx.$$

This yields

$$\|A\|_{\alpha, p, \infty} \leqslant c\, \|\Omega\|^{\alpha}_{\alpha p} < +\infty,$$

and completes the proof when $0 < p \leqslant 1$.

Let us now consider $1 < p < +\infty$. By (10.3), the matrix A is bounded on $\mathbf{f}_{\infty}^{\alpha_1 \infty}$ if and only if

$$\|A\|_{\alpha_1, \infty, \infty} \equiv \sup_{Q} \sum_{P:\, Q \in N(3P)} (|P|/|Q|)^{\alpha_1/n} < +\infty.$$

For a fixed Q, we have $l(Q) \leqslant 2l(P)$ if $Q \in N(3P)$. So, there are at most a fixed number of cubes P of any given sidelength such that $Q \in N(3P)$. Hence, the series sums and $\|A\|_{\alpha_1, \infty, \infty} < +\infty$ exactly when $\alpha_1 < 0$. Therefore, if $\Omega \in D_s$, A is bounded on $\mathbf{f}_1^{\alpha_0 \infty}$ with $\alpha_0 = s$, by the first part of the proof, and on $\mathbf{f}_{\infty}^{\alpha_1 \infty}$ for any $\alpha_1 < 0$. By interpolation (again only Proposition 8.1 and Theorem 8.2 are needed), A is bounded on $\mathbf{f}_p^{\alpha \infty}$ with $1/p = (1-\theta)$ and $\alpha = (1-\theta)\, s + \theta \alpha_1$. Taking $\alpha_1 < 0$ sufficiently close to 0, we obtain boundedness of A for an arbitrary $\alpha < s/p$. This proves the theorem in the case $q = +\infty$.

In the general case, note that our assumptions (13.17)–(13.18) guarantee that we are always in the case where no vanishing moments are required for smooth molecules for $\mathbf{F}_p^{\alpha q}$. Hence, writing $\chi f = \sum t_Q h_Q$ as before (where $s_P = 0$ for $P \notin \mathscr{C}$, as before also), each h_Q is a smooth atom for $\mathbf{F}_p^{\alpha q}$ for Q and, similar to the case $q = +\infty$, it suffices to prove

$$\|t\|_{\mathbf{f}_p^{\alpha q}} \leqslant c\, \|s\|_{\mathbf{f}_p^{\alpha q}}. \tag{13.22}$$

By (13.9), the cubes in $\mathscr{F}$ are pairwise disjoint; therefore,

$$\|t\|_{\mathbf{f}_p^{\alpha q}} = \|t\|_{\mathbf{f}_p^{\alpha \infty}}. \tag{13.23}$$

By the case $q = +\infty$, $\|t\|_{\mathbf{f}_p^{\alpha\infty}} \leqslant c\, \|s\|_{\mathbf{f}_p^{\alpha\infty}}$. The trivial imbedding $\mathbf{f}_p^{\alpha q} \to \mathbf{f}_p^{\alpha \infty}$ now yields (13.22), completing the proof. ∎

Assume for a moment that $|\partial\Omega| = 0$. Then $\chi_{\partial\Omega} = 0$ in $\mathscr{S}'$ and hence the operator $f \to \chi_{\partial\Omega} f$ is zero on $\mathbf{F}_p^{\alpha q}$. If $\tilde{\Omega}$ is the domain complementary to Ω, i.e., $\tilde{\Omega} = (\Omega^c)^0$, then $\chi_\Omega \in \mathbf{MF}_p^{\alpha q}$ if and only if $\chi_{\tilde{\Omega}} \in \mathbf{MF}_p^{\alpha q}$. This indicates that the D_s condition can be symmetrized. We define a class $\tilde{D}_s$ for $s>0$ consisting of all domains such that

$$\sup_{Q \text{ dyadic}, \, l(Q) \leqslant 1} \min\{\rho_s(Q, \Omega), \rho_s(Q, \tilde{\Omega})\} < +\infty.$$

(The function δ in $\rho_s(Q, \tilde{\Omega})$ is the distance to Ω.) Then Theorem 13.3 and the following two corollaries hold with $\tilde{D}_s$ in place of D_s. To see this, partition the set $\mathscr{C}$ from the proof into two sets $\mathscr{C}_0$ and $\mathscr{C}_1$ so that $\sup_{Q \in \mathscr{C}_0} \rho_s(Q, \Omega) < +\infty$, and $\sup_{Q \in \mathscr{C}_1} \rho_s(Q, \tilde{\Omega}) < +\infty$.

We can use duality and interpolation in a standard way to extend the range of indices in (13.17)–(13.18).

COROLLARY 13.4. *Suppose $1 < p, q \leqslant +\infty$, and $\Omega \in D_s$ for some $s > 0$. Let $\chi = \chi_\Omega$. Then $\chi \in \mathbf{MF}_p^{\alpha q}$ if either $p < +\infty$ and $s(1/p - 1) < \alpha < 0$ or $p = +\infty$ and $-s \leqslant \alpha < 0$.*

Proof. These cases follow from Theorem 13.3 and duality ($\mathbf{M}X = \mathbf{M}X^*$ if $X \subsetneq \mathscr{S}'(\mathbb{R}^n)$ and X is a Banach space), via Theorem 5.13 and Remark 5.14. ∎

COROLLARY 13.5. *Suppose $1 < p < +\infty$, $0 < q \leqslant +\infty$, and $\Omega \in D_s$ for some $s > 0$. Then $\chi = \chi_\Omega \in \mathbf{MF}_p^{\alpha q}$ if*

$$s(1/p - 1) + n(1/q - 1)_+ < \alpha < s/p.$$

Proof. For $q > 1$, only the case $\alpha = 0$ remains, which follows by interpolation, e.g., Corollary 8.3. For $0 < q \leqslant 1$, we apply the interpolation property, justified by, say, Theorem 8.5. Now let $\varepsilon > 0$ be sufficiently small, and set $q_0 = 1 + \varepsilon$, $p_0 = p + \varepsilon$, and $\alpha_0 = s(1/p_0 - 1) + \varepsilon$. Let $p_1 = 1$ and let $\theta = \theta(\varepsilon) = \varepsilon/p(p + \varepsilon - 1)$, so that $1/p = (1 - \theta)/p_0 + \theta/p_1$. Let q_1 satisfy $1/q = (1 - \theta)/q_0 + \theta/q_1$ (hence $q_1 < 1$) and set $\alpha_1 = n(1/q_1 - 1) + \varepsilon$. Then χ is bounded on $\mathbf{F}_{p_0}^{\alpha_0 q_0}$ and $\mathbf{F}_{p_1}^{\alpha_1 q_1}$ by Corollary 13.5 and Theorem 13.3, respectively. Hence χ is bounded on $\mathbf{F}_p^{\alpha q}$ for

$$\alpha = (1 - \theta)\,\alpha_0 + \theta\alpha_1 = (1 - \theta)\, s\left(\frac{1}{p + \varepsilon} - 1\right) + n\left(\frac{1}{q} - \frac{1 - \theta}{1 + \varepsilon} - \theta\right) + \varepsilon.$$

Taking $\varepsilon > 0$ sufficiently small (and hence $\theta(\varepsilon)$ small) yields the result. ∎

Earlier we noted that $\Omega = \mathbb{R}_+^n \in D_s$ for $0 < s < 1$. A similar computation show that whenever $\Phi\colon \mathbb{R}^{n-1} \to \mathbf{R}^1$ is Lipschitz if order 1 (i.e., $|\Phi(x) - \Phi(y)| \leqslant M\,|x - y|$ for all $x, y \in \mathbb{R}^{n-1}$), then

$$\Omega = \{(x_1, ..., x_n) \in \mathbb{R}^n \colon x_n > \Phi(x_1, ..., x_{n-1})\}$$

belongs to D_s for $0<s<1$. Since the D_s condition is essentially local, it is also true that $\Omega \in D_s$ for $0<s<1$ if Ω is a bounded Lipschitz domain. Hence we have the following.

COROLLARY 13.6. *Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded Lipschitz domain, and $0<q\leqslant +\infty$. Then $\chi=\chi_\Omega \in \mathbf{MF}_p^{\alpha q}$ if either*

$$0<p<1 \qquad \textit{and} \qquad J-n<\alpha<1/p,$$

or

$$1\leqslant p<\infty \qquad \textit{and} \qquad (1/p-1)+J-n<\alpha<1/p.$$

Proof. Take s close enough to 1 and apply Theorem 13.3 and Corollary 13.5. ∎

These results agree with those in Triebel [Tr2, p. 158], for the case $\Omega=\mathbb{R}^n_+$, and overlap with results independently obtained by Strichartz [Stri] for potentials of Hardy spaces.

Finally, we note that there are non-Lipschitz domains which belong to D_s for $0<s<1$, and hence for which $\chi_\Omega \in \mathbf{MF}_p^{\alpha q}$ (for the indices allowed in Corollary 13.6). One example is

$$\Omega=\{(x,y)\in\mathbb{R}^2 : y>|x|^\varepsilon\},$$

for $0<\varepsilon<1$.

Note the independence of q in (13.23) for the sequence t in the proof of Theorem 13.3. This is a trivial conseqence of the disjointness of the Whitney cubes. Recall also that in this proof, the function f_1 consists of the terms corresponding to the boundary in the decomposition of f. For a "reasonable" domain there will be an analogous independence of q for f_1, i.e., $\|f_1\|_{\mathbf{F}_p^{\alpha\infty}} \approx \|f_1\|_{\mathbf{F}_p^{\alpha q}}$ for all q. Our next theorem gives a precise characterization of the domains for which this happens. This characterization describes exactly under what geometrical conditions on the domain the technique used in the trace problem (Theorem 11.1) can be applied.

Recall that

$$\mathscr{C}=\{Q \text{ dyadic}: l(Q)\leqslant 1 \text{ and } (3Q)^\circ \cap \partial\Omega \neq \varnothing\}.$$

For a sequence $s=\{s_Q\}_{l(Q)\leqslant 1}$, let $\tilde{s}=\{\tilde{s}_Q\}_{l(Q)\leqslant 1}$ be defined by $\tilde{s}_Q=s_Q$ if $Q\in\mathscr{C}$ and $\tilde{s}_Q=0$ if not. For $\alpha\in\mathbb{R}$ and $0<p, q\leqslant +\infty$, set

$$\|s\|_{\mathbf{f}_p^{\alpha q}(\partial\Omega)}=\|\tilde{s}\|_{\mathbf{f}_p^{\alpha q}}.$$

We say $\Omega\in\mathrm{NST}$ (not so terrible) if there exists $\mu\in\mathbb{Z}$, $\mu>0$, with the property that for any dyadic cube Q with $l(Q)\leqslant 1$ satisfying $\bar{Q}\cap\partial\Omega\neq\varnothing$, there exists a dyadic cube $P \subseteq Q$ with $l(P) = 2^{-\mu} l(Q)$ such that $P^\circ \cap \partial\Omega = \emptyset$.

THEOREM 13.7. *Suppose* $0 < p < +\infty$ *and* $\alpha \in \mathbb{R}$. *Then the following are equivalent*:

(i) $\Omega \in \mathrm{NST}$;

(ii) *for all* $q, \tilde{q}$ *with* $0 < q, \tilde{q} \leqslant +\infty$, $\|s\|_{\mathbf{f}_p^{\alpha q}(\partial\Omega)} \approx \|s\|_{\mathbf{f}_p^{\alpha \tilde{q}}(\partial\Omega)}$;

(iii) *there exist* q *and* $\tilde{q}$ *with* $0 < q < \tilde{q} \leqslant +\infty$ *such that*

$$\|s\|_{\mathbf{f}_p^{\alpha q}(\partial\Omega)} \leqslant c\, \|s\|_{\mathbf{f}_p^{\alpha \tilde{q}}(\partial\Omega)}.$$

Here the constants in (ii) *and* (iii) *are independent of* s.

Proof. First suppose $\Omega \in \mathrm{NST}$, and let μ be the number given by the definition. Note that for any $Q \in \mathscr{C}$, we can select a dyadic cube $E_Q \subseteq Q$ such that $l(E_Q) = 2^{-\mu-2} l(Q)$, and $(3E_Q)^\circ \cap \partial\Omega = \emptyset$. To see this, first select $P \subseteq Q$, P dyadic, with $l(P) = 2^{-\mu} l(Q)$, such that $P^\circ \cap \partial\Omega = \emptyset$; this is possible by assumption if $P \cap \partial\Omega \neq \emptyset$, and trivial otherwise. Then let E_Q be any dyadic cube satisfying $l(E_Q) = l(P)/4$ and $\bar{E}_Q \subseteq P^\circ$.

We define $\mu + 2$ pairwise disjoint families of dyadic cubes as follows. Let

$$A_i = \{Q \in \mathscr{C} : l(Q) = 2^{-(\mu+2)k - i} \text{ for some } k \in \mathbb{Z}, k \geqslant 0\},$$

for $i = 0, 1, ..., \mu + 1$. The main observation is that for each i, the cubes belonging to $\{E_Q : Q \in A_i\}$ have pairwise disjoint interiors. To see this, suppose to the contrary that for some i, $Q_1, Q_2 \in A_i$, $Q_1 \neq Q_2$, $l(Q_1) \leqslant l(Q_2)$, say, and $E^\circ_{Q_1} \cap E^\circ_{Q_2} \neq \emptyset$. Since the cubes are dyadic, and $Q_1^\circ \cap Q_2^\circ \neq \emptyset$, we must have $l(Q_1) < l(Q_2)$. But then by definition of A_i, we must have $l(Q_1) \leqslant 2^{-(\mu+2)} l(Q_2) = l(E_{Q_2})$. Then since $Q_1^\circ \cap E^\circ_{Q_2} \neq \emptyset$ and the cubes are dyadic, it follows that $Q_1 \subseteq E_{Q_2}$. But $(3E_{Q_2})^\circ \cap \partial\Omega = \emptyset$, contradicting $Q_1 \in \mathscr{C}$.

For $i = 0, 1, ..., \mu + 1$, define $\tilde{s}_i = \{(\tilde{s}_i)_Q\}_{l(Q) \leqslant 1}$ by $(\tilde{s}_i)_Q = \tilde{s}_Q$ if $Q \in A_i$, and $(\tilde{s}_i)_Q = 0$ otherwise. Since $|E_Q| = 2^{-(\mu+2)n} |Q|$, Proposition 2.7 and the disjointness of the E_Q's for $Q \in A_i$ yield, for any q and $\tilde{q}$ satisfying $0 < q, \tilde{q} \leqslant +\infty$,

$$\begin{aligned}\|\tilde{s}\|_{\mathbf{f}_p^{\alpha q}} &\approx \left\| \left(\sum_Q (|Q|^{-\alpha/n} |(\tilde{s}_i)_Q| \, \tilde{\chi}_{E_Q})^q \right)^{1/q} \right\|_{L^p} \\ &= \left\| \left(\sum_Q (|Q|^{-\alpha/n} |(\tilde{s}_i)_Q| \, \tilde{\chi}_{E_Q})^{\tilde{q}} \right)^{1/\tilde{q}} \right\|_{L^p} \approx \|\tilde{s}_i\|_{\mathbf{f}_p^{\alpha \tilde{q}}}.\end{aligned}$$

Hence

$$\|\tilde{s}\|_{\mathbf{f}_p^{\alpha q}} \approx \max_{0 \leqslant i \leqslant \mu+1} \|\tilde{s}_i\|_{\mathbf{f}_p^{\alpha q}} \approx \max_{0 \leqslant i \leqslant \mu+1} \|\tilde{s}_i\|_{\mathbf{f}_p^{\alpha \tilde{q}}} \approx \|\tilde{s}\|_{\mathbf{f}_p^{\alpha \tilde{q}}}.$$

Therefore (i) implies (ii). Obviously (ii) implies (iii).

Now suppose (iii) holds, and suppose (i) fails, i.e., $\Omega \notin \mathrm{NST}$. Then for arbitrarily large μ, there exists Q dyadic with $l(Q) \leqslant 1$, and $\bar{Q} \cap \partial\Omega \neq \varnothing$, such that for all dyadic cubes $P \subseteq Q$ with $l(P) = 2^{-\mu} l(Q)$, we have $P^\circ \cap \partial\Omega \neq \varnothing$. Define a sequence $s = \{s_P\}_{l(P) \leqslant 1}$ by setting $s_P = |P|^{1/2 + \alpha/n}$ if $P \subseteq Q$ and $l(P) \geqslant 2^{-\mu} l(Q)$, and $s_P = 0$ otherwise. Then

$$\|s\|_{\mathbf{f}_p^{\alpha q}(\partial\Omega)} = \left\| \left(\sum_{\substack{P \subseteq Q \\ l(P) \geqslant 2^{-\mu} l(Q)}} \chi_P^q \right)^{1/q} \right\|_{L^p} = (\mu+1)^{1/q} |Q|^{1/p}.$$

Making the same observation for $\tilde{q}$ contradicts (iii) if μ is sufficiently large. Hence (iii) implies (i). ∎

We remark that the D_s and NST conditions are not strictly comparable. Indeed, there exists $\Omega \in D_s$, for all $0 < s < 1$, such that $\Omega \notin \mathrm{NST}$. On the other hand, if $\Omega \in \mathrm{NST}$ with $\mu = 1$, then $\Omega \in D_s$ for $0 < s < -\log_2(1 - 2^{-n})$, but not necessarily for $s \geqslant -\log_2(1 - 2^{-n})$. For the first assertion, we may construct an example as follows. Let $n(k)$ grow sufficiently rapidly as $k \to +\infty$. For each dyadic $Q \subset [0, 1]^n$, with $l(Q) = 2^{-k}$, let $P(Q)$ be any subcube of Q with sidelength $2^{-n(k)}$. Then let $\Omega = \bigcup \{P(Q)^0 : Q$ is dyadic, $Q \subset [0, 1]^n\}$. We omit the details of the verification. For the second assertion, the case which gives the largest D_s "norm" is particularly easy to determine because of the assumption that $\mu = 1$. An explicit calculation, which we omit, gives the result.

14. Conclusion

We conclude with a brief description of some directions that further research along the lines of this paper either has been taken or could be taken.

As we have mentioned in the introduction, there are a variety of approaches to nonorthogonal decompositions that have been studied, as in [DGM] and the references given there. In [DGM] the relation of these approaches to mathematical physics and the theory of coherent states is stressed. Also, the φ-transform and the wavelet decomposition ([Le-M; Co-M] as discussed in Section 1) are being studied for possible computational applications in engineering and applied mathematics.

More abstract generalizations have been considered by Feichtinger and Gröchenig; see, e.g., [Fei-G]. Their approach stresses the underlying group structure (in our case, the group of translations and dilations on $\mathbb{R}^n$, see Remark 3.2). From this perspective, they consider other group structures, leading to a unified perspective on various types of decompositions. Their techniques in some ways resemble ours in Section 4; in particular, a Neumann series is summed to invert an operator, to obtain the decomposition. This method has been used previously, e.g., in [Co-R; DJS].

Another direction has been considered in [HJTW]. They consider a definition of $\dot{\mathbf{F}}_p^{\alpha q}$ that avoids dependence on an underlying translation structure. Namely, it is seen in [HJTW] that an equivalent definition of $\dot{\mathbf{F}}_p^{\alpha q}$ can be obtained under appropriate restrictions by replacing the convolution operators $\varphi_\nu * f$ with kernel operators of the form

$$T_\varepsilon f(x) = \int K_\varepsilon(x, y) f(y)\, dy,$$

where $\{T_\varepsilon\}_{\varepsilon > 0}$ is an "ε-family of operators" (cf. [CJ]), i.e. the kernels K_ε satisfy natural size and smoothness assumptions. This approach may allow adaptation of our results to cases where there is a natural dilation structure, but no translation structure.

The geometrical aspects of the $\dot{\mathbf{f}}_p^{\alpha q}$-spaces may be useful for studying potential theory and capacities. The results in [Ja-P-W] are examples of this. There it is shown that the positive cone in $\dot{\mathbf{F}}_p^{\alpha q}$ is independent of q, for $\alpha < 0$ (this result is due to David Adams for most values of p and q; cf. [Ja-P-W] for a precise statement and references); this fact is directly related to the close connection between potentials and fractional maximal operators.

Possible settings which may allow full or partial adaptation of our results include homogenous groups, spaces of homogeneous type (in the sense of Coifman and Weiss [Co-W1]), and the polydisk. We have noted previously [Fr-J1, Section 8] that Folland and Stein [Fo-S, p. 47] have constructed a resolution of the δ-function on homogeneous groups, which yields a version of the Calderón reproducing formula. Calderón's formula was, of course, the starting point of our work here. For spaces of homogeneous type, one special case of particular interest is that of Lipschitz domains in $\mathbb{R}^n$. For this setting, and others, the generalized φ-transform from Section 4 may be a useful tool.

APPENDIX A: Proofs of Lemmas 2.3 and 2.5

To prove Lemma 2.3 we need the Fefferman–Stein vector-valued maximal inequality. Let M be the Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y)|\, dy,$$

where the sup is taken over all cubes (not necessarily dyadic) with sides parallel to the axes.

THEOREM A.1 (Fefferman and Stein [Fef-S1]). *Suppose* $1 < p < +\infty$ *and* $1 < q \leqslant +\infty$. *Then*

$$\left\| \left(\sum_{i=1}^{\infty} |Mf_i|^q \right)^{1/q} \right\|_{L^p} \leqslant c_{p,q} \left\| \left(\sum_{i=1}^{\infty} |f_i|^q \right)^{1/q} \right\|_{L^p}.$$

The sequence s_r^* can be majorized by M; this is essentially just a special case of the standard fact that the convolution with a radial, decreasing L^1-function can be majorized (pointwise) by M.

LEMMA A.2. *Suppose* $0 < a \leqslant r < +\infty$, *and* $\lambda > nr/a$. *Fix* $\mu, \nu \in \mathbb{Z}$ *with* $\mu \leqslant \nu$. *For each* dyadic cube Q *with* $l(Q) = 2^{-\nu}$ *and each* $x \in Q$,

$$\left(\sum_{l(P) = 2^{-\mu}} |s_P|^r / (1 + l(P)^{-1} |x_P - x_Q|)^{\lambda} \right)^{1/r} \leqslant c \left(M \left(\sum_{l(P) = 2^{-\mu}} |s_P|^a \chi_P \right)(x) \right)^{1/a},$$

where c *depends only on* n *and* $\lambda - nr/a$.

Proof. We may assume $x_Q = 0$. Let $A_0 = \{P \text{ dyadic} : l(P) = 2^{-\mu}$ and $|x_P|/l(P) \leqslant 1\}$ and, for $k = 1, 2, 3, \ldots$, let $A_k = \{P \text{ dyadic} : l(P) = 2^{-\mu}$ and $2^{k-1} < |x_P|/l(P) \leqslant 2^k\}$. Then

$$\begin{aligned} &\sum_{P \in A_k} |s_P|^r / (1 + |x_P|/l(P))^{\lambda} \\ &\quad \leqslant c 2^{-k\lambda} \sum_{P \in A_k} |s_P|^r \\ &\quad \leqslant c 2^{-k\lambda} \left(\sum_{P \in A_k} |s_P|^a \right)^{r/a} \leqslant c 2^{-k\lambda} 2^{n\mu r/a} \left(\int \sum_{P \in A_k} |s_P|^a \chi_P \right)^{r/a} \\ &\quad \leqslant c 2^{-k(\lambda - nr/a)} \left(M \left(\sum_{P \in A_k} |s_P|^a \chi_P \right)(x) \right)^{r/a}. \end{aligned}$$

Summing over k and taking the rth roots yields the result. ∎

Remark A.3. With the same restrictions on a, r, and λ, we have a more general estimate. For each dyadic cube Q with $l(Q) = 2^{-\nu}$ and each $x \in Q$,

$$\left(\sum_{l(P)=2^{-\mu}} |s_P|^r \Big/ \left(1+\frac{|x_Q-x_P|}{\max(l(P),l(Q))}\right)^{\lambda}\right)^{1/r}$$
$$\leqslant c2^{(\mu-\nu)_+ n/a}\left(M\left(\sum_{l(P)=2^{-\mu}} |s_P|^a \chi_P\right)(x)\right)^{1/a},$$

where c depends only on n and $\lambda - nr/a$; here $(\mu-\nu)_+ = \max(\mu-\nu, 0)$. To prove this when $\mu > \nu$, replace A_0 in the above by $B_0 = \{P \text{ dyadic}: l(P) = 2^{-\mu}$ and $|x_P|/2^{-\nu} \leqslant 1\}$ and A_k by $B_k = \{P \text{ dyadic}: l(P) = 2^{-\mu}$, and $2^{k-1} < |x_P|/2^{-\nu} \leqslant 2^k\}$.

Proof of Lemma 2.3. Let $r = \min(p, q)$ and $\varepsilon = -1 + \lambda/n > 0$. Let $a = r/(1+\varepsilon/2)$. Then $0 < a < r$, and $\lambda > nr/a$. Hence, by Lemma A.3 with $\mu = \nu$,

$$\sum_{l(Q)=2^{-\nu}} (s_r^*)_Q \tilde{\chi}_Q \leqslant c\left(M\left(\sum_{l(P)=2^{-\nu}} |s_P|\, \tilde{\chi}_P\right)^a\right)^{1/a},$$

for all $\nu \in \mathbb{Z}$. Hence,

$$\|s_r^*\|_{\dot{f}_p^{\alpha q}} \leqslant c \left\| \left(\sum_{\nu\in\mathbb{Z}} \left(M\left(\sum_{l(P)=2^{-\nu}} |P|^{-\alpha/n} |s_P|\, \tilde{\chi}_P\right)^a\right)^{q/a}\right)^{a/q} \right\|_{L^{p/a}}^{1/a}.$$

Since p/a, $q/a > 1$, the Fefferman–Stein inequality (Theorem A.1) allows us to remove M in the last expression, to obtain $\|s_r^*\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|s\|_{\dot{f}_p^{\alpha q}}$. The other direction is trivial since $|s_Q| \leqslant (s_r^*)_Q$ for all Q. ∎

To prove Lemma 2.5 we need two additional lemmas; the first is an adaptation of Peetre's mean-value theorem estimate for φ_ν^{**} [P2].

Lemma A.4. *Suppose* $f \in \mathcal{S}'$ *and* $\operatorname{supp} \hat{f}(\xi) \subseteq \{\xi : |\xi| \leqslant 2\}$. *Let* $\gamma \in \mathbb{Z}$, *with* $\gamma \geqslant 0$. *For Q dyadic, let* $a_Q = \sup_{y\in Q} |f(y)|$ *and* $b_{Q,\gamma} = \max\{\inf_{y\in\tilde{Q}} |f(y)| : l(\tilde{Q}) = 2^{-\gamma} l(Q), \tilde{Q} \subseteq Q\}$. *Let* $a = \{a_Q\}_Q$ *and* $b = \{b_{Q,\gamma}\}_Q$. *If* $0 < r < +\infty$, $l(Q) = 1$ *and γ is sufficiently large, then*

$$(a_r^*)_Q \approx (b_r^*)_Q,$$

with constants independent of f and Q.

Proof. We may assume $Q = Q_{00}$. First, suppose $f \in \mathcal{S}$ and $\operatorname{supp} \hat{f} \subseteq \{\xi : |\xi| \leqslant 3\}$. By the mean-value theorem,

$$a_P \leqslant b_{P,\gamma} + c2^{-\gamma} \sup_{y\in P} |\nabla f(y)|,$$

if $l(P)=1$. Let $d_P = \sup_{y \in P} |\nabla f(y)|$ and $d = \{d_P\}_P$. Then

$$(a_r^*)_Q \leqslant c(b_r^*)_Q + c2^{-\gamma}(d_r^*)_Q.$$

Let $g \in \mathscr{S}$ satisfy $\hat{g}(\xi) = 1$ if $|\xi| \leqslant 3$ and $\operatorname{supp} \hat{g}(\xi) \subseteq \{\xi : |\xi| < \pi\}$. Writing $f = f * g = (\hat{f} \cdot \hat{g})^{\vee}$ and proceeding as in the proof of Lemma 2.1 in [Fr-J1], we obtain

$$f(x) = \sum_{k \in \mathbb{Z}^n} f(k)\, g(x-k) = \sum_{l(J)=1} f(x_J)\, g(x - x_J).$$

Hence,

$$\begin{aligned} d_P &\leqslant \sup_{y \in P} \sum_{l(J)=1} |f(x_J)|\, |\nabla g(y - x_J)| \\ &\leqslant \sup_{y \in P} \sum_{l(J)=1} |f(x_J)|\, |\nabla g(y - x_J)|(1 + |x_J - x_P|)^{\lambda/r} \\ &\quad \times (1 + |x_P|)^{\lambda/r}/(1 + |x_J|)^{\lambda/r}. \end{aligned}$$

Since $g \in \mathscr{S}$, we have $\sup_{y \in P} |\nabla g(y - x_J)| \leqslant c_M (1 + |x_J - x_P|)^{-M}$ for any $M \in \mathbb{Z}$. Therefore, taking L sufficiently large,

$$\begin{aligned} (d_r^*)_Q &\leqslant c_L \left(\sum_{l(P)=1} \left(\sum_{l(J)=1} |f(x_J)|/(1 + |x_J|)^{\lambda/r}\, (1 + |x_J - x_P|)^L \right)^r \right)^{1/r} \\ &\leqslant c(a_r^*)_Q \end{aligned}$$

by Minkowski's inequality if $r \geqslant 1$, or by the r-triangle inequality followed by Minkowski's inequality if $r < 1$. Since $f \in \mathscr{S}$, we have $(a_r^*)_Q < +\infty$. Thus, taking γ sufficiently large above yields $(a_r^*)_Q \leqslant c(b_r^*)_Q$.

More generally, let $f \in \mathscr{S}'$ with $\operatorname{supp} \hat{f} \subseteq \{\xi : |\xi| \leqslant 2\}$. Then f is slowly increasing and infinitely differentiable (e.g., [Hör2, p. 21]). We now apply a standard regularization argument (see, e.g., [Tr2, p. 22]). Let $g \in \mathscr{S}$ satisfy $\operatorname{supp} \hat{g}(\xi) \subseteq \{\xi : |\xi| \leqslant 1\}$, $\hat{g}(\xi) \geqslant 0$, and $g(0) = 1$. Then $|g(x)| \leqslant 1$ for all x, by Fourier inversion. For $0 < \delta < 1$, let $f_\delta(x) = f(x)\, g(\delta x)$. Then $\operatorname{supp} \hat{f}_\delta \subseteq \{\xi : |\xi| \leqslant 3\}$, $f_\delta \in \mathscr{S}$, $|f_\delta| \leqslant |f|$, and $f_\delta \to f$ as $\delta \to 0$, uniformly on compact sets. Applying our result to f_δ, noting that $\sup_{y \in P} |f_\delta(y)| \leqslant \sup_{y \in P} |f(y)|$ for all P, and letting $\delta \to 0$, we obtain $(a_r^*)_Q \leqslant c(b_r^*)_Q$.

The converse estimate is trivial. ∎

The next lemma is very simple; let us recall that $\inf_{Q,\gamma}(f) = |Q|^{1/2} \max\{\inf_{y \in \tilde{Q}} |\tilde{\varphi}_\nu * f(y)| : l(\tilde{Q}) = 2^{-\gamma} l(Q), \tilde{Q} \subseteq Q\}$.

LEMMA A.5. *For $f \in \mathscr{S}'/\mathscr{P}$, $\alpha \in \mathbb{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$,*

$$\|\inf_\gamma(f)\|_{\dot{\mathbf{f}}_p^{\alpha q}} \leqslant c_{\gamma,n,p,q} \|f\|_{\dot{\mathbf{F}}_p^{\alpha q}}.$$

Proof. Let $t = \{t_J\}_J$ be defined by

$$t_J = |J|^{1/2} \inf_{y \in J} |\tilde{\varphi}_{\mu-\gamma} * f(y)| \qquad \text{if} \quad l(J) = 2^{-\mu}.$$

Then for $0 < r < +\infty$,

$$\inf_{Q,\gamma}(f)\, \tilde{\chi}_Q \leqslant C_{n,r} 2^{\gamma\lambda/r} \sum_{\substack{J \subseteq Q \\ l(J) = 2^{-\gamma} l(Q)}} (t_r^*)_J\, \tilde{\chi}_J$$

for Q dyadic and λ as in the definition of s_r^*. Picking $r = \min(p, q)$ and applying Lemma 2.3,

$$\begin{aligned} \|\inf_\gamma(f)\|_{\dot{f}_p^{\alpha q}} &\leqslant c 2^{\gamma(\lambda/r-\alpha)} \|t_r^*\|_{\dot{f}_p^{\alpha q}} \leqslant c 2^{\gamma(\lambda/r-\alpha)} \|t\|_{\dot{f}_p^{\alpha q}} \\ &\leqslant c 2^{\gamma(\lambda/r-\alpha)} \left\| \left(\sum_{\nu \in \mathbb{Z}} (2^{\nu\alpha} |\tilde{\varphi}_{\nu-\gamma} * f|)^q \right)^{1/q} \right\|_{L^p} \\ &\leqslant c 2^{\gamma\lambda/r} \|f\|_{\dot{F}_p^{\alpha q}}. \quad \blacksquare \end{aligned}$$

Proof of Lemma 2.5. The estimate $\|f\|_{\dot{F}_p^{\alpha q}} \leqslant c\, \|\sup(f)\|_{\dot{f}_p^{\alpha q}}$ follows from the definitions. Applying Lemma A.4 to each of the functions $\tilde{\varphi}_\nu * f(2^{-\nu}x)$ leads to the inequality

$$(\sup(f)_r^*)_Q \leqslant c(\inf_\gamma(f)_r^*)_Q$$

if $r = \min(p, q)$ and $l(Q) = 2^{-\nu}$. Then Lemma 2.3 gives

$$\|\sup(f)\|_{\dot{f}_p^{\alpha q}} \leqslant c\, \|\inf_\gamma(f)\|_{\dot{f}_p^{\alpha q}}.$$

Finally, Lemma A.5 completes the proof. ∎

APPENDIX B: PROOFS OF LEMMAS 3.6 AND 3.8

We need the following two technical lemmas.

LEMMA B.1. *Suppose* $R > n$, $0 < \theta \leqslant 1$, $j, k \in \mathbb{Z}$, $j \geqslant k$, $L \in \mathbb{Z}$, $L \geqslant 0$, $S > L + n + \theta$, *and* $x_1 \in \mathbb{R}^n$. *Suppose* g, $h \in L^1(\mathbb{R}^n)$ *satisfy*

$$|\partial^\gamma g(x)| \leqslant 2^{j(n/2+|\gamma|)}(1 + 2^j |x|)^{-R} \qquad \text{if} \quad |\gamma| \leqslant L, \tag{B.1}$$

$$|\partial^\gamma g(x) - \partial^\gamma g(y)| \leqslant 2^{j(n/2+L+\theta)} |x-y|^\theta \sup_{|z| \leqslant |y-x|} (1 + 2^j |z-x|)^{-R} \tag{B.2}$$

if $|\gamma| = L$,

$$|h(x)| \leqslant 2^{kn/2}(1 + 2^k |x - x_1|)^{-\max(R, S)}, \tag{B.3}$$

and

$$\int x^\gamma h(x)\,dx = 0 \qquad \textit{if} \quad |\gamma| \leqslant L. \tag{B.4}$$

Then

$$|g * h(x)| \leqslant c2^{-(k-j)(L+\theta+n/2)}(1+2^j|x-x_1|)^{-R}, \tag{B.5}$$

where c is independent of k, j, x, and x_1.

Proof. Using translations and dilations, we may assume $x_1 = 0$ and $j = 0$. Let $A = \{y : |y-x| \leqslant 3\}$, $B = \{y : |y-x| > 3$ and $|y| \leqslant |x|/2\}$, and $C = \{y : |y-x| > 3$ and $|y| > |x|/2\}$. Then

$$\begin{aligned} |g * h(x)| &\leqslant \int \left| \left(g(y) - \sum_{|\beta| \leqslant L} \partial^\beta g(x)(y-x)^\beta/\beta! \right) h(x-y) \right| dy \\ &\equiv \int_A + \int_B + \int_C. \end{aligned}$$

For $y \in A$, (B.2) implies that

$$\begin{aligned} \left| g(y) - \sum_{|\beta| \leqslant L} \partial^\beta g(x)(y-x)^\beta/\beta! \right| &\leqslant c\,|x-y|^{L+\theta} \sup_{|z| \leqslant 3} (1+|z-x|)^{-R} \\ &\leqslant c\,|x-y|^{L+\theta}(1+|x|)^{-R}, \end{aligned}$$

since $1 + |z-x| \geqslant \frac{1}{8}(1+|x|)$ if $|z| \leqslant 3$. Using this and (B.3), we see that

$$\begin{aligned} \int_A &\leqslant c2^{kn/2}(1+|x|)^{-R} \int_A |x-y|^{L+\theta}(1+2^k|x-y|)^{-S}\,dy \\ &\leqslant c2^{-k(L+\theta+n/2)}(1+|x|)^{-R}, \end{aligned}$$

since $S > L+n+\theta$.

For $y \in B$, $|x|/2 \leqslant |y-x| \leqslant 3|x|/2$, so $2^k|x-y| \geqslant \frac{1}{4}2^k(1+|x|)$, since $|y-x| \geqslant 3$. Therefore,

$$\begin{aligned} \int_B &\leqslant c \int_B \left[\frac{1}{(1+|y|)^R} + \sum_{|\beta| \leqslant L} \frac{|x-y|^{|\beta|}}{(1+|x|)^R} \right] \\ &\quad \times 2^{kn/2}2^{-kS}(1+|x|)^{-\max(R,S)}\,dy \\ &\leqslant c2^{-k(S-n/2)}(1+|x|)^{-R} \\ &\quad \times \left[\int (1+|y|)^{-R}\,dy + \frac{|x|^L}{(1+|x|)^S} \int_{\{|y| \leqslant |x|/2\}} dy \right] \\ &\leqslant c2^{-k(S-n/2)}(1+|x|)^{-R}, \end{aligned}$$

as needed, since $S > L+n+\theta$.

For $y \in C$, $1+|y| \geqslant \frac{1}{2}(1+|x|)$, so

$$\int_C \leqslant c \int_C \left[\frac{1}{(1+|y|)^R} + \sum_{|\beta| \leqslant L} \frac{|x-y|^{|\beta|}}{(1+|x|)^R}\right] 2^{kn/2}(1+2^k|x-y|)^{-S}\,dy$$

$$\leqslant c2^{kn/2}(1+|x|)^{-R} \int_3^\infty \frac{r^{L+n-1}}{(1+2^k r)^S}\,dr \leqslant c2^{-k(S-n/2)}(1+|x|)^{-R},$$

yielding (B.5). ∎

The case in which no vanishing moments on h are assumed (formally $L=-1$) is very simple.

LEMMA B.2. *Suppose* $R>n$, j, $k \in \mathbb{Z}$, $j \leqslant k$, *and* $x_1 \in \mathbb{R}^n$. *Suppose* g, $h \in L^1(\mathbb{R}^n)$ *satisfy*

$$|g(x)| \leqslant 2^{jn/2}(1+2^j|x|)^{-R}, \tag{B.6}$$

and

$$|h(x)| \leqslant 2^{kn/2}(1+2^k|x-x_1|)^{-R}. \tag{B.7}$$

Then

$$|g * h(x)| \leqslant c2^{-(k-j)n/2}(1+2^j|x-x_1|)^{-R}. \tag{B.8}$$

Proof. Again we may assume $x_1=0$ and $j=0$. Let A, B, and C be as in the proof of Lemma B.1. For $y \in A \cup C$, $1+|y| \geqslant \frac{1}{8}(1+|x|)$, so

$$\int_{A \cup C} |g(y)|\,|h(x-y)|\,dy \leqslant c(1+|x|)^{-R}\,2^{kn/2} \int_{\mathbb{R}^n} (1+2^k|x-y|)^{-R}\,dy$$

$$\leqslant c2^{-kn/2}(1+|x|)^{-R}.$$

For $y \in B$, $2^k|x-y| \geqslant \frac{1}{4}2^k(1+|x|)$, so

$$\int_B |g(y)|\,|h(x-y)|\,dy \leqslant c2^{-k(R-n/2)}(1+|x|)^{-R} \int_{\mathbb{R}^n} (1+|y|)^{-R}\,dy,$$

yielding (B.8), since $R>n$. ∎

By choosing all numbers and the functions g, h properly, Lemmas B.1 and B.2 yield the following corollary, which includes both Lemmas 3.6 and 3.8.

COROLLARY B.3. *Let* $M>J$, $(J-\alpha)^* < \rho \leqslant 1$, *and* $\alpha^* < \delta \leqslant 1$. *Suppose that* $\{m_Q\}_Q$ *is a family of smooth molecules satisfying* (3.3)–(3.6), *and that*

$\{b_Q\}_Q$ *is a family of functions satisfying* (3.7)–(3.10). *Then there exist* ε_1 *and a constant* C, *independent of* P *and* Q, *such that*

$$|\langle m_P, b_Q\rangle| \leqslant C\omega_{QP}(\varepsilon) \qquad \textit{if} \quad \varepsilon \leqslant \varepsilon_1.$$

Proof. Possibly reducing δ, ρ, or M, we may assume that $\delta - \alpha^* = (M-J)/2 = \rho - (J-\alpha)^* > 0$.

If $l(Q) = 2^{-\nu} \leqslant 2^{-\mu} = l(P)$, and $\alpha \geqslant 0$, we apply Lemma B.1 with $R = M$, $\theta = \delta$, $k = \nu$, $j = \mu$, $L = [\alpha]$, $S = M + n + \alpha - J$, $x_1 = x_Q$, $g(x) = \overline{m_P(x_P - x)}$, and $h = b_Q$. We obtain

$$|\langle m_P, b_Q\rangle| = |g * h(x_P)| \leqslant c2^{-(\nu-\mu)([\alpha]+\delta+n/2)}(1 + 2^\mu\,|x_P - x_Q|)^{-M},$$

yielding the desired estimate, since $M \geqslant J + \varepsilon$ and $[\alpha] + \delta + n/2 \geqslant \varepsilon/2 + \alpha + n/2$ if $\varepsilon > 0$ is small enough. Similarly, when $\alpha < 0$, Lemma B.2 gives

$$|\langle m_P, b_Q\rangle| \leqslant c2^{-(\nu-\mu)n/2}(1 + 2^\mu\,|x_P - x_Q|)^{-M},$$

which again is satisfactory since $n/2 \geqslant n/2 + \alpha + \varepsilon/2$ if $0 < \varepsilon \leqslant -2\alpha$, say.

If $l(P) = 2^{-\mu} \leqslant 2^{-\nu} = l(Q)$, and $N \geqslant 0$, we apply Lemma B.1 with $R = M$, $\theta = \rho$, $k = \mu$, $j = \nu$, $L = N$, $S = M - \alpha$, $x_1 = x_P$, $g(x) = \overline{b_Q(x_Q - x)}$, and $h = m_P$. We obtain

$$|\langle m_P, b_Q\rangle| \leqslant |g * h(x_P)| \leqslant c2^{-(\mu-\nu)(N+\rho+n/2)}(1 + 2^\nu\,|x_P - x_Q|)^{-M},$$

which is as required because $N + \rho + n/2 \geqslant \varepsilon/2 + J - \alpha - n/2$, again if $\varepsilon > 0$ is small enough. Similarly, when $N = -1$, Lemma B.2 gives

$$|\langle m_P, b_Q\rangle| \leqslant c2^{-(\mu-\nu)n/2}(1 + 2^\nu\,|x_P - x_Q|)^{-M},$$

as desired, since $N = -1$ implies $n + \alpha > J$, so that $n/2 > -\alpha + n/2 + \varepsilon/2 + J - n$ if $\varepsilon \leqslant 2(n + \alpha - J)$.

Remark B.4. As we pointed out in Section 3, some care is sometimes necessary when interpreting expressions like $\langle f, b_Q\rangle$ for $f \in \mathscr{S}'/\mathscr{P}$. For this, we briefly recall Peetre's discussion on pp. 52–56 of [P3].

For $f \in \mathscr{S}'/\mathscr{P}$ and φ and ψ satisfying (2.1)–(2.4), $\sum_{\nu=1}^{\infty} \tilde{\varphi}_\nu * \psi_\nu * f$ converges in $\mathscr{S}'$, where as usual $\tilde{\varphi}_\nu(x) = \overline{\varphi_\nu(-x)}$. Using standard estimates (see e.g., [P3, p. 54; TR2, pp. 17–18]), if $f \in \dot{\mathbf{F}}_p^{\alpha q}$, $\alpha \in \mathbb{R}$, $0 < p < +\infty$, and $0 < q \leqslant +\infty$, we have

$$\begin{aligned}\|\partial^\beta \tilde{\varphi}_\nu * \psi_\nu * f\|_{L^\infty} &\leqslant \|\partial^\beta \psi_\nu\|_{L^1}\,\|\tilde{\varphi}_\nu * f\|_{L^\infty}\\ &\leqslant c2^{\nu(|\beta| + n/p)}\,\|\tilde{\varphi}_\nu * f\|_{L^p}.\end{aligned}$$

Hence, if $|\beta| > \alpha - n/p$ (or if $|\beta| = \alpha - n/p$ and $q \leqslant 1$),

$$\sum_{\nu=-\infty}^{1} \|\partial^\beta \tilde{\varphi}_\nu * \psi_\nu * f\|_{L^\infty} \leqslant c \, \|f\|_{\dot{F}_p^{\alpha q}}.$$

It follows that there exists a sequence of polynomials $\{P_N\}_{N=1}^\infty$ with degree $\leqslant L$ for all N, where $L = [\alpha - n/p]$, such that

$$g \equiv \lim_{N \to \infty} \left(\sum_{\nu=-N}^{\infty} \tilde{\varphi}_\nu * \psi_\nu * f + P_N \right)$$

exists in $\mathscr{S}'$. By (2.4), if f_0 is any representative of the equivalence class $f + \mathscr{P}$, then $\operatorname{supp}(\hat{g} - \hat{f}_0) = \{0\}$, so g is another representative of this class.

Now suppose, for $i = 1, 2$, that $\tilde{\varphi}^i$, ψ^i, $\{P_N^i\}_{N=1}^\infty$, and g^i are as in the previous paragraph. Since g^1 and g^2 represent the same equivalence class, $g^1 - g^2$ is a polynomial. For $i = 1, 2$, denote $\tilde{\varphi}_\nu^i * \psi_\nu^i * f$ by f_ν^i. If $|\beta| > L$, and $\eta \in \mathscr{S}$, then

$$\begin{aligned} &\langle \partial^\beta (g^1 - g^2), \eta \rangle \\ &= \lim_{N \to \infty} \left\langle \sum_{\nu=-N}^{\infty} f_\nu^1 + P_N^1 - \sum_{\nu=-N}^{\infty} f_\nu^2 - P_N^2, (-1)^{|\beta|} \partial^\beta \eta \right\rangle. \end{aligned}$$

However, by (2.1)–(2.4), $\operatorname{supp}(\sum_{\nu=-N}^\infty (f_\nu^1 - f_\nu^2)^\wedge) \subseteq \{\xi : |\xi| \leqslant 2^{-N+1}\}$. Let $\chi \in \mathscr{S}$ satisfy $\hat{\chi}(\xi) = 1$ for $|\xi| \leqslant 2$ and $\hat{\chi}(\xi) = 0$ for $|\xi| > 4$, and set $\chi_\nu(x) = 2^{\nu n} \chi(2^\nu x)$ for $\nu \in \mathbb{Z}$. Then

$$\begin{aligned} \sum_{\nu=-N}^{\infty} (f_\nu^1 - f_\nu^2) &= \chi_{-N} * \sum_{\nu=-N}^{\infty} (f_\nu^1 - f_\nu^2) \\ &= \sum_{\nu=-N}^{-N+2} (\chi_{-N} * f_\nu^1 - \chi_{-N} * f_\nu^2), \end{aligned}$$

by (2.2). Hence,

$$\begin{aligned} |\langle \partial^\beta (g^1 - g^2), \eta \rangle| &= \left| \lim_{N \to \infty} \left\langle \sum_{\nu=-N}^{-N+2} \partial^\beta (\chi_{-N} * f_\nu^1 - \chi_{-N} * f_\nu^2), \eta \right\rangle \right| \\ &\leqslant \lim_{N \to \infty} \sum_{\nu=-N}^{-N+2} \|\chi_{-N}\|_{L^1} (\|\partial^\beta f_\nu^1\|_{L^\infty} + \|\partial^\beta f_\nu^2\|_{L^\infty}) \|\eta\|_{L^1} = 0, \end{aligned}$$

for $|\beta| > L$, by the estimates above. Hence, $\deg(g^1 - g^2) \leqslant L$.

Thus, we see that for $f \in \dot{F}_p^{\alpha q}$, the representative g above is well defined modulo $\mathscr{P}_L$, the set of polynomials of degree $\leqslant L = [\alpha - n/p]$. In other words, by identifying the equivalence class $f + \mathscr{P}$ for $f \in \dot{F}_p^{\alpha q}$ with its "canonical" representative g above, the elements of $\dot{\mathbf{F}}_p^{\alpha q}$ can be regarded as equivalence classes of tempered distributions modulo $\mathscr{P}_L$. (Note that for $\alpha < n/p$, the above discussion shows that if $f \in \dot{\mathbf{F}}_p^{\alpha q}$, then $\sum_{v \in \mathbb{Z}} \tilde{\varphi}_v * \psi_v * f$ converges in $\mathscr{S}'$ and represents f in $\mathscr{S}'/\mathscr{P}$. This is useful when considering equivalences of $\dot{\mathbf{F}}_p^{\alpha q}$ with other spaces; for example, the identification of $\dot{\mathbf{F}}_p^{02}$ with H^p, $0 < p < +\infty$, is obtained by identifying $f \in \dot{\mathbf{F}}_p^{02}$ with its canonical representative $\sum_{v \in \mathbb{Z}} \tilde{\varphi}_v * \psi_v * f$. Note that in a certain sense, this representative is chosen to be minimal at infinity.)

Note that if b_Q satisfies (3.7)–(3.8) for some $M > J$, then $\int x_\gamma b_Q(x)\, dx = 0$ for $|\gamma| \leqslant L$. In this case, for $f \in \dot{\mathbf{F}}_p^{\alpha q}$ let $\{P_N\}_{N=1}^{\infty}$ be any sequence of polynomials of degree $\leqslant L$ such that $\sum_{v=-N}^{\infty} \tilde{\varphi}_v * \psi_v * f + P_N$ converges in $\mathscr{S}'$ as $N \to +\infty$. We define

$$\begin{aligned}\langle f, b_Q \rangle &\equiv \lim_{N \to \infty} \left\langle \sum_{v=-N}^{N} \tilde{\varphi}_v * \psi_v * f + P_N, b_Q \right\rangle \\ &= \sum_{v \in \mathbb{Z}} \langle \tilde{\varphi}_v * \psi_v * f, b_Q \rangle,\end{aligned}$$

whenever this last sum converges absolutely. In the case we are considering in Theorem 3.7 and Lemma 3.8, we have $\tilde{\varphi}_v * \psi_v * f = \sum_{l(P) = 2^{-v}} s_P \psi_P$. Hence,

$$\sum_{v \in \mathbb{Z}} |\langle \tilde{\varphi}_v * \psi_v * f, b_Q \rangle| \leqslant \sum_P |s_P|\, |\langle \psi_P, b_Q \rangle|,$$

and our argument in the proof of Lemma 3.8 guarantees absolute convergence.

APPENDIX C: Proof of Proposition 7.1

We break up the proof of Proposition 7.1 into two (known) lemmas. For a pair of "quasi-normed Abelian groups" (X_0, X_1) and the E-functional $E(t, x; \bar{X})$, defined in Section 6, we set

$$|x|_{\theta, q; E} = \left(\sum_{v \in \mathbb{Z}} (2^{v\theta/(1-\theta)} E(2^v, x; \bar{X}))^{(1-\theta)q} \right)^{1/q}.$$

We have

LEMMA C.1. *Let* $0 < \theta < 1$ *and* $0 < q \leqslant +\infty$. *Then* $(X_0, X_1)_{\theta, q} = (X_0, X_1)_{\theta, q; E}$.

Proof. This is an immediate consequence of the facts that $K(t) \approx K_\infty(t)$ and that $K_\infty(t)/t$ and $E(s)/s$ are inverse (cf. (6.2)); making the change of variables $t \to E(s)/s$ we find

$$\|x\|_{\theta,q} \approx \left(\int_0^\infty (t^{-\theta} K_\infty(t, x; X_0, X_1))^q \frac{dt}{t}\right)^{1/q}$$
$$= c_{\theta q} \left(\int_0^\infty (s^{\theta/(1-\theta)} E(s, x; X_0, X_1))^{q(1-\theta)} \frac{ds}{s}\right)^{1/q}.$$

This proves the lemma modulo the discretization, which is trivial since E is nonincreasing. ▌

For more details we refer to [Be-L, Theorem 7.1.7; Ja-R-W, Section 2].

The second lemma requires a straightforward (known) modification of the so-called Fundamental Lemma (see [Be-L]); we need it for the E- and e-functionals instead of the standard K and J.

LEMMA C.2. *Let* $0 < \theta < 1$ *and* $0 < q \leqslant +\infty$. *Then*

$$\|x\|_{\theta,q;E} \approx \|x\|_{\theta,q;e}.$$

Proof. We give the proof in the normed case, with the usual modifications (in particular, the use of Lemma 3.10.2 in [Be-L]) required for the adaptation to the quasi-normed case.

Suppose first that $x = \sum_{v \in \mathbb{Z}} x_v$ with $e(2^v, x_v; \bar{X}) < +\infty$ for each $v \in \mathbb{Z}$. Then $\|x_v\|_{X_1} \leqslant 2^v$ and $e(2^v, x_v; \bar{X}) = \|x_v\|_{X_0}$. Therefore, for each $\mu \in \mathbb{Z}$, $\|\sum_{v=-\infty}^{\mu-1} x_v\|_{X_1} \leqslant 2^\mu$, and hence

$$E\left(2^\mu, \sum_v x_v; \bar{X}\right) \leqslant \left\| \sum_{v=\mu}^\infty x_v \right\|_{X_0} \leqslant \sum_{v=\mu}^\infty \|x_v\|_{X_0} = \sum_{v=\mu}^\infty e(2^v, x_v; \bar{X}).$$

Consequently, using Minkowski's inequality,

$$\|x\|_{\theta,q;E} \leqslant \left(\sum_{\mu \in \mathbb{Z}} \left(\sum_{v=\mu}^\infty 2^{(\mu-v)\theta/(1-\theta)} 2^{v\theta/(1-\theta)} e(2^v, x_v; \bar{X})\right)^{(1-\theta)q}\right)^{1/q}$$
$$\leqslant c\, \|x\|_{\theta,q;e}.$$

For the converse inequality, we have, for each $v \in \mathbb{Z}$, elements $x_{0,v}$ and $x_{1,v}$ such that $x = x_{0,v} + x_{1,v}$, $\|x_{1,v}\|_{X_1} \leqslant 2^v$, and $\|x_{0,v}\|_{X_0} \leqslant cE(2^v, x; \bar{X})$. Let $u_v = x_{1,v} - x_{1,v-1} = x_{0,v-1} - x_{0,v}$. Then $\|u_v\|_{X_1} \leqslant 2^v + 2^{v-1} < 2^{v+1}$, so

$$e(2^{v+1}, u_v; \bar{X}) = \|u_v\|_{X_0} \leqslant \|x_{0,v-1}\|_{X_0} + \|x_{0,v}\|_{X_0} \leqslant cE(2^{v-1}, x; \bar{X}),$$

since E is nonincreasing. If $\|x\|_{\theta,q;E} < +\infty$, then $E(2^{\nu}, x; \bar{X})$ is finite for all ν and $\lim_{\nu\to\infty} E(2^{\nu}, x; \bar{X}) = 0$. This readily implies that $x = \sum u_{\nu}$. Letting $x_{\nu} = u_{\nu-1}$ we get the desired result. ∎

Of course, Proposition 7.1 immediately follows by combining these two lemmas.

APPENDIX D. Proofs of Theorem 9.1 and Lemma 9.14

We first consider the proof of Theorem 9.1. For this, we need the following fact.

Lemma D.1. *Suppose $l(S) \leqslant l(T)$, $r \in \mathbb{Z}$, and $M > n$. For $x \in \mathbb{R}^n$, let*

$$g_{S,T,M,r}(x) = \sum_{l(R)=2^{-r}} \left(1 + \frac{|x_R - x_T|}{\max(l(R), l(T))}\right)^{-M} \times \left(1 + \frac{|x_R - x|}{\max(l(R), l(S))}\right)^{-M}.$$

Then with $l(R) = 2^{-r}$,

$$g_{S,T,M,r}(x) \leqslant c_{M,n} \left(1 + \frac{|x_T - x|}{\max(l(R), l(T))}\right)^{-M} \max\left(1, \frac{l(S)}{l(R)}\right)^{n}.$$

Proof. We first note the following simple estimate. Suppose $2^{-r} = l(R) \leqslant l(U)$. Then for all $x \in \mathbb{R}^n$,

$$\sum_{l(R)=2^{-r}} \left(1 + \frac{|x_R - x|}{l(U)}\right)^{-M} \leqslant c_{M,n} \left(\frac{l(U)}{l(R)}\right)^{n}. \tag{D.1}$$

This and the trivial fact $(1 + |x_R - x_T|/\max(l(R), l(T)))^{-M} \leqslant 1$ yield the desired estimate in the case $|x - x_T| \leqslant 100\sqrt{n}\max(l(R), l(T))$. Suppose now that $|x - x_T| \geqslant 100\sqrt{n}\max(l(R), l(T))$. We let

$$A_r = \{R : l(R) = 2^{-r} \text{ and } |x_R - x_T| < \tfrac{1}{2}|x - x_T|\},$$

and

$$A_r^c = \{R : l(R) = 2^{-r} \text{ and } |x_R - x_T| \geqslant \tfrac{1}{2}|x - x_T|\}.$$

Note that for $R \in A_r$ we have $|x - x_R| \geqslant \frac{1}{2}|x - x_T|$. Write

$$g_{S,T,M,r}(x) = \sum_{R \in A_r} + \sum_{R \in A_r^c} = \mathrm{I} + \mathrm{II}.$$

We easily get the desired estimate for II from (D.1), since

$$\left(1+\frac{|x_R-x_T|}{\max(l(R),l(T))}\right)^{-M}\leqslant c\left(1+\frac{|x-x_T|}{\max(l(R),l(T))}\right)^{-M}$$

for $R\in A_r^c$. For $R\in A_r$ we have

$$\begin{aligned}&\left(1+\frac{|x-x_R|}{\max(l(R),l(S))}\right)^{-M}\\&\qquad\leqslant c\left(1+\frac{|x-x_T|}{\max(l(R),l(T))}\right)^{-M}\left(\frac{\max(l(R),l(S))}{\max(l(R),l(T))}\right)^n,\end{aligned}$$

since $l(S)\leqslant l(T)$, $M>n$, and $|x-x_T|\geqslant c\max(l(R),l(T))$. However, using (D.1) again, we have

$$\sum_{l(R)=2^{-r}}\left(1+\frac{|x_R-x_T|}{\max(l(R),l(T))}\right)^{-M}\leqslant c_{M,n}\left(\frac{\max(l(R),l(T))}{l(R)}\right)^n.$$

Putting these estimates together, we obtain the result. ∎

The previous result is used to establish a technical estimate which we present next. It is convenient to introduce some notation. Let $J=n/\min(1,p,q)$. We set

$$\begin{aligned}\omega_{QP}(\beta,\gamma)=&\left(\frac{l(Q)}{l(P)}\right)^{\alpha}\left(1+\frac{|x_Q-x_P|}{\max(l(P),l(Q))}\right)^{-J-\beta}\\&\times\min\left[\left(\frac{l(Q)}{l(P)}\right)^{(n+\gamma)/2},\left(\frac{l(P)}{l(Q)}\right)^{(n+\gamma)/2+J-n}\right],\end{aligned}$$

and

$$W_{QP}(\beta,\gamma_1,\gamma_2)=\sum_R\omega_{QR}(\beta,\gamma_1)\,\omega_{RP}(\beta,\gamma_2).$$

THEOREM D.2. *Suppose* $\beta,\gamma_1,\gamma_2>0$, $\gamma_1\neq\gamma_2$, *and* $\gamma_1+\gamma_2>2\beta$. *Then there exists a constant* $c=c_{n,J,\beta,\gamma_1,\gamma_2}$ *such that*

$$W_{QP}(\beta,\gamma_1,\gamma_2)\leqslant c\,\omega_{QP}(\beta,\min(\gamma_1,\gamma_2)).$$

Proof. Let $\gamma=\min(\gamma_1,\gamma_2)$. We first consider the case $l(P)\leqslant l(Q)$. We may assume that $\alpha=0$ since the terms $l(R)^{\alpha}$ cancel in the sum defining W_{QP} leaving $(l(Q)/l(P))^{\alpha}$. We write

$$\begin{aligned}W_{QP}(\beta,\gamma_1,\gamma_2)&=\sum_{l(R<l(P)\leqslant l(Q)}+\sum_{l(P)\leqslant l(R)\leqslant l(Q)}+\sum_{l(P)\leqslant l(Q)<l(R)}\\&=\mathrm{I}+\mathrm{II}+\mathrm{III}.\end{aligned}$$

If $l(R)=2^{-r}$, $l(P)=2^{-p}$, and $l(Q)=2^{-q}$, then

$$I=\sum_{r=p+1}^{\infty}\sum_{l(R)=2^{-r}}\left(1+\frac{|x_Q-x_R|}{l(Q)}\right)^{-J-\beta}\left(\frac{l(R)}{l(Q)}\right)^{(n+\gamma_1)/2+J-n}$$
$$\times\left(1+\frac{|x_R-x_P|}{l(P)}\right)^{-J-\beta}\left(\frac{l(R)}{l(P)}\right)^{(n+\gamma_2)/2}$$
$$=2^{q((n+\gamma_1)/2+J-n)}2^{p(n+\gamma_2)/2}\sum_{r=p+1}^{\infty}2^{-r((\gamma_1+\gamma_2)/2+J)}g_{P,Q,J+\beta,r}(x_P).$$

By using Lemma D.1, this can be estimated by

$$c2^{q((n+\gamma_1)/2+J-n)}2^{-p(n-\gamma_2)/2}\left(1+\frac{|x_P-x_Q|}{l(Q)}\right)^{-J-\beta}\sum_{r=p+1}^{\infty}2^{-r((\gamma_1+\gamma_2)/2+J-n)}$$
$$\leqslant c\left(1+\frac{|x_P-x_Q|}{l(Q)}\right)^{-J-\beta}\left(\frac{l(P)}{l(Q)}\right)^{(n+\gamma)/2+J-n},$$

which is the required estimate. II can also be estimated by this quantity. The proof of this is essentially the same, starting from the identity

$$\mathrm{II}=2^{q((n+\gamma_1)/2)+J-n)}2^{-p((n+\gamma_2)/2+J-n)}\sum_{r=p}^{q}2^{-r(\gamma_1-\gamma_2)/2}g_{P,Q,J+\beta,r}(x_P),$$

and using Lemma D.1. Finally,

$$\mathrm{III}=2^{-q(n+\gamma_1)/2}2^{-p((n+\gamma_2)/2+J-n)}\sum_{r=-\infty}^{q-1}2^{r((\gamma_1+\gamma_2)/2+J)}g_{P,Q,J+\beta,r}(x_P).$$

Here, using Lemma D.1 again,

$$g_{P,Q,J+\beta,r}(x_P)\leqslant c\left(1+\frac{|x_P-x_Q|}{l(R)}\right)^{-J-\beta}$$
$$\leqslant c\left(1+\frac{|x_P-x_Q|}{l(Q)}\right)^{-J-\beta}\left(\frac{l(Q)}{l(R)}\right)^{-J-\beta}.$$

Inserting this in the sum, we readily get

$$\mathrm{III}\leqslant c\left(1+\frac{|x_P-x_Q|}{l(Q)}\right)^{-J-\beta}\left(\frac{l(P)}{l(Q)}\right)^{-J-\beta},$$

and this completes the proof in the case $l(P)\leqslant l(Q)$.

The case $l(Q)\leqslant l(P)$ follows by symmetry; we apply the previous case with P and Q interchanged and α replaced by $-\alpha+J-n$. Alternatively, we may give a direct proof, virtually the same as the one just completed. ∎

Proof of Theorem 9.1. We prove (i) first. If $A=\{a_{QP}\}_{Q,P}$, $B=\{b_{QP}\}_{Q,P}\in \mathbf{ad}_p^{\alpha q}$, then by definition there are $\varepsilon_A, \varepsilon_B>0$ such that $|a_{QP}|\leqslant c\omega_{QP}(\varepsilon_A)$ and $|b_{QP}|\leqslant c\omega_{QP}(\varepsilon_B)$. Since $\omega_{QP}(\varepsilon)$ is a nonincreasing function of ε, we may assume that $\varepsilon_A>\varepsilon_B$. By the definitions, we have $\omega_{QP}(\varepsilon)=\omega_{QP}(\varepsilon,\varepsilon)$ and $\omega_{QP}(\varepsilon)\leqslant\omega_{QP}(\beta,\gamma)$ if $\beta,\gamma\leqslant\varepsilon$. Hence, Theorem D.2 implies that

$$\begin{aligned}|(AB)_{QP}| &= \Big|\sum_R a_{QR}b_{RP}\Big| \leqslant c\sum_R \omega_{QR}(\varepsilon_A)\,\omega_{RP}(\varepsilon_B)\\ &\leqslant C\sum_R \omega_{QR}(\varepsilon_B,\varepsilon_A)\,\omega_{RP}(\varepsilon_B,\varepsilon_B)\\ &\leqslant C\omega_{QP}(\varepsilon_B,\varepsilon_B)=C\omega_{QP}(\varepsilon_B).\end{aligned}$$

This proves (i).

It remains to show (ii). Let $\tilde{A}=(\mathrm{I}-A)$. Suppose we have $|\tilde{A}_{QP}|\leqslant\delta\omega_{QP}(\varepsilon)$ for some $\varepsilon,\delta>0$. Clearly, $\omega_{QP}(\varepsilon)\leqslant\omega_{QP}(\tilde{\varepsilon},\varepsilon)\leqslant\omega_{QP}(\tilde{\varepsilon},\tilde{\varepsilon})$ for any fixed $0<\tilde{\varepsilon}<\varepsilon$. The proof of (i), with A replaced by $\tilde{A}$, B by $\tilde{A}^{n-1}$, ε_A by ε, and ε_B by $\tilde{\varepsilon}$, shows that $|(\tilde{A}^n)_{QP}|\leqslant(\delta C)^n\,\omega_{QP}(\tilde{\varepsilon},\tilde{\varepsilon})$ for any nonnegative integer n and some constant C independent of n and δ. For a sufficiently small $\delta>0$ we have $\delta C<1$, and, hence, the Neumann series $\sum_{n\geqslant 0}(\tilde{A})^n$ converges to $(\mathrm{I}-\tilde{A})^{-1}=A^{-1}$. Furthermore, $(A^{-1})_{QP}\leqslant(1-\delta C)^{-1}\,\omega_{QP}(\tilde{\varepsilon})$ and $A^{-1}\in\mathbf{ad}_p^{\alpha q}$. ∎

We now turn to the proof of Lemma 9.14.

Proof of Lemma 9.14. To estimate $|m_Q(x)|$, with $l(Q)=2^{-q}$, write

$$|m_Q(x)|\leqslant\sum_{l(P)\leqslant l(Q)}\omega_{PQ}(\varepsilon)|g_P(x)|+\sum_{l(P)>l(Q)}\omega_{PQ}(\varepsilon)|g_P(x)|\equiv\mathrm{I}+\mathrm{II}.$$

Then, in the notation of Lemma D.1, with $l(R)=2^{-r}$ replaced by $l(P)=2^{-p}$,

$$\begin{aligned}I &\leqslant\sum_{p=q}^{\infty}\sum_{l(P)=2^{-p}}\left(1+\frac{|x_Q-x_P|}{l(Q)}\right)^{-J-\varepsilon}\left(\frac{l(P)}{l(Q)}\right)^{(n+\varepsilon)/2}\\ &\qquad\times|P|^{-1/2}\left(1+\frac{|x-x_P|}{l(P)}\right)^{-J-\tilde{\varepsilon}}\\ &\leqslant 2^{q(n+\varepsilon)/2}\sum_{p=q}^{\infty}2^{-p\varepsilon/2}g_{P,Q,J+\tilde{\varepsilon},p}(x)\\ &\leqslant C\,|Q|^{-1/2}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-J-\tilde{\varepsilon}}.\end{aligned}$$

Similarly, for II we have, by Lemma D.1,

$$\begin{aligned}\mathrm{II} &\leqslant 2^{-q(J+(\varepsilon-n)/2)} \sum_{p=-\infty}^{q-1} 2^{p(J+\varepsilon/2)} g_{P,Q,J+\tilde{\varepsilon},p}(x)\\ &\leqslant C\,|Q|^{-1/2}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-J-\tilde{\varepsilon}}.\end{aligned}$$

This yields

$$|m_Q(x)| \leqslant C\,|Q|^{-1/2}\left(1+\frac{|x-x_Q|}{l(Q)}\right)^{-J-\tilde{\varepsilon}}. \tag{D.2}$$

Now note that our proof of (D.2) shows that for $|\gamma| \leqslant [J-n]$,

$$\int |x|^{\gamma} \sum_P |a_{QP}|\,|g_P(x)|\,dx < +\infty.$$

Therefore the vanishing moment condition

$$\int x^{\gamma} m_Q(x)\,dx = 0 \qquad \text{if} \quad |\gamma| \leqslant [J-n]$$

is inherited by m_Q from the corresponding condition on the g_P's.

It remains only to verify that

$$\begin{aligned}&|m_Q(x)-m_Q(y)|\\ &\quad\leqslant C\,|Q|^{-1/2-\delta/n}\,|x-y|^{\delta} \sup_{|z|\leqslant|y-x|}\left(1+\frac{|x-z-x_Q|}{l(Q)}\right)^{-J-\tilde{\varepsilon}}.\end{aligned} \tag{D.3}$$

To prove this, we may assume $|x-y| < l(Q)$, since otherwise (D.1) and the trivial estimate $|m_Q(x)-m_Q(y)| \leqslant |m_Q(x)| + |m_Q(y)|$ yield (D.3).

With $|x-y| < l(Q)$, we have

$$\begin{aligned}|m_Q(x)-m_Q(y)| &\leqslant \sum_{l(P)<|x-y|} \omega_{PQ}(\varepsilon)(|g_P(x)|+|g_P(y)|)\\ &\quad+ \sum_{|x-y|\leqslant l(P)\leqslant l(Q)} \omega_{PQ}(\varepsilon)|g_P(x)-g_P(y)|\\ &\quad+ \sum_{l(Q)<l(P)} \omega_{PQ}(\varepsilon)|g_P(x)-g_P(y)|\\ &\equiv \mathrm{III}+\mathrm{IV}+\mathrm{V}.\end{aligned}$$

To estimate III, let $k \in \mathbb{Z}$ be such that $2^{-k} < |x-y| \leqslant 2^{-k+1}$. Then, similarly to I above,

$$\begin{aligned}\sum_{l(P)<|x-y|} &\omega_{PQ}(\varepsilon)|\, g_P(y)| \\ &\leqslant c\left(1+\frac{|y-x_Q|}{l(Q)}\right)^{-J-\tilde{\varepsilon}} 2^{q(n+\varepsilon)/2} \sum_{p=k}^{\infty} 2^{-p\varepsilon/2} \\ &\leqslant c\,|Q|^{-1/2-\varepsilon/2n}\,|x-y|^{\varepsilon/2}\left(1+\frac{|y-x_Q|}{l(Q)}\right)^{-J-\tilde{\varepsilon}},\end{aligned}$$

which is dominated by the right-hand side of (D.3) since $\delta < \varepsilon/2$. The same estimate holds for $\sum_{l(P)<|x-y|} \omega_{PQ}(\varepsilon)|\, g_P(x)|$, and hence for III. For IV and V we use the estimate

$$\begin{aligned}|\, g_P(x)-g_P(y)| &\leqslant |P|^{-1/2-\delta/n}\,|x-y|^{\delta} \sup_{|z|\leqslant|y-x|}\left(1+\frac{|x-z-x_P|}{l(P)}\right)^{-J-\tilde{\varepsilon}} \\ &\leqslant |P|^{-1/2-\delta/n}\,|x-y|^{\delta}\left(1+\frac{|x-x_P|}{l(P)}\right)^{-J-\tilde{\varepsilon}},\end{aligned}$$

since $|z| \leqslant |y-x| \leqslant l(P)$ gives $1+|x-z-x_P|/l(P) \approx 1+|x-x_P|/l(P)$. With this, the estimates for IV and V are almost the same as for I and II. For IV we obtain $\sum_{p=q}^{k-1} 2^{p(\delta-\varepsilon/2)}$, which converges, since $\delta < \varepsilon/2$. For V we get $\sum_{p=-\infty}^{q-1} 2^{p(\delta+\varepsilon/2-\tilde{\varepsilon})}$, which is better than in II. These estimates yield (D.3) and complete the proof. ∎

APPENDIX E. Proofs of Lemma 10.8, Theorems 10.11–10.12, and Proposition 10.14

Proof of Lemma 10.8. Since T_m is a Fourier multiplier operator, $a_{QP} = \langle T\psi_P, \varphi_Q\rangle = 0$ unless $\frac{1}{2} \leqslant l(Q)/l(P) \leqslant 2$. When $a_{QP} \neq 0$ we have, as in Example 9.19, $a_{QP} = (2\pi)^{-n}\, 2^{-(\mu+\nu)n/2}\, h_{\mu\nu}(x_Q - x_P)$, where $l(P) = 2^{-\mu}$, $l(Q) = 2^{-\nu}$, and $h_{\mu\nu}(x) = (2\pi)^n\, 2^{\nu n}(m(2^\nu \cdot)\,\chi_j(\cdot))^\vee\,(2^\nu x)$. With these notations, we will estimate

$$\begin{aligned}I \equiv \sup_{\nu}\ \sup_{l(Q)=2^{-\nu}} \sum_{\mu=\nu-1}^{\nu+1}\ &\sum_{l(P)=2^{-\mu}} \\ &\times((1+2^{\mu}\,|x_Q-x_P|)^{\delta_2 - \nu n}\,|h_{\mu\nu}(x_Q-x_P)|)^r.\end{aligned}$$

Now, supp $\hat{h}_{\mu\nu} \subset \{\xi : |\xi| \leqslant 2^{\mu+1}\}$, so $h_{\mu\nu}$ is of exponential type. By the proof of (2.11) in [Fr-J1] (essentially the Plancherel-Pólya theorem), we have

$$2^{-\nu n}\,|h_{\mu\nu}(x_Q-x_P)|^r \leqslant c_L \sum_{l\in\mathbb{Z}^n} (1+|l|)^{-L} \int_{S_{P,Q,l}} |h_\nu(x)|^r\,dx, \tag{E.1}$$

for each L, where $S_{P,Q,l}=\{x_Q+2^{-\mu}l-x\colon x\in P\}$, and $h_\nu(x)=(m\bar{\tilde{\varphi}}_\nu)^\vee(x)$. For $x\in S_{P,Q,l}$

$$\begin{aligned}(1+2^\mu\,|x_Q-x_P|) &\approx 1+2^\mu\,|x-2^{-\mu}l| \leqslant 1+2^\mu\,|x|+|l|\\ &\leqslant 2(1+2^\nu\,|x|)(1+|l|),\end{aligned} \tag{E.2}$$

for $|\mu-\nu|\leqslant 1$. Substituting (E.1) and (E.2) into I, taking the sum on l outside, noting that $\bigcup_P S_{P,Q,l}=\mathbb{R}^n$ for each Q and l, and picking L sufficiently large, we obtain

$$I \leqslant C \sup_\nu \sum_{\mu=\nu-1}^{\nu+1} 2^{-\nu n(r-1)} \int (1+2^\nu\,|x|)^{\delta}\,|h_\nu(x)|)^r\,dx.$$

By a change of variables this last expression is dominated by the right-hand side in Lemma 10.8. ▮

Remark E.1. Suppose more generally that $\Phi(t)$, $t\geqslant 0$, is a nondecreasing function with $\Phi(2t)\leqslant C\Phi(t)$, and $\Phi(0)=1$. In Corollary 10.10, we need the analogue of Lemma 10.8 for $\Phi(|x|)^{1/2}$ in place of $(1+|x|)^\delta$. The proof in this more general case is almost the same. We first notice that there is a $\beta>0$ such that $\Phi(|tx|)\leqslant C(1+t)^\beta\,\Phi(|x|)$, $t>0$. To estimate $\Phi(2^\mu\,|x_P-x_Q|)$ we argue as in the proof above. For $x\in S_{P,Q,l}$, we have

$$\begin{aligned}\Phi(2^\mu\,|x_P-x_Q|) &\leqslant C\Phi(2^\mu x-l) \leqslant C\Phi((1+2^\nu\,|x|)(1+|l|))\\ &\leqslant C(1+|l|)^\beta\,\Phi(1+2^\nu\,|x|) \leqslant C(1+|l|)^\beta\,\Phi(2^\nu\,|x|).\end{aligned}$$

Now the proof can be continued as before.

Proof of Theorem 10.11. We first show that (ii) and (iii) are equivalent. Assume that (ii) holds. Given a function $v(x)\in L^r$ we define the sequence $t=\{t_Q\}_Q$ by $t_Q=\int_Q |v(x)|\,dx$. We have $\|t\|_{\dot{\mathbf{f}}_r^{\beta\infty}}=\|M^d v\|_{L^r}$, where M^d denotes the "dyadic" maximal operator (i.e., the supremum being taken over averages only over dyadic cubes). Since $r>1$, this and the maximal theorem imply that $t\in\dot{\mathbf{f}}_r^{\beta\infty}$. Now (ii) provides us with a sequence $\tau=\{\tau_Q\}_Q$ and we define $w(x)=\sup_{x\in Q}\,|\,|Q|^{-1}\,\tau_Q\chi_Q(x)|$. Clearly, $v(x)\leqslant M^d v(x)\leqslant w(x)$, and $\|w\|_{L^r}=\|\tau\|_{\dot{\mathbf{f}}_r^{\beta\infty}}\leqslant c\,\|t\|_{\dot{\mathbf{f}}_r^{\beta\infty}}\leqslant c\,\|v\|_{L^r}$. Furthermore,

$$\|As\|_{\dot{f}_q^{\alpha q}(v)} = \left(\sum_Q (|Q|^{-\alpha/n - 1/2} (As)_Q)^q \, t_Q\right)^{1/q}$$
$$\leqslant C\left(\sum_Q (|Q|^{-\alpha/n - 1/2} |s_Q|)^q \, \tau_Q\right)^{1/q} \leqslant C \, \|s\|_{\dot{f}_q^{\alpha q}(w)}.$$

This is (iii) except for the fact that we now have different weights on the left and right. However, using the inductive argument in the proof of Theorem A′ in [RdF2], this gives us the full statement.

The proof that (ii) is a consequence of (iii) is similar. Given the sequence $t = \{t_Q\}_Q$ we put $v(x) = \sup_Q | |Q|^{-1} t_Q \chi_Q(x)|$ and $\tau_Q = \int_Q w(x)\, dx$, and obtain

$$\left(\sum_Q (|Q|^{-\alpha/n - 1/2} (As)_Q)^q \, t_Q\right)^{1/q} \leqslant C\left(\sum_Q (|Q|^{-\alpha/n - 1/2} |s_Q|)^q \, \tau_Q\right)^{1/q}.$$

Again the argument in [RdF2] shows that this implies (ii).

That either (ii) or (iii) implies (i) is a simple argument based on duality. For instance, let us assume (ii). Since $(\dot{f}_{r'}^{-\beta 1})^* \approx \dot{f}_r^{\beta\infty}$ (Remark 5.11), there is a nonnegative sequence $t = \{t_Q\}_Q \in \dot{f}_r^{\beta\infty}$, $\|t\|_{\dot{f}_r^{\beta\infty}} = 1$, such that

$$\|As\|^p_{\dot{f}_p^{\alpha q}} = \|\{|Q|^{-\alpha/n - 1/2} |(As)_Q|\}\|^{r'}_{\dot{f}_{r'}^{-\beta 1}}$$
$$\leqslant c\left(\sum_Q (|Q|^{-\alpha/n - 1/2} |(As)_Q|)^q \, t_Q\right)^{p/q}.$$

By (ii) there is a sequence $\tau = \{\tau_Q\}_Q$ so that this is dominated by

$$\left(\sum_Q (|Q|^{-\alpha/n - 1/2} |(As)_Q|)^q \, \tau_Q\right)^{p/q} \leqslant C\left(\sum_Q (|Q|^{-\alpha/n - 1/2} |s_Q|)^q \, \tau_Q\right)^{p/q}.$$

Using the duality once more and that $\|\tau\|_{\dot{f}_r^{\beta\infty}} \leqslant c$ yields

$$\|As\|^p_{\dot{f}_p^{\alpha q}} \leqslant C \, \|s\|^p_{\dot{f}_p^{\alpha q}}.$$

The converse implication, that (i) implies (ii) or (iii), is harder. Here we simply rely on Section 3 of [RdF2]. According to the results there, we only have to check that the lattice $\dot{f}_p^{\alpha q}$ is q-convex, which is trivial.

Theorem 10.12 follows by the duality $(l^s)^* \approx l^{s'}$ and Theorem 10.11, cf. [RdF2].

Proof of Proposition 10.14. The proof proceeds as the proof of Theorem 2.2 with two minor modifications. First, instead of the usual Fefferman-Stein vector-valued maximal inequality (Theorem A.1) we use a version for doubling measures $w dx$. We let M_w be the weighted Hardy–Littlewood maximal operator,

$$M_w f(x) = \sup_{x \in Q} w(Q)^{-1} \int_Q |f(y)| \, w(y) \, dy.$$

When $w \equiv 1$, this is just the ordinary maximal operator M. Now for $1 < p < +\infty$ and $1 < q \leqslant +\infty$, we have

$$\left\| \left(\sum |M_w f_i|^q \right)^{1/q} \right\|_{L^p(w\,dx)} \leqslant c_{p,q} \left\| \left(\sum |f_i|^q \right)^{1/q} \right\|_{L^p(w\,dx)}.$$

For a proof of this, see [AJ; or Ja-T1].

The second modification we need is a variant of Lemma A.2. It is easy to see that w satisfies the doubling condition if and only if there is an $\alpha > 0$ such that

$$\frac{w(Q)}{w(E)} \leqslant C \left(\frac{|Q|}{|E|} \right)^{\alpha}$$

for each cube Q and each subcube $E \subset Q$. Using this and weighted averages in the proof of Lemma A.2, we again get a geometric series, which this time can be summed as long as $\lambda > n\alpha r/a$. In this way we obtain the inequality

$$\begin{aligned} &\left(\sum_{l(P) = 2^{-\mu}} |s_P|^r / (1 + l(P)^{-1} |x_P - x_Q|)^{\lambda} \right)^{1/r} \\ &\qquad \leqslant c \left(M_w \left(\sum_{l(P) = 2^{-\mu}} |s_P|^a \chi_P \right)(x) \right)^{1/a} \end{aligned}$$

for each $x \in Q$ with $l(Q) = 2^{-\nu}$ and $\mu \leqslant \nu$. Here $0 < a \leqslant r < +\infty$ and $\lambda > n\alpha r/a$. With these two modifications, we can proceed as in the proof of Theorem 2.2 (there is no difficulty in adapting the proof, since λ may be taken arbitrarily large in Appendix A). ∎

APPENDIX F: NOTATION

We generally define our notation as it is introduced, but for convenience we list here our more commonly used conventions. The Fourier transform $\hat{f}$ of a function on $\mathbb{R}^n$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) \, e^{-ix \cdot \xi} \, dx.$$

The inverse Fourier transform is denoted $^\vee$. Here n always refers to the Euclidean dimension, which is fixed.

$\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing test functions, $\mathscr{S}'$ its dual, and $\mathscr{S}_0 = \{f \in \mathscr{S} : 0 \notin \operatorname{supp} \hat{f}\}$. Throughout, supp f denotes the closed support of f. $\mathscr{P}$ is the space of all polynomials.

For $k \in \mathbb{Z}^n$ and $\nu \in \mathbb{Z}$, $Q_{\nu k}$ is the dyadic cube

$$Q_{\nu k} = \{(x_1, ..., x_n) : k_i \leqslant 2^\nu x_i < k_i + 1 \text{ for } i = 1, ..., n\}.$$

We let $x_Q = 2^{-\nu}k$ (for $Q = Q_{\nu k}$) be the "lower left corner" of Q. The sidelength of any cube Q is denoted $l(Q)$, and for $r > 0$, rQ is the cube concentric with Q having sidelength $rl(Q)$. We often let Q represent an index set, as in $\sum_Q$ or $\{\cdot\}_Q$. This means that the index set is the collection of all dyadic cubes in $\mathbb{R}^n$. For $\nu \in \mathbb{Z}$, f_ν is the $2^{-\nu}$-dilate of f, i.e., $f_\nu(x) = 2^{\nu n} f(2^\nu x)$. $\tilde{\chi}_Q$ is the L^2-normalized characteristic function of Q, i.e., $\tilde{\chi}_Q(x) = |Q|^{-1/2}$ if $x \in Q$ and $\tilde{\chi}_Q(x) = 0$ otherwise.

The pairing $\langle f, g \rangle$, is always linear in f and conjugate linear in g. In the case when f is a distribution and g is a test function, this means that $\langle f, g \rangle = f(\bar{g})$. As usual, $\langle f, g \rangle = \int f\bar{g}$ when $f, g \in L^2$.

For $x \in \mathbb{R}$, we let $[x]$ be the greatest integer in x (the integer satisfying $x - 1 < [x] \leqslant x$), $x^* = x - [x]$, and $x_+ = \max(x, 0)$. In Section 3 we introduce the quantities J, N, and M, which are referred to frequently. Here

$$J = n/\min(1, p, q), \qquad N = \max([J - n - \alpha], -1),$$

and M is some fixed real number greater than J. (Only N is necessarily an integer.)

We use the standard multi-index notation. For $\gamma = (\gamma_1, ..., \gamma_n)$, where $\gamma_i \in \mathbb{Z}$, $\gamma_i \geqslant 0$ for each $i = 1, ..., n$, we have $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, $\partial^\gamma = \partial^{\gamma_1}/\partial x_1^{\gamma_1} \cdots \partial^{\gamma_n}/\partial x_n^{\gamma_n}$, $\gamma! = \gamma_1! \cdots \gamma_n!$, and $|\gamma| = \gamma_1 + \cdots + \gamma_n$.

In general, the notation $\|f\|_X \approx \|f\|_Y$ (or just $\|\cdot\|_X \approx \|\cdot\|_Y$) means that the quasi-norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent; i.e., there exists a constant c such that $c^{-1}\|f\|_X \leqslant \|f\|_Y \leqslant c\|f\|_X$ for all f with $\|f\|_X$ or $\|f\|_Y$ finite. As usual, c and C will in general represent various constants at various times.

The weighted Lebesgue space $L^p(w\,dx)$ is equipped with the norm $(\int |f(x)|^p w(x)\,dx)^{1/p}$. For a measurable subset $E \subset \mathbb{R}^n$, $|E|$ is the Lebesgue measure of E.

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