直角アルティン群と
向き付け不可能曲面の曲線グラフ

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Plan

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§1. Preliminaries

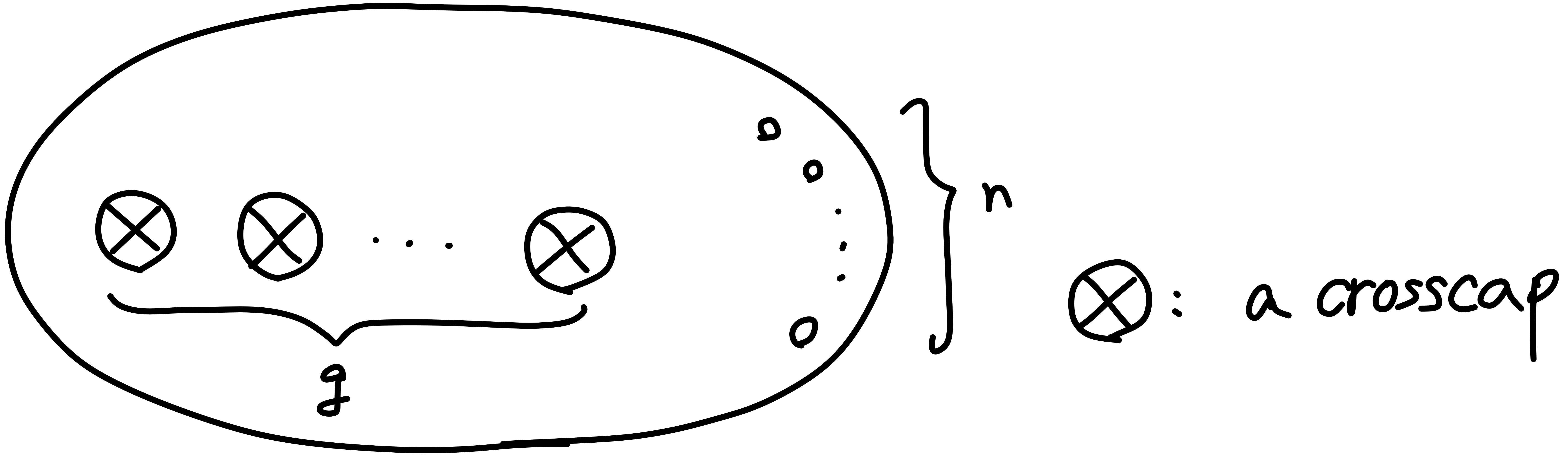
§1.1. Mapping class groups

$S = S_{g,n}$: a conn. ori. surf. of genus $g \geq 0$ with n punctures.

$N = N_{g,n}$: 〃 nonori. 〃 $g \geq 1$ 〃 .

$F = S$ or N.

$N_{g,n} =$



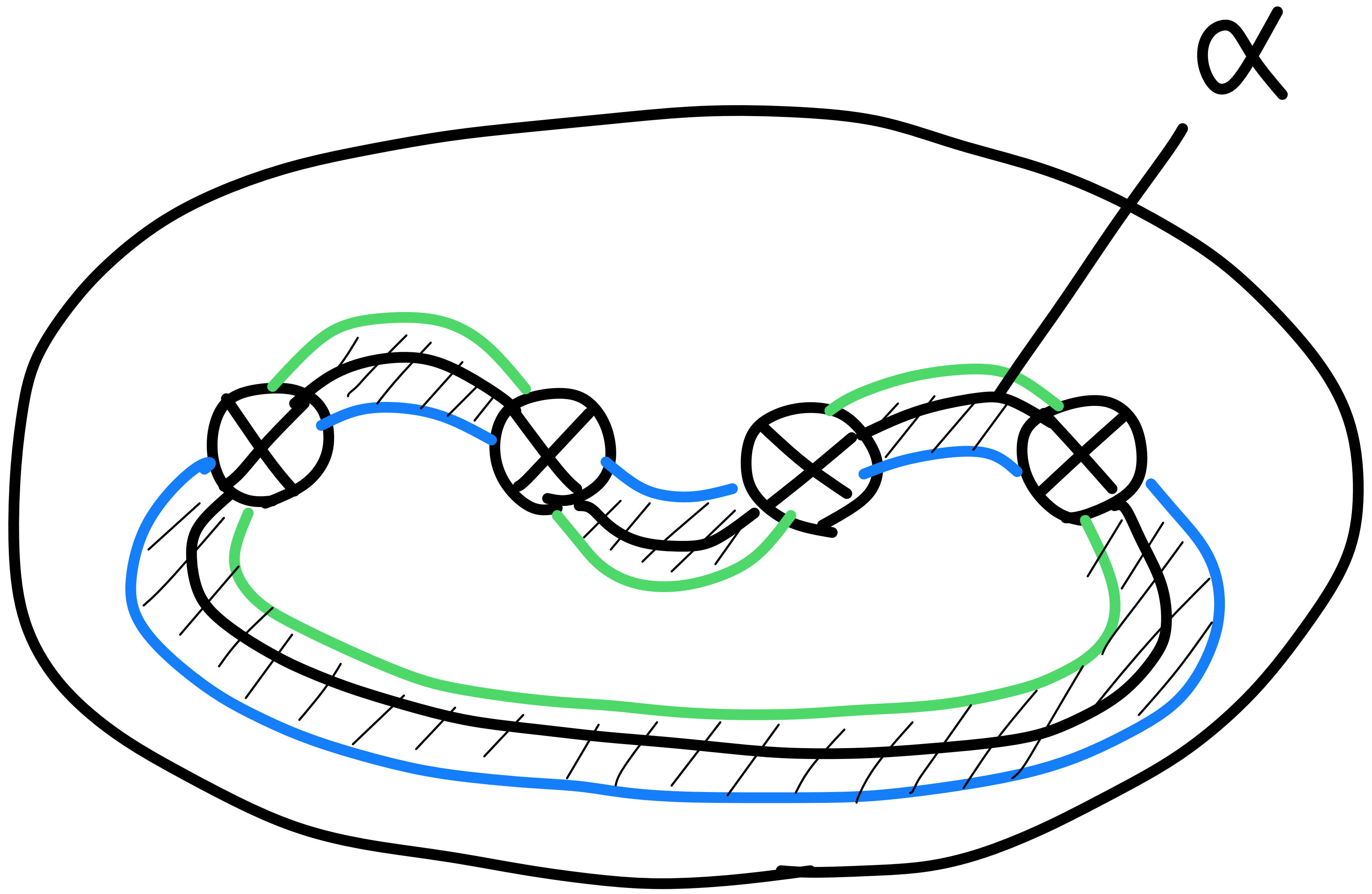
$\mathrm{Mod}(F) = \{f : F \to F : (\text{ori-pres if } F = S) \text{ homeo}\} / \text{isotopy}$

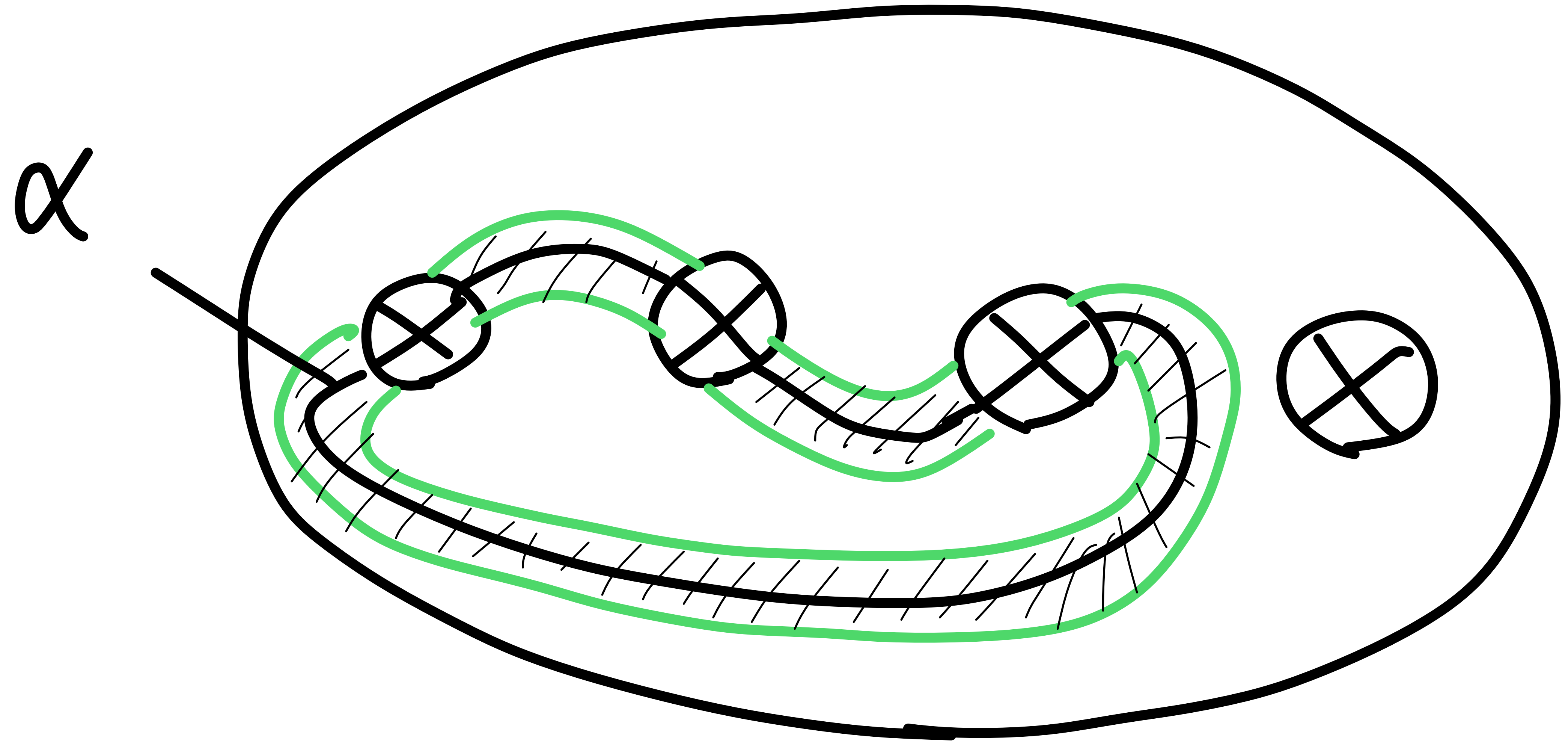
: the mapping class group of F.

A simple closed curve (s.c.c., curve) α is

- two-sided if the reg. nbd. of α is an annulus
- one-sided if the reg. nbd. of α is a Möbius band.

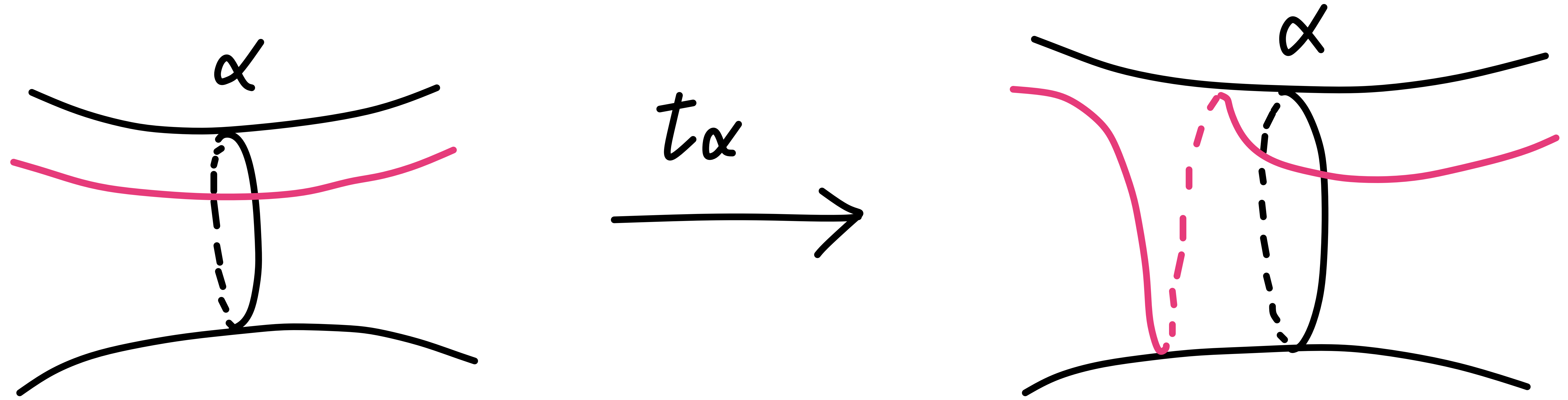
e.g.



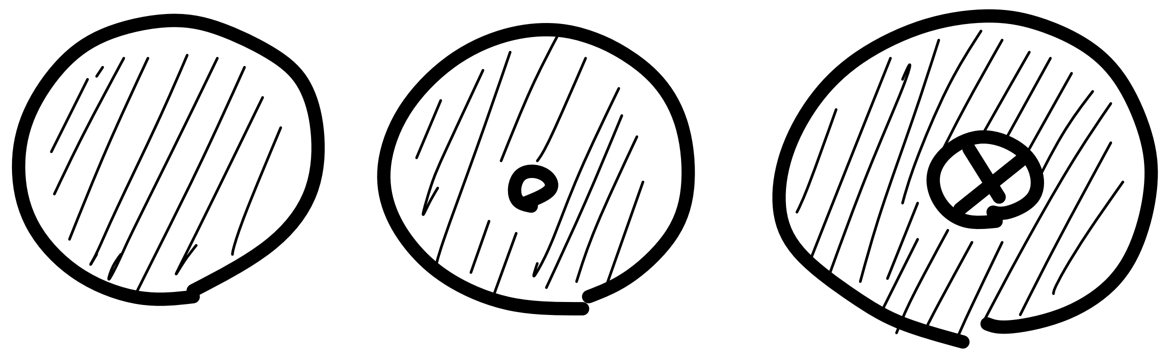


An important element of Mod(F):

A Dehn twist along a two-sided s.c.c. α



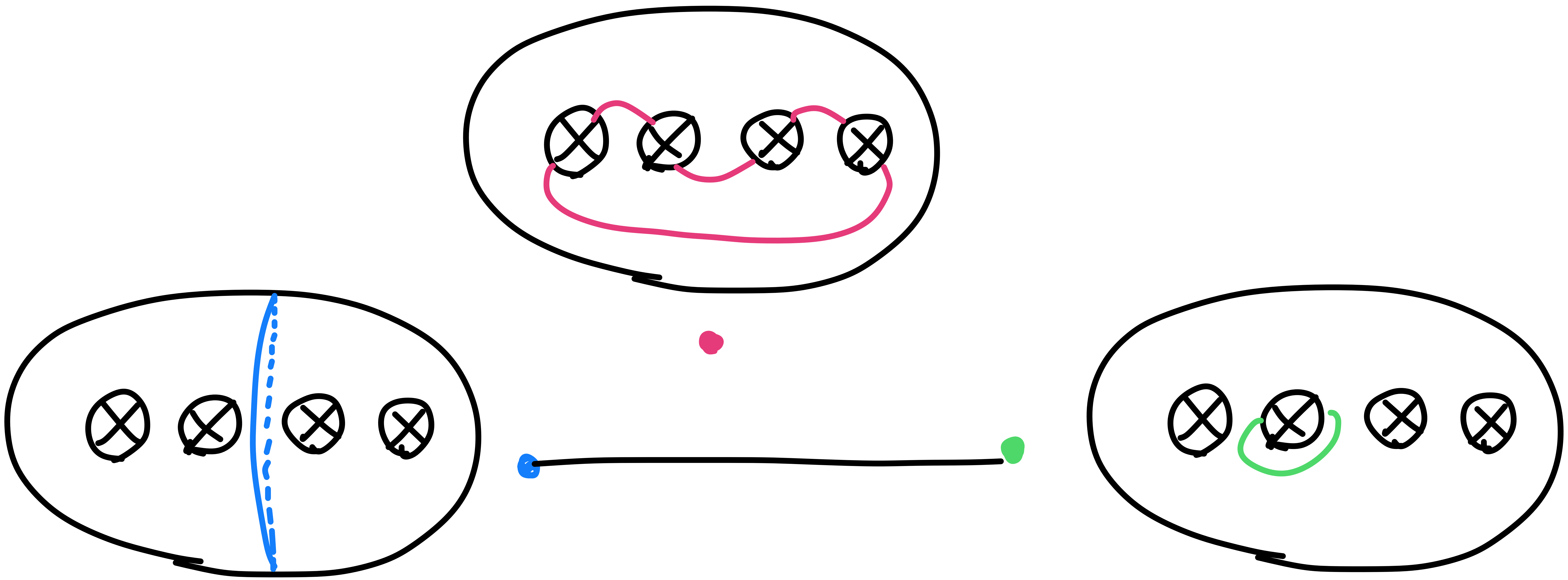
§ 1.2 Curve graphs

NOT: 

$C(F)$: the curve graph of F

$\overset{\Longleftrightarrow}{\text{def}}$
- a vertex: an isotopy class of essential s.c.c.
- an edge $\{a, b\}$: a pair of vertices which can be realized disj.

e.g.



§ 1.3. Right-angled Artin groups (RAAG)

Γ : a finite graph without loops and multi-edges

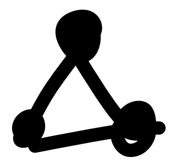
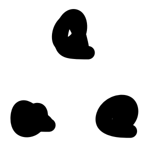
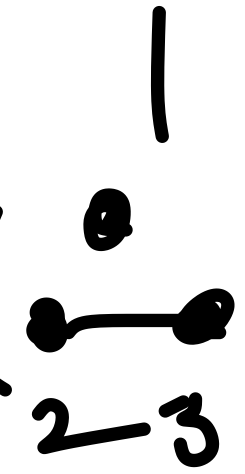
$V(\Gamma)$: the vertex set of Γ.

$E(\Gamma)$: the edge set of Γ.

The right-angled Artin group (RAAG) on Γ :

$$A(\Gamma) = \langle V(\Gamma) \mid v_i v_j = v_j v_i \text{ iff } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

e.g.

$A($  $) \cong \mathbb{Z}^3$, $A($  $) \cong F_3$, $A($ 

$) = \langle 1, 2, 3 \mid 23 = 32 \rangle \cong \mathbb{Z}^2 * \mathbb{Z}$.

§2. Previous researches and Main results

Thm. (Koberda, 2012)
$\chi(S) < 0$. If Γ is a full-subgraph of $\mathcal{C}(S)$ ($\Gamma \leq \mathcal{C}(S)$),
then $A(\Gamma)$ is a subgroup of $\mathrm{Mod}(S)$ ($A(\Gamma) \leq \mathrm{Mod}(S)$).

$$\begin{array}{ccc} A(\Gamma) & \hookrightarrow & \mathrm{Mod}(S) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v^{\text{-gen.}} & \longmapsto & T_v^{N} \end{array}$$

N: independent of generators of $A(\Gamma)$

Thm. (Kim–Koberda, 2016)
$\xi(S) = 3g-3+n \geq 4$. $\exists\, \Gamma$ s.t. $A(\Gamma) \leq \mathrm{Mod}(S)$ but $\Gamma \not\leq \mathcal{C}(S)$.

Q. $\Gamma \le C(N) \overset{?}{\Longrightarrow} A(\Gamma) \le \mathrm{Mod}(S)$?

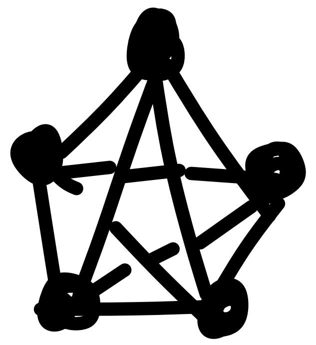
A. NO !

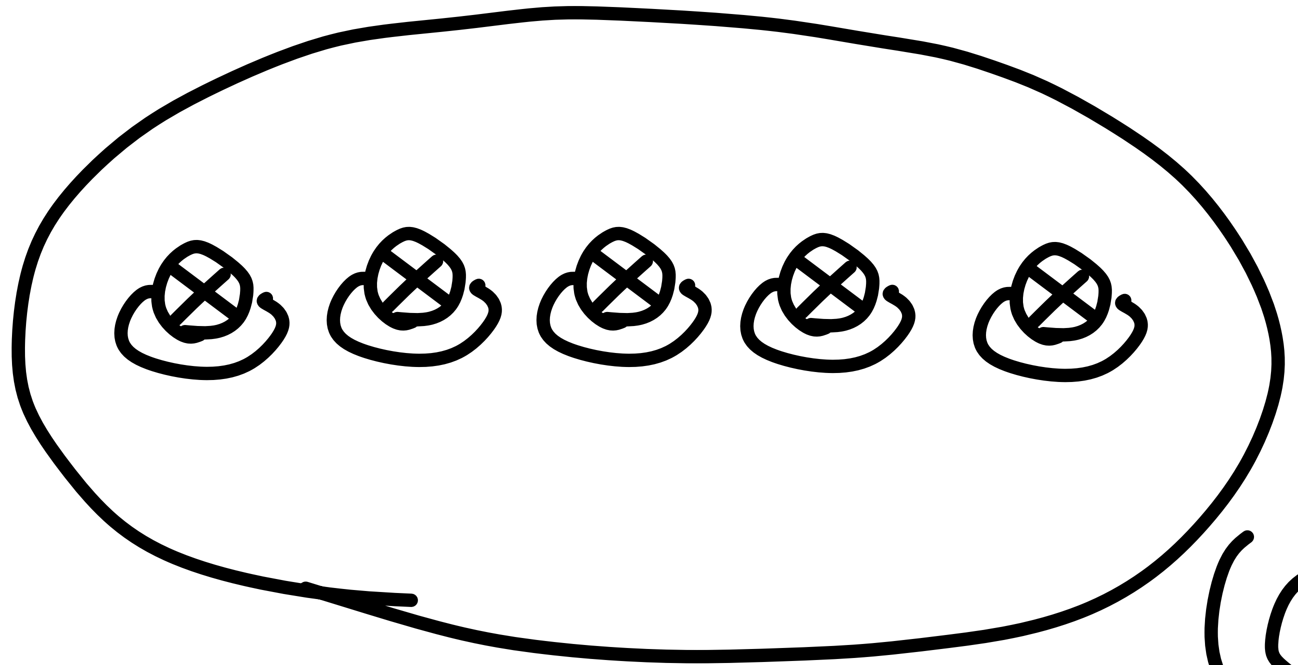
Prop. (K. 2019)

$\chi(N) < 0$. The maximal rank of free abel. subgp. of $\mathrm{Mod}(N)$ is

$$\begin{cases} \cdot\ \frac{3}{2}(g-1) + n - 2 & \text{if } g \text{ is odd,} \\ \cdot\ \frac{3}{2}g + n - 3 & \text{if } g \text{ is even.} \end{cases}$$

e.g.

$K_5 =$  $\le C(N_{5,0})$, but

$N_{5,0} =$ 

$A(K_5) \cong \mathbb{Z}^5 \not\le \mathrm{Mod}(N_{5,0})$

($\because$ max. rank $= \frac{3}{2}(5-1) + 0 - 2 = 4$.)

$\mathcal{C}_{two}(N) \leq \mathcal{C}(N)$

: consists of all vertices represented by two-sided s.c.c.s.

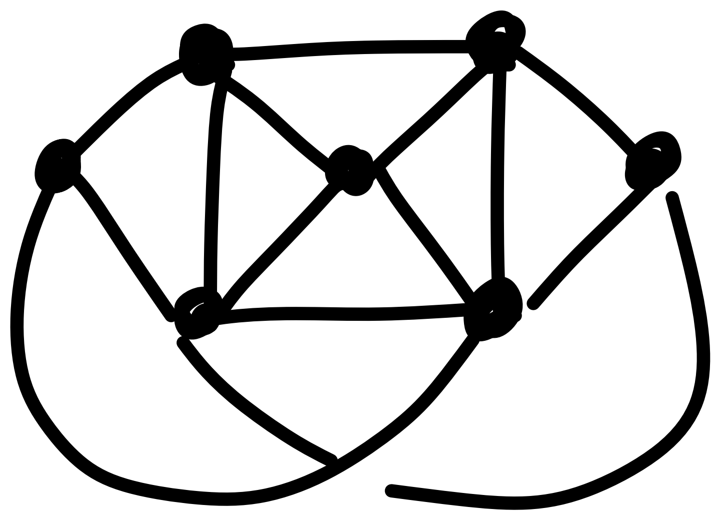
<u>Thm.</u> 1 (Katayama – K, 2021, arXiv)

$\chi(N) < 0$. If $\Gamma \leq \mathcal{C}_{two}(N)$, then $A(\Gamma) \leq \mathrm{Mod}(N)$.

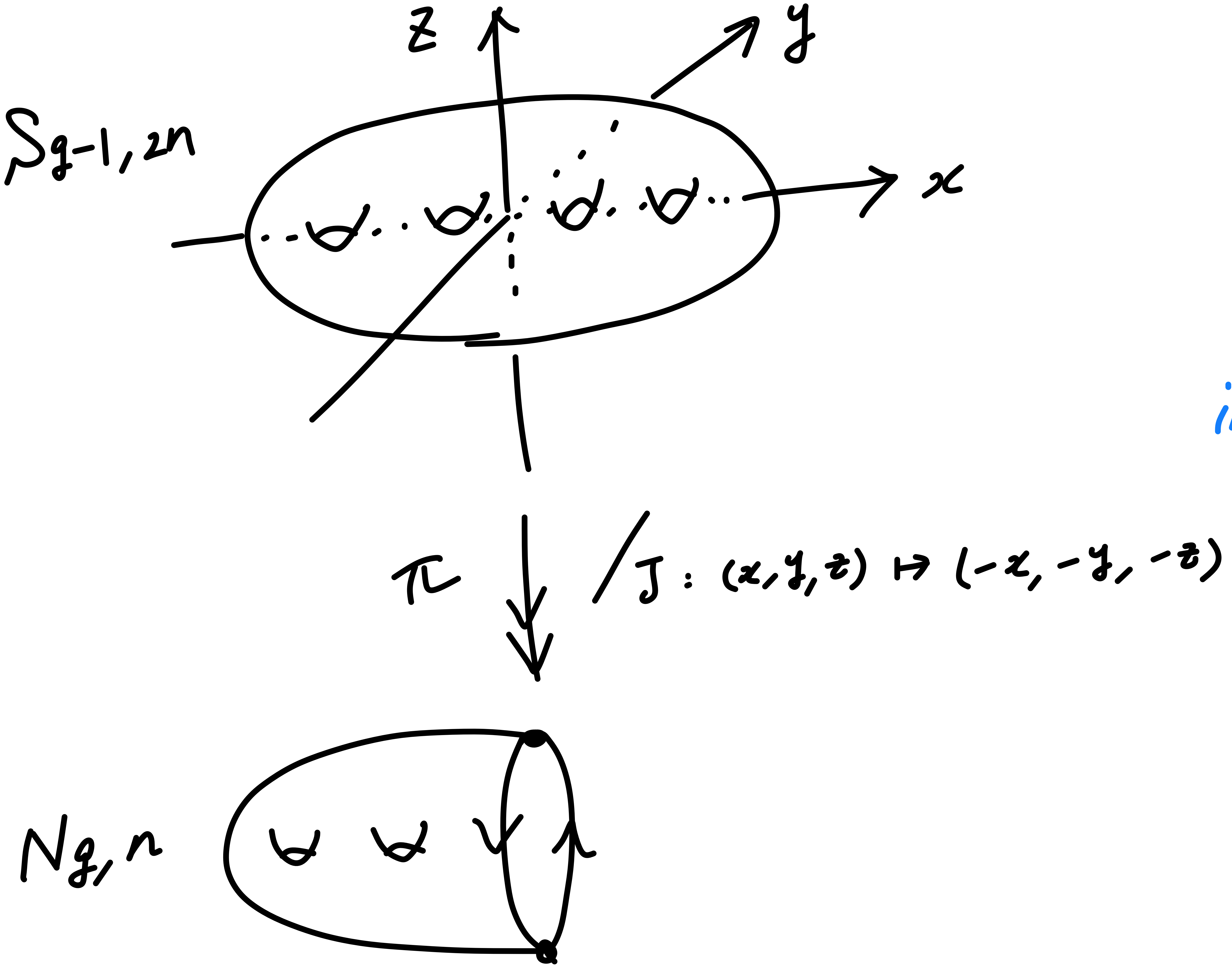
<u>Thm.</u> 2 (Katayama – K, 2021, arXiv)

$N = N_{1,6}, N_{3,3}, N_{5,0}$. $\exists \Gamma = \Gamma_0$ s.t. $A(\Gamma_0) \leq \mathrm{Mod}(N)$, but

$\Gamma_0 \not\leq \mathcal{C}_{two}(N)$.

$\Gamma_0 =$ 

§3. Idea of proof



induces $\rightsquigarrow$

consists of all lifts of Γ

$$\begin{array}{c} \tilde{\Gamma} \le \mathcal{C}(S) \\ \Big\downarrow p \\ \Gamma \le \mathcal{C}_{two}(N) \end{array}$$

Key idea 1:

For $p: \tilde{\Gamma} \to \Gamma$, $\varphi: A(\Gamma) \to A(\tilde{\Gamma})$ is a gp. inj. hom.

$$\underset{\text{gen.}}{u} \mapsto v \cdot J(v) \quad (v \in p^{-1}(u))$$

$$\tilde{\Gamma} \le C(S) \overset{\text{Koberda}}{\Longrightarrow} \exists \psi: A(\tilde{\Gamma}) \hookrightarrow \mathrm{Mod}(S) \text{ is a gp. inj. hom.}$$

$$\underset{\text{gen}}{v} \mapsto t_v^N$$

$$\therefore \quad A(\Gamma) \le \mathrm{Mod}(\underset{=}{S}).$$

double covering ori. surf.

Key idea 2:

Any $g \in \mathrm{Im}(\psi \circ \varphi)$ is commutative with J.

$$
\begin{array}{ccccc}
A(\Gamma) & \xrightarrow{\varphi} & A(\tilde{\Gamma}) & \xrightarrow{\psi} & \mathrm{Mod}(S) \\
\Downarrow & & \Downarrow & & \Downarrow \\
u^{gen} & \longmapsto & v \cdot J(v) & \longmapsto & t_v^N \cdot t_{J(v)}^N
\end{array}
$$

$$
\left(
\begin{aligned}
\because\ J(t_v^N \cdot t_{J(v)}^N)J^{-1} &= J t_v J^{-1} J t_v J^{-1} J \cdots J t_{J(v)} J^{-1} \\
&= t_{J(v)}^N \cdot t_v^N \\
&= t_v^N \cdot t_{J(v)}^N
\end{aligned}
\right)
$$

$\therefore\ A(\Gamma) \leq Z([J])$.

$\therefore \quad A(T) \leq \mathrm{Mod}(N)$

$\Bigg($ $\because$ Fact 1 : $\mathrm{Mod}(N) < \mathcal{Z}([J])$: index 2

Fact 2 : $A(T) \leq G \not\Rightarrow \forall H < G$: f. i. subgp., $A(T) \leq H$. $\Bigg)$

$\square$