On the stable (co)homology of the IA-automorphism groups of free groups

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Outline

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- The (co)homology of the IA-automorphism group IA_n
- The structure of the stable Albanese homology of IA_n

Introduction

The IA-automorphism group IA_n of F_n

$F_n = \langle x_1, \dots, x_n \rangle$: free group of rank n
$\mathrm{Aut}(F_n)$: the automorphism group of F_n

$F_n \twoheadrightarrow \mathbb{Z}^n$: the abelianization map
$\mathrm{IA}_n = \ker(\mathrm{Aut}(F_n) \twoheadrightarrow \mathrm{GL}(n, \mathbb{Z}))$: the **IA-automorphism group** of F_n

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z}) \to 1$$

Torelli groups of surfaces

$\Sigma_{g,1}$: a surface of genus g with one boundary component
$\mathcal{M}_{g,1} = \mathrm{Homeo}_+(\Sigma_{g,1})/\text{isotopy}$: the mapping class group of $\Sigma_{g,1}$

$\mathcal{M}_{g,1}$ acts on $H_1(\Sigma_{g,1}, \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2g}$
$\mathcal{I}_{g,1} = \ker(\mathcal{M}_{g,1} \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z}))$: the **Torelli group** of $\Sigma_{g,1}$

Then we have

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{I}_{g,1} & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \mathrm{Sp}(2g, \mathbb{Z}) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & \mathrm{IA}_{2g} & \longrightarrow & \mathrm{Aut}(F_{2g}) & \longrightarrow & \mathrm{GL}(2g, \mathbb{Z}) & \longrightarrow & 1.
\end{array}$$

The stable (co)homology of $\mathrm{Aut}(F_n)$ and $\mathrm{GL}(n, \mathbb{Z})$. I

$$F_n \hookrightarrow F_{n+1}, \quad x_i \mapsto x_i \quad (1 \le i \le n)$$

induces

$$\cdots \to H_*(\mathrm{Aut}(F_n), \mathbb{Q}) \to H_*(\mathrm{Aut}(F_{n+1}), \mathbb{Q}) \to \cdots$$

Hatcher–Vogtmann:

$H_*(\mathrm{Aut}(F_n), \mathbb{Q})$ **stabilizes**,

i.e., $H_*(\mathrm{Aut}(F_n), \mathbb{Q}) \xrightarrow{\cong} H_*(\mathrm{Aut}(F_{n+1}), \mathbb{Q})$ for $n \gg *$.

Galatius:

$$\varinjlim_n H_*(\mathrm{Aut}(F_n), \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0 \\ 0 & * \ge 1 \end{cases}$$

The stable (co)homology of $\mathrm{Aut}(F_n)$ and $\mathrm{GL}(n,\mathbb{Z})$. II

$$\mathrm{GL}(n,\mathbb{Z}) \hookrightarrow \mathrm{GL}(n+1,\mathbb{Z}), \quad A \mapsto A \oplus 1$$

induces

$$\cdots \leftarrow H^*(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q}) \leftarrow H^*(\mathrm{GL}(n+1,\mathbb{Z}),\mathbb{Q}) \leftarrow \cdots$$

Borel:

$H^*(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q})$ **stabilizes**,

$$\varprojlim_n H^*(\mathrm{GL}(n,\mathbb{Z}),\mathbb{Q}) \cong \bigwedge\nolimits_{\mathbb{Q}}(x_1,x_2,\dots), \quad \deg x_i = 4i+1$$

The (co)homology of IA_n

The first homology of IA_n. I

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}(F_n) \to \mathrm{GL}(n,\mathbb{Z}) \to 1$$

$H_*(\mathrm{IA}_n,\mathbb{Q})$: a $\mathrm{GL}(n,\mathbb{Z})$-representation

$H := H_1(F_n,\mathbb{Q}) \cong \mathbb{Q}^{\oplus n}$: the standard $\mathrm{GL}(n,\mathbb{Z})$-representation

Theorem (Formanek 1990, Cohen–Pakianathan, Farb, Kawazumi 2005)

The **Johnson homomorphism**

$$\tau : \mathrm{IA}_n \to \mathrm{Hom}(H, \bigwedge^2 H)$$

induces an isomorphism of $\mathrm{GL}(n,\mathbb{Z})$*-representations*

$$\tau : H_1(\mathrm{IA}_n,\mathbb{Q}) \xrightarrow{\cong} \mathrm{Hom}(H, \bigwedge^2 H).$$

The first homology of IA_n. II

$$H_1(\mathrm{IA}_n, \mathbb{Q}) \cong \mathrm{Hom}(H, \bigwedge^2 H) \cong H^* \otimes \bigwedge^2 H,$$

$$\dim_{\mathbb{Q}} H_1(\mathrm{IA}_n, \mathbb{Q}) = \frac{n^2(n-1)}{2}$$

$H_1(\mathrm{IA}_n, \mathbb{Q})$ **does not** stabilize as a vector space
but stabilizes as a $\mathrm{GL}(n, \mathbb{Z})$-representation.
(cf. **representation stable**)

The second homology of IA_n

Theorem (Bestvina–Bux–Margalit 2007)

For $n = 3$, $H_2(\mathrm{IA}_3, \mathbb{Q})$ is infinite dimensional.

For $n \geq 4$, it is not known whether $H_2(\mathrm{IA}_n, \mathbb{Q})$ is finite dimensional.

Theorem (Day–Putman 2017)

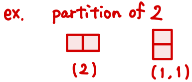
$H_2(\mathrm{IA}_n, \mathbb{Q})$ is finitely generated as a $\mathrm{GL}(n, \mathbb{Z})$-representation.

Algebraic $GL(n, \mathbb{Z})$-representations. I

A finite-dimensional $GL(n, \mathbb{Z})$-representation (ρ, V) is **algebraic** if the group homomorphism $\rho : GL(n, \mathbb{Z}) \to GL(V)$ is described by rational functions on entries of matrices of $GL(n, \mathbb{Z})$.

Algebraic $GL(n, \mathbb{Z})$**-representations** have the following properties

- **Completely reducible**;
  For any algebraic $GL(n, \mathbb{Z})$-rep. V, we have $V \cong \bigoplus_{\underline{\lambda}} V_{\underline{\lambda}}^{\oplus m_{\underline{\lambda}}}$
- $\{\text{irreps } V_{\underline{\lambda}}\} \cong \{\text{bipartitions } \underline{\lambda} = (\lambda^+, \lambda^-)\text{: pairs of partitions}\}$
  $V_{\underline{\lambda}}$ is realized in $H^{\otimes p} \otimes (H^*)^{\otimes q}$



Algebraic $\mathrm{GL}(n,\mathbb{Z})$-representations. II

For example,

$$H = H_1(F_n,\mathbb{Q}) = V_{1,0},$$
$$H_1(\mathrm{IA}_n,\mathbb{Q}) \cong \mathrm{Hom}(H, \bigwedge^2 H) \cong H^* \otimes \bigwedge^2 H$$
$$\cong V_{0,1} \otimes V_{1^2,0} \cong V_{1^2,1} \oplus V_{1,0}.$$

For $n \gg *$, $H_*(\mathrm{IA}_n,\mathbb{Q})$ is conjectured to be algebraic and moreover **representation-stable**,
i.e., $H_*(\mathrm{IA}_n,\mathbb{Q}) \cong \bigoplus_{\underline{\lambda}} V_{\underline{\lambda}}^{\oplus m_{\underline{\lambda}}}$, where $m_{\underline{\lambda}}$ is independent of n.

Albanese homology of IA_n

$U := H_1(\mathrm{IA}_n, \mathbb{Q}) \cong \mathrm{Hom}(H, \bigwedge^2 H)$: algebraic

The **Albanese homology** of IA_n is defined by

$$H_*^A(\mathrm{IA}_n, \mathbb{Q}) = \mathrm{im}(H_*(\mathrm{IA}_n, \mathbb{Q}) \xrightarrow{\tau_*} H_*(U, \mathbb{Q})).$$

We have

$$\tau_* : H_*(\mathrm{IA}_n, \mathbb{Q}) \twoheadrightarrow H_*^A(\mathrm{IA}_n, \mathbb{Q}) \hookrightarrow H_*(U, \mathbb{Q}) \cong \bigwedge^* U.$$

algebraic

The Albanese homology of IA_n is algebraic.

- $H_1^A(\mathrm{IA}_n, \mathbb{Q}) \cong H_1(\mathrm{IA}_n, \mathbb{Q})$.
- $H_2^A(\mathrm{IA}_n, \mathbb{Q})$ is determined by [Pettet 2005].
- $H_i^A(\mathrm{IA}_n, \mathbb{Q})$ is determined for $n \geq 3i$ by [K. 2024].

The Albanese cohomology of IA_n

The **Albanese cohomology** of IA_n is defined by

$$H^*_A(\mathrm{IA}_n, \mathbb{Q}) = \mathrm{im}(H^*(U, \mathbb{Q}) \xrightarrow{\tau^*} H^*(\mathrm{IA}_n, \mathbb{Q})).$$

We have

$$\tau^* : H^*(U, \mathbb{Q}) \cong \bigwedge^*(U^*) \twoheadrightarrow H^*_A(\mathrm{IA}_n, \mathbb{Q}) \hookrightarrow H^*(\mathrm{IA}_n, \mathbb{Q}).$$

The Albanese cohomology of IA_n is dual to the Albanese homology:

$$H^*_A(\mathrm{IA}_n, \mathbb{Q}) \cong H^A_*(\mathrm{IA}_n, \mathbb{Q})^*.$$

The stable rational cohomology of IA_n

Habiro–K. proposed a conjectural structure of $H^*(\mathrm{IA}_n, \mathbb{Q})$ for $n \gg *$.

Conjecture (Habiro–K. 2022)

We have

$$H^*(\mathrm{IA}_n, \mathbb{Q}) \cong H^*_A(\mathrm{IA}_n, \mathbb{Q}) \otimes H^*(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{GL}(n,\mathbb{Z})},$$
$$H^*(\mathrm{IA}_n, \mathbb{Q})^{\mathrm{GL}(n,\mathbb{Z})} \cong \mathbb{Q}[y_1, \cdots], \quad \deg y_i = 4i$$

for $n \gg *$.

We proved this conjecture under the assumption that $H^*(\mathrm{IA}_n, \mathbb{Q})$ is stably algebraic.

cf. [Lindell 2024]

The stable Albanese homology of IA_n

The stable Albanese homology of IA_n

The main theorem is the following.

Theorem (K. 2024)

We have an isomorphism of $\mathrm{GL}(n, \mathbb{Q})$*-representations*

$$F_i : H_i^A(\mathrm{IA}_n, \mathbb{Q}) \xrightarrow{\cong} W_i$$

for $n \geq 3i$.

The Albanese homology of IA_n is **representation-stable** in $n \geq 3i$.

Traceless part W_*. I

$U_* := \bigoplus_{i\geq 1} U_i, \quad U_i = \mathrm{Hom}(H, \bigwedge^{i+1} H)$
$S^*(U_*)$: the graded-symmetric algebra of U_*
$W_* := \widetilde{S}^*(U_*)$: the **traceless part** of $S^*(U_*)$

For example,

$$\begin{aligned}
S^*(U_*)_1 &= U_1 = W_1 \\
&\cong V_{1^2,1} \oplus V_{1,0}, \\
S^*(U_*)_2 &= U_2 \oplus (U_1 \wedge U_1) \\
&\cong V_{1^4,1^2} \oplus V_{21^2,2} \oplus V_{2^2,1^2} \oplus V_{1^3,1}^{\oplus 3} \oplus V_{21,1}^{\oplus 2} \oplus V_{1^2,0}^{\oplus 4}, \\
W_2 &= U_2 \oplus (U_1 \mathbin{\widetilde{\wedge}} U_1) \\
&\cong V_{1^4,1^2} \oplus V_{21^2,2} \oplus V_{2^2,1^2} \oplus V_{1^3,1}^{\oplus 2} \oplus V_{21,1} \oplus V_{1^2,0}^{\oplus 2}.
\end{aligned}$$

Traceless part W_*. II

The **traceless part** $T_{p,q}$ of $H^{p,q} = H^{\otimes p} \otimes (H^*)^{\otimes q}$ is defined by

$$T_{p,q} = \bigcap_{\substack{k \in \{1,\dots,p\} \\ l \in \{1,\dots,q\}}} \ker(c_{k,l} : H^{p,q} \to H^{p-1,q-1}),$$

where $c_{k,l}$ denotes the contraction map for the k-th component of $H^{\otimes p}$ and the l-th component of $(H^*)^{\otimes q}$.

The **traceless tensor product** $V_{\underline{\lambda}} \widetilde{\otimes} V_{\underline{\mu}}$ is defined by

$$V_{\underline{\lambda}} \widetilde{\otimes} V_{\underline{\mu}} = (V_{\underline{\lambda}} \otimes V_{\underline{\mu}}) \cap T_{|\lambda^+|+|\mu^+|,|\lambda^-|+|\mu^-|}$$

for $\underline{\lambda} = (\lambda^+, \lambda^-), \underline{\mu} = (\mu^+, \mu^-)$.

Traceless part W_*. III

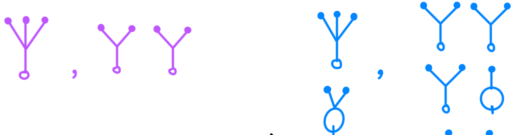
Graphical interpretation:

● : element of H
○ : element of H^*
s.t. traceless

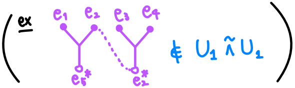
$\mathrm{Hom}(H, \Lambda^2 H)$

$$S^*(U_*)_1 = U_1 = W_1$$

$$S^*(U_*)_2 = U_2 \oplus (U_1 \wedge U_1), \quad W_2 = U_2 \oplus (U_1 \mathbin{\widetilde{\wedge}} U_1)$$



● : element of H
○ : element of H^*



The lower bound of the Albanese homology of IA_n

To detect W_*, we constructed **abelian cycles**.

ϕ: an i-tuple of mutually commutative elements of IA_n
$\Phi : \mathbb{Z}^i \to \mathrm{IA}_n$: the group homomorphism induced by ϕ

The abelian cycle corresponding to ϕ is the image of the fundamental class under

$$\Phi_* : H_i(\mathbb{Z}^i, \mathbb{Q})(\cong \mathbb{Q}) \to H_i(\mathrm{IA}_n, \mathbb{Q}).$$

Theorem (K. 2022)

We have a morphism of $\mathrm{GL}(n, \mathbb{Q})$-representations

$$F_i : H_i^A(\mathrm{IA}_n, \mathbb{Q}) \to S^*(U_*)_i$$

such that $F_i(H_i^A(\mathrm{IA}_n, \mathbb{Q})) \supset W_i$ for $n \geq 3i$.

The upper bound of the Albanese (co)homology of IA_n

The Johnson homomorphism $\tau : \mathrm{IA}_n \to U$ induces a surjective $\mathrm{GL}(n, \mathbb{Q})$-equivariant morphism of graded algebras

$$\tau^* : H^*(U, \mathbb{Q})/\langle R_2 \rangle \twoheadrightarrow H_A^*(\mathrm{IA}_n, \mathbb{Q}),$$

where $R_2 := \ker(\tau^* : H^2(U, \mathbb{Q}) \to H_A^2(\mathrm{IA}_n, \mathbb{Q}))$.

Theorem (Pettet 2005)

For $n \geq 3$, we have

$$R_2 \cong V_{1,21} \oplus V_{0,1^2}.$$

Graphical interpretations

We need to prove that $(W_i)^* \twoheadrightarrow (H^*(U,\mathbb{Q})/\langle R_2\rangle)_i$.
$\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}$: the wheeled PROP constructed in [Kawazumi–Vespa]

Proposition (K.)

For $n \geq \max(3i, p+q)$, we have

$$[(W_i)^* \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})} \cong \mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}(p,q)_i.$$

Proposition (Lindell 2024)

We have morphisms of graded vector spaces

$$\mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}(p,q) \twoheadrightarrow [(H^*(U,\mathbb{Q})/\langle R_2\rangle) \otimes H^{p,q}]^{\mathrm{GL}(n,\mathbb{Z})}.$$

Therefore, we have for $n \geq 3i$,

$$[(W_i)^* \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})} \cong \mathcal{C}_{\mathcal{P}_0^{\circlearrowright}}(2i,i)_i \twoheadrightarrow [(H^*(U,\mathbb{Q})/\langle R_2\rangle)_i \otimes H^{2i,i}]^{\mathrm{GL}(n,\mathbb{Z})}.$$