Asymptotic behavior of compressible non-isothermal nematic liquid crystal flow in infinite layer

Yuka Teramoto (Ehime University)

We consider the flow of compressible nematic liquid crystal in an infinite layer, which is governed by the following system based on simplified Ericksen-Leslie system:

$$\begin{aligned}
\partial_t\rho + \mathrm{div}\,(\rho\boldsymbol{u}) &= 0, && (1)\\
\partial_t(\rho\boldsymbol{u}) + \mathrm{div}\,(\rho\boldsymbol{u}\otimes\boldsymbol{u}) &= \mathrm{div}\,\mathbb{S}_C, && (2)\\
\partial_t(\rho\theta) + \mathrm{div}\,(\rho\theta\boldsymbol{u}) + \mathrm{div}\,\boldsymbol{q} &= \mathbb{S}_C : \nabla\boldsymbol{u}, && (3)\\
\partial_t\boldsymbol{d} + \boldsymbol{u}\cdot\nabla\boldsymbol{d} &= \tau^*(\Delta\boldsymbol{d} + |\nabla\boldsymbol{d}|^2\boldsymbol{d}). && (4)
\end{aligned}$$

Here the unknown variables ρ, $\boldsymbol{u}$, θ denote the density, velocity and temperature of fluid, respectively, $\boldsymbol{d}$ the orientation field for averaged macroscopic molecular directions of nematic liquid crystal, and hence, $|\boldsymbol{d}| = 1$. $\boldsymbol{q}$ is the heat flux given by $\boldsymbol{q} = -\kappa^*\nabla\theta$. Moreover, $\mathbb{S}_C = \mathbb{S}_N - \eta^*(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d} - \frac{1}{2}|\nabla\boldsymbol{d}|^2\mathbb{I}) - P\mathbb{I}$, where $P = P(\rho,\theta)$ is pressure which satisfies $\partial_\rho P \geq 0$, $\partial_\theta P \geq 0$, and $\mathbb{S}_N = \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top) + \mu'(\mathrm{div}\,\boldsymbol{u})\mathbb{I}$. η^*, μ, μ', κ^* and τ^* describe some positive constants. $\mathbb{I}$ is the identity matrix. We set $(\boldsymbol{u}\otimes\boldsymbol{u})_{ij} = u^i u^j$, $(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})_{ij} = \partial_{x_i}\boldsymbol{d}\cdot\partial_{x_j}\boldsymbol{d}$ and $\mathbb{S}_C : \nabla\boldsymbol{u} = \sum_{i,j}(\mathbb{S}_C)_{ij}\partial_{x_i}u^j$. (1)–(4) is considered in $\Omega_h = \{x = (x', x_3);\, x' = (x_1, x_2),\, 0 < x_3 < h\}$. The boundary condition is

$$\boldsymbol{u}|_{x_3=0,h} = \boldsymbol{0},\ \theta|_{x_3=0} = \theta_0^*,\ \theta|_{x_3=h} = \theta_1^*,\ \frac{\partial\boldsymbol{d}}{\partial x_3}|_{x_3=0,h} = \boldsymbol{0}, \qquad (5)$$

for given positive constants θ_0^*, θ_1^*. We consider density ρ and temperature θ around ρ^* and θ_0^*, respectively, where ρ^* is given positive constant.

The Ericksen-Leslie system is produced by Ericksen [1, 2, 3] and Leslie [4, 5]. The isothermal simplified model is first proposed by Lin [6] and many works have done for this isothermal model. On the other hand, there are few results for non-isothermal model of nematic liquid crystals for global classical solutions.

The problem (1)–(5) has a stationary solution $u_s = {}^\top(\rho_s, \boldsymbol{u}_s, \theta_s, \boldsymbol{d}_s) = {}^\top(\rho_s, \boldsymbol{0}, \theta_s, \boldsymbol{d}^*)$. Here ρ_s and θ_s are smooth function of x_3 satisfying $|\rho_s - \rho^*| \ll 1$ and $|\theta_s - \theta_0^*| \ll 1$. The perturbation equation from this motionless state u_s is written in the following non-dimensionalized form:

$$\begin{aligned}
\partial_t\phi + \mathrm{div}\,((\rho_s - 1)\boldsymbol{u}) + \mathrm{div}\,\boldsymbol{u} &= f, && (6)\\
\partial_t\boldsymbol{u} - \nu\Delta\boldsymbol{u} - (\nu + \nu')\nabla\mathrm{div}\,\boldsymbol{u} + \nabla\theta + \nabla\phi &= \boldsymbol{g}, && (7)\\
\partial_t\theta - \kappa\Delta\theta + \beta(P(\rho_s,\theta_s) - 1)\mathrm{div}\,\boldsymbol{u} + \beta\mathrm{div}\,\boldsymbol{u} &= h, && (8)\\
\partial_t\boldsymbol{d} - \tau\Delta\boldsymbol{d} &= \boldsymbol{k}, && (9)\\
\boldsymbol{u}|_{x_3=0,1} = \theta|_{x_3=0,1} = 0,\ \frac{\partial\boldsymbol{d}}{\partial x_3}|_{x_3=0,1} &= \boldsymbol{0}, && (10)\\
u|_{t=0} &= u_0, && (11)
\end{aligned}$$

where $u_0 = {}^\top(\phi_0, \boldsymbol{u}_0, \theta_0, \boldsymbol{d}_0)$ and f, $\boldsymbol{g}$, h, $\boldsymbol{k}$ are nonlinear terms.

Theorem 1. *There exist positive constants δ_1, δ_2, δ_3 and ϵ_0 such that if $u_0 \in (H^3(\Omega))^3 \times H^4(\Omega)$ satisfies some compatibility conditions and if $\|u_0\|_{H^3}$, $\|\boldsymbol{d}_0\|_{H^4} \le \epsilon_0$, $\|\rho_s - 1\|^2_{H^6} \le \delta_1$, $\|\rho_s\theta_s - 1\|^2_{H^6} \le \delta_2$ and $\|\theta_s - 1\|^2_{H^6} \le \delta_3$, then (6)–(11) has a unique solution $u(t)$ and it satisfies the estimates*

$$\|u(t)\|_{(H^3(\Omega))^3\times H^4(\Omega)} \le C\|u_0\|_{(H^3(\Omega))^3\times H^4(\Omega)}.$$

We also consider the asymptotic behavior for this model.

Theorem 2. *In addition to the assumption of Theorem 1, we assume that $u_0 \in (L^1(\Omega))^4$, then the solution $u(t)$ satisfies the following estimates:*

(i)

$$\|\partial^l_{x_3}\partial^{l'}_{x'}u(t)\|_{L^2(\Omega)} \le C(1+t)^{-\frac{1}{2}-\frac{|l'|}{2}}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)},$$

for $0 \le l$, $|l'| \le 1$. Furthermore,

$$\|\partial^l_{x_3}\partial^{l'}_{x'}(\boldsymbol{u},\theta)(t)\|_{L^2(\Omega)} \le C(1+t)^{-1}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)},$$

$$\|\partial^l_{x_3}\partial^{l'}_{x'}\boldsymbol{d}(t)\|_{L^2(\Omega)} \le C(1+t)^{-\frac{1}{2}-\frac{l+|l'|}{2}}\|u_0\|_{H^3(\Omega)\cap L^1(\Omega)}$$

for $0 \le l$, $|l'| \le 1$, $l + |l'| \le 1$.

(ii)

$$\|u(t) - \tilde{\sigma}_0(t)\|_{L^2(\Omega)} = o(t^{-\frac{1}{2}}) \quad \text{as } t \to \infty,$$

where

$$\tilde{\sigma}_0(t) = {}^\top(\tilde{\phi}_{low}(t), \mathbf{0}, 0, \tilde{\boldsymbol{d}}_{low}(t)),$$

$$\tilde{\phi}_{low}(t) = \alpha_0 G_0(x',t),\ \ \tilde{\boldsymbol{d}}_{low} = {}^\top(\beta_1 G_1(x',t), \beta_2 G_2(x',t), \beta_3 G_3(x',t)),$$

$$\alpha_0 = \left(\int_\Omega \phi_0(y)\,dy\right)\frac{1}{\partial_\rho P(\rho_s,\theta_s)},\ \ \beta_j = \int_\Omega d^j_0(y)\,dy + \int_0^\infty\int_\Omega k^j(\boldsymbol{u},\boldsymbol{d})(y,s)\,dyds,$$

$$G_j(t) = (4\pi\kappa_j t)^{-1}\mathrm{e}^{-\frac{|x'|^2}{4\kappa_j t}} \quad (j = 0,1,2,3,\ \kappa_j > 0).$$

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