Keenness for bridge splittings of links

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第8回岡潔女性数学者セミナー
2025年 9月 10日

knot theory

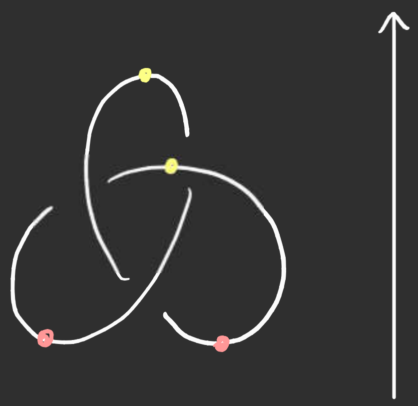
◎ $K = \mathrm{Im}(S^1 \hookrightarrow \underset{\mathbb{R}^3 \cup \{\infty\}}{\underset{\shortparallel}{S^3}})$: knot

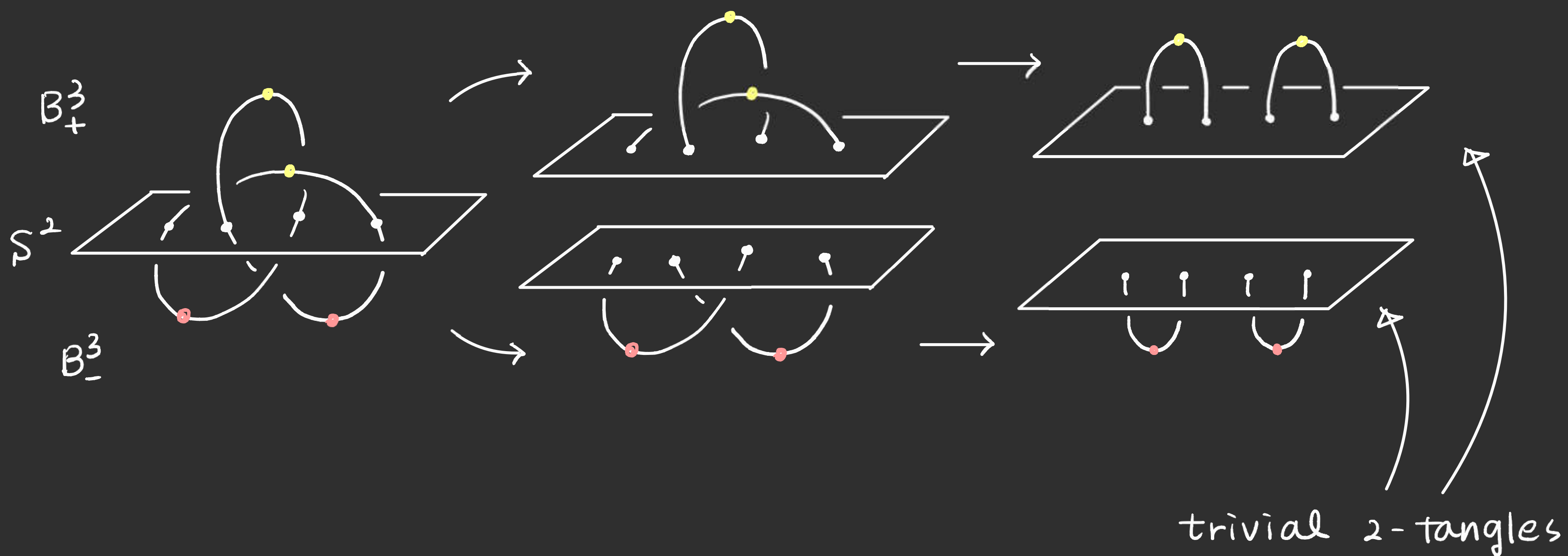
or

$\mathrm{Im}(S^1 \sqcup \cdots \sqcup S^1 \hookrightarrow S^3)$: link

◎ $K \cong K' \overset{\text{def}}{\iff} \exists h : S^3 \longrightarrow S^3$
(o-p) homeomorphism
s.t. $h(K) = K'$.

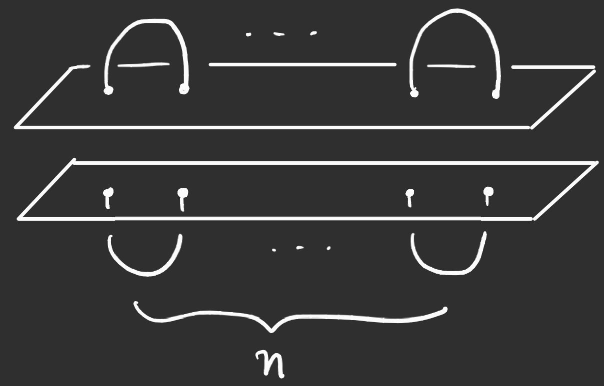
Bridge splittings of links





- $(S^3, K) = (B^3_+, T_+) \cup_{(S^2, P)} (B^3_-, T_-)$

  : n-bridge splitting of K.



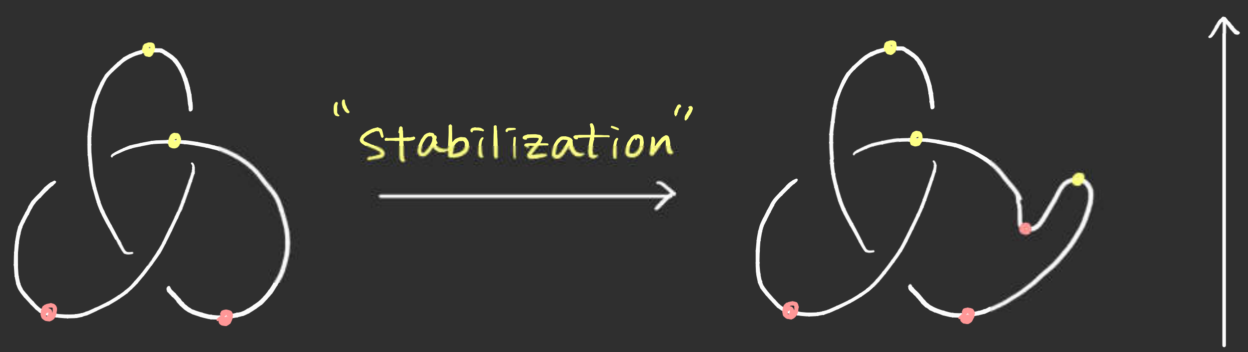
$\overset{\text{def}}{\Longleftrightarrow}$
- $(B^3_\pm, T_\pm)$: trivial n-tangles
- $(B^3_+, T_+) \cup (B^3_-, T_-) = (S^3, K)$
- $(B^3_+, T_+) \cap (B^3_-, T_-) = (\partial B^3_\pm, \partial T_\pm) = (S^2, P)$ ← 2n points

  ↑ n-bridge sphere

<u>Fact</u> (1) Any link admits an n-bridge splitting for some n.

(2) K admits an n-bridge splitting

$\Rightarrow$ K admits an (n+1)-bridge splitting.



Problems

- Decide $b(K) := \min\{n \mid K \text{ admits an } n\text{-bridge splitting}\}$.

  (– decided for "torus links", "Montesinos links", ⋯
  [Schubert '54] [Boileau-Zieschang '85]

- Classify links which admits an n-bridge splitting.

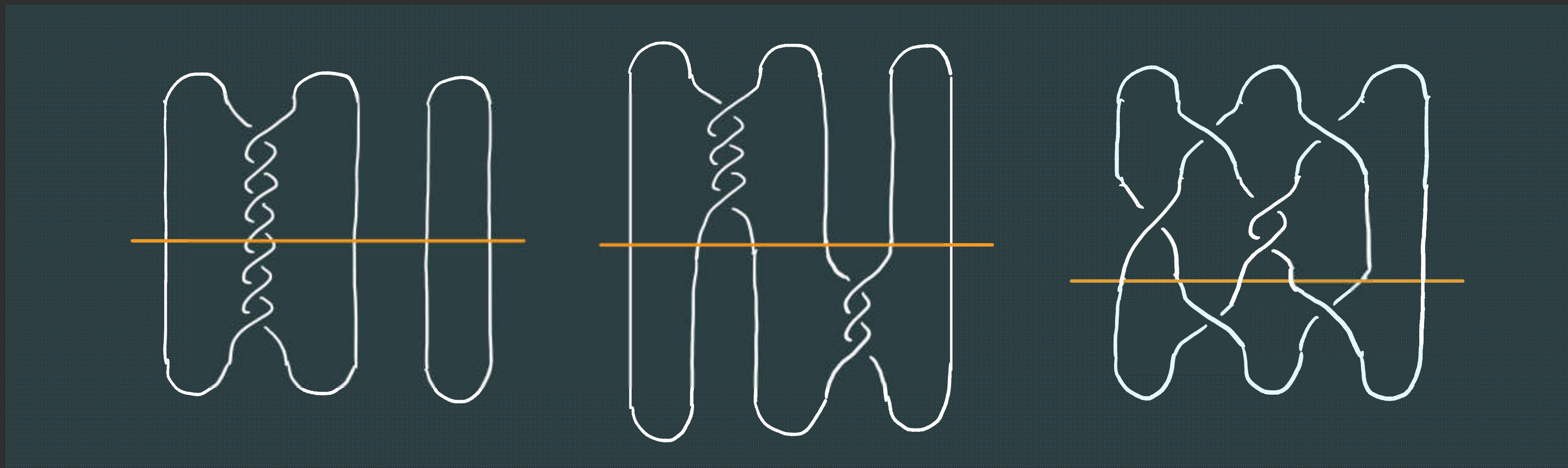
  – $b(K) = 1 \iff K \cong \bigcirc$.
  – 2-bridge links are classified by [Schubert '56]
  – 3-bridge "arborescent" links ⋯ [J. '11]

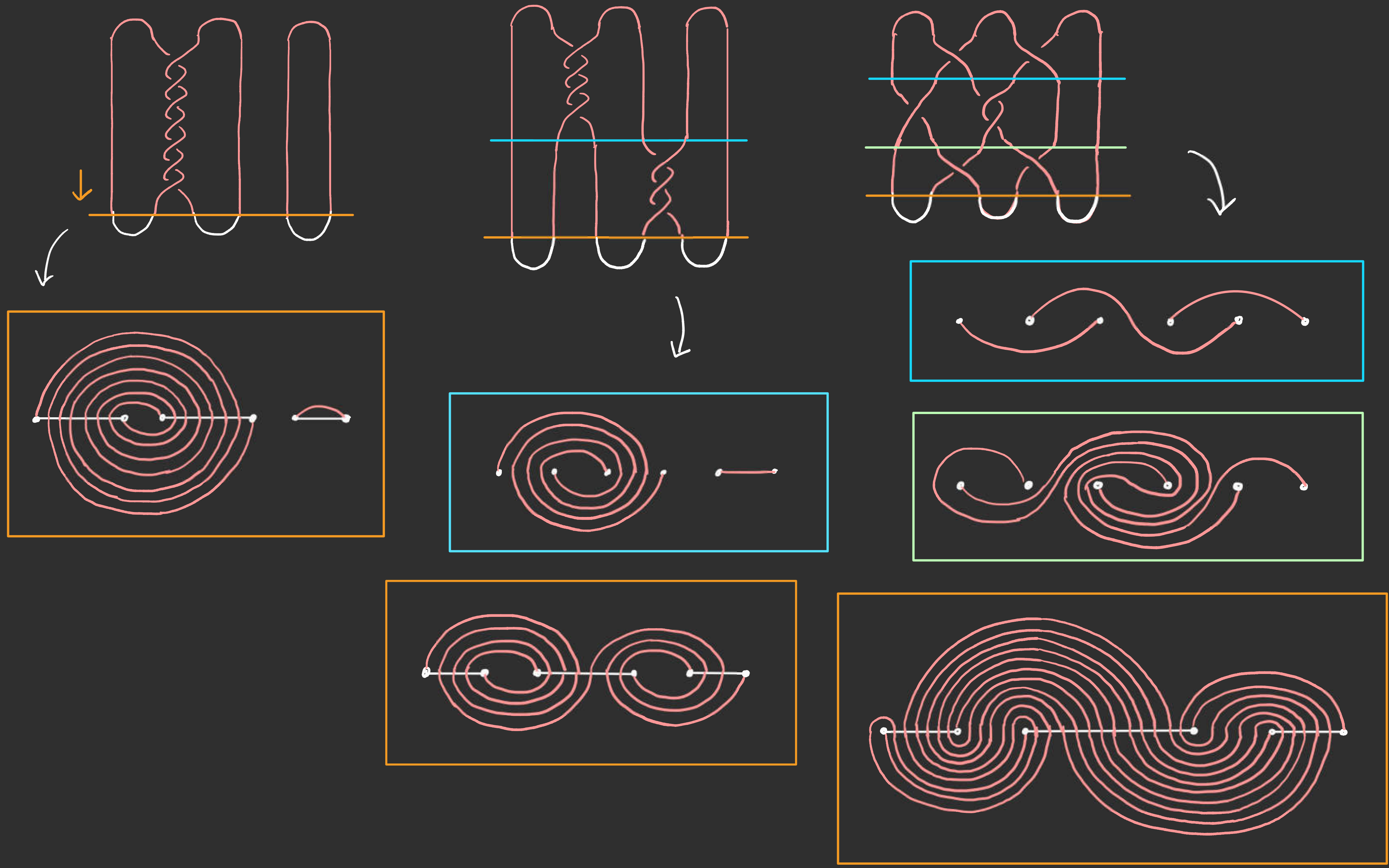
- Classify minimal bridge splittings of a link.

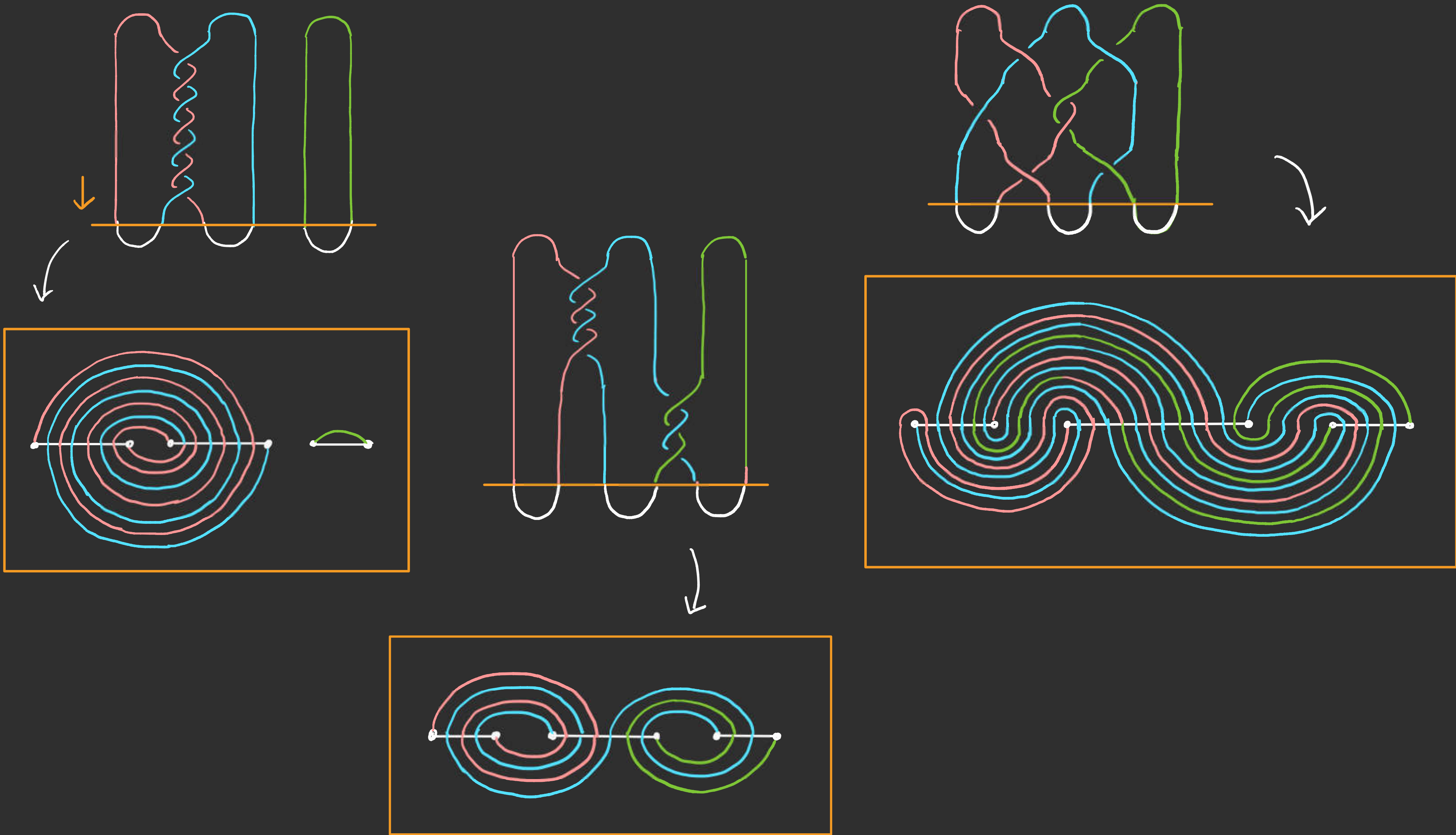
  – unique for $\bigcirc$, 2-bridge links, torus links, ⋯
  – not unique for some links
  [Birman '76],
  [Montesinos '76],
  [J. '10, '13]

"Complexity" of n-bridge splittings

Which splitting looks more / most complicated ?







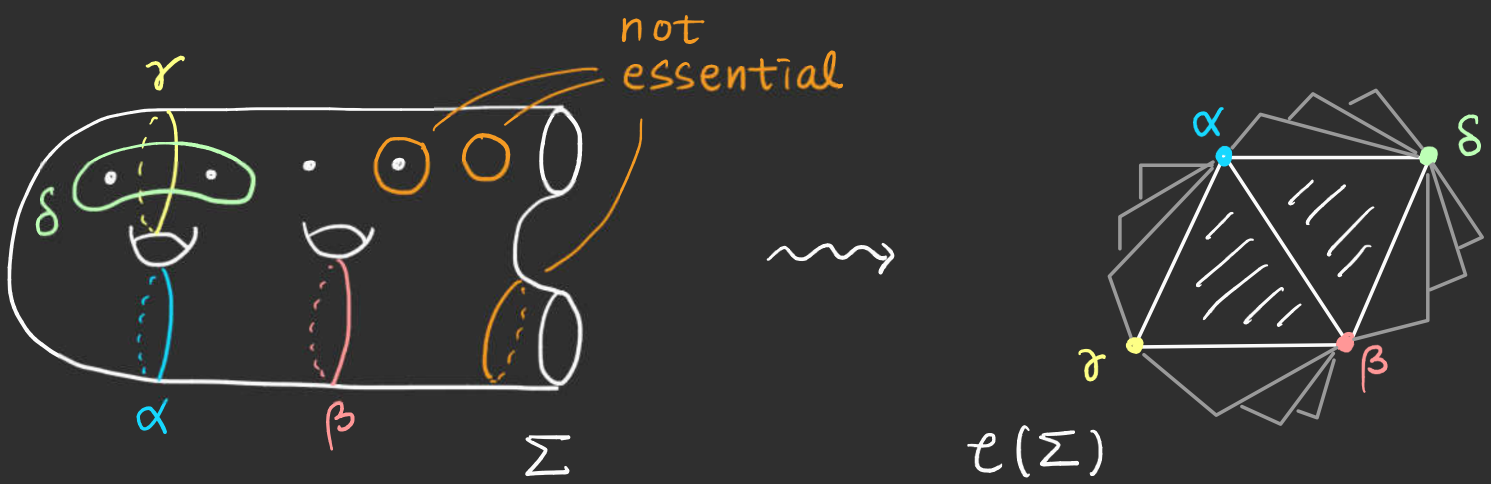
Curve complex

Σ : orientable surface of genus g
with c boundary components & p punctures

(Assume : $3g+c+p>4$)

The curve complex $\mathcal{C}(\Sigma)$ of Σ is the simplicial complex s.t.

- 0-simplex $\longleftrightarrow$ (isotopy class of) an essential s.c.c. on Σ
- n-simplex $\longleftrightarrow$ mutually disjoint $(n+1)$ s.c.c.'s on Σ $(n\geq 1)$

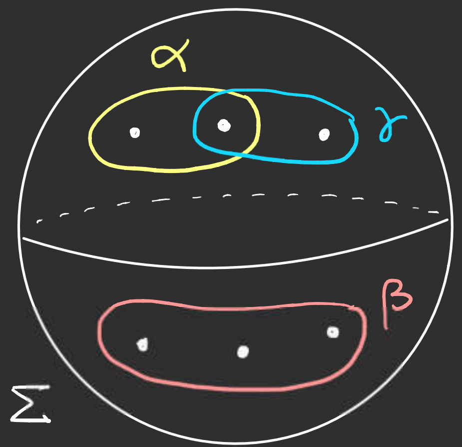


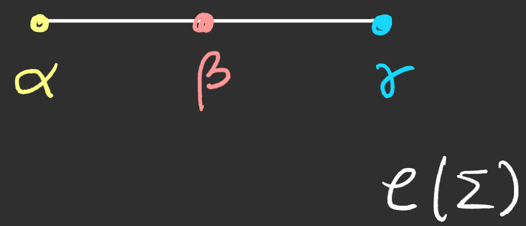
$\mathcal{C}^0(\Sigma)$: 0-skeleton (i.e., vertex set) of $\mathcal{C}(\Sigma)$

Define $d_\Sigma : \mathcal{C}^0(\Sigma) \times \mathcal{C}^0(\Sigma) \longrightarrow \mathbb{Z}_{\geq 0}$ by

$d_\Sigma(\alpha, \beta) :=$ minimal number of edges in any edge path in $\mathcal{C}(\Sigma)$ connecting α & β.

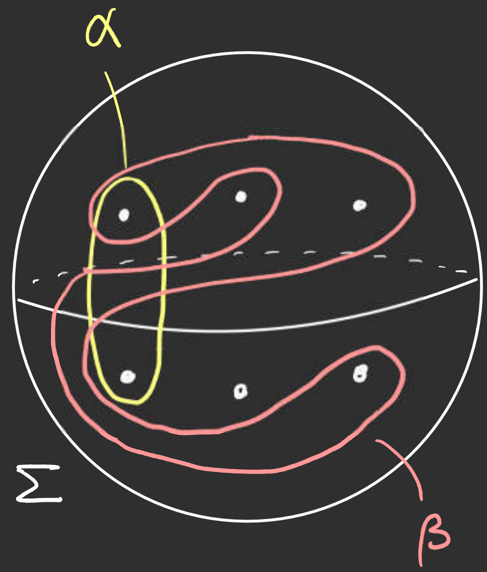
ex)





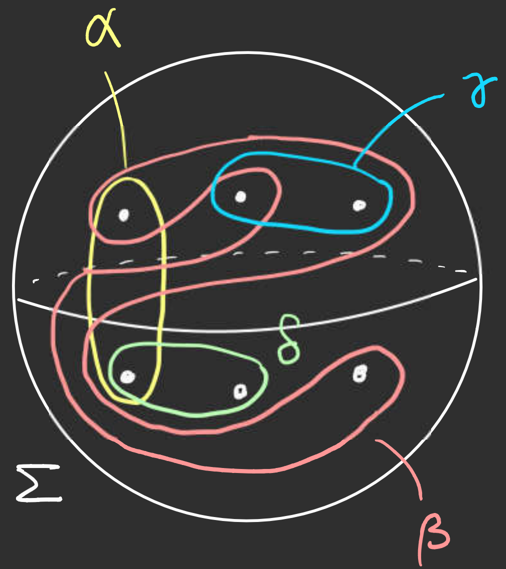
$$d_\Sigma(\alpha, \alpha) = 0.$$
$$d_\Sigma(\alpha, \beta) = 1.$$
$$d_\Sigma(\alpha, \gamma) = 2.$$

ex)

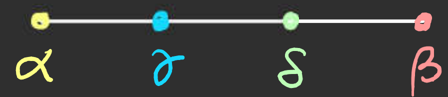


$d_\Sigma(\alpha, \beta) = ?$

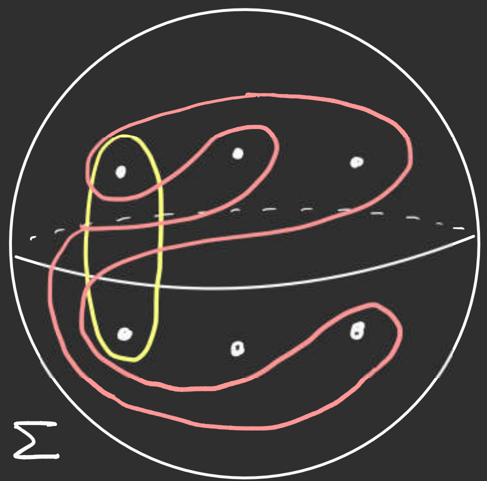
ex)



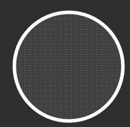
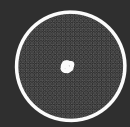
$d_\Sigma(\alpha, \beta) = ?$



$\rightsquigarrow$ $d_\Sigma(\alpha, \beta) \leq 3 \quad \cdots$ ①



Every component of $\Sigma \setminus (\alpha \cup \beta)$ is homeomorphic to

 or  .

$\rightsquigarrow$ $\nexists$ essential s.c.c. disjoint from $\alpha \cup \beta$.

$\rightsquigarrow$ $d_\Sigma(\alpha, \beta) > 2 \quad \cdots$ ②

$\therefore$ $d_\Sigma(\alpha, \beta) = 3$ by ① & ②.

Hempel distance of n-bridge splitting

$(S^3, K) = (B^3_+, T_+) \cup_{(S^2, P)} (B^3_-, T_-)$: n-bridge splitting of K

$\mathcal{C}(S^2 \setminus P)$: the curve complex of $S^2 \setminus P$ (2n-punctured sphere)

$\cup$

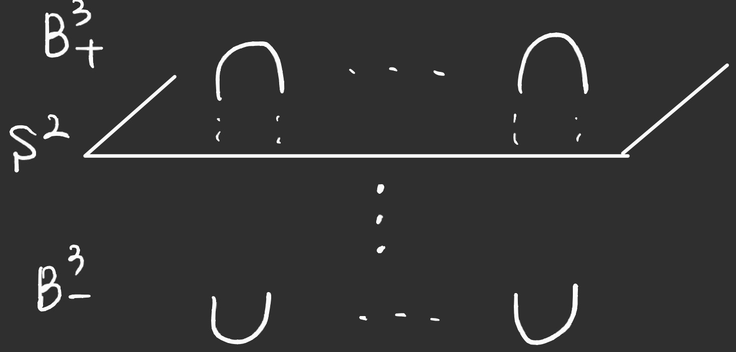
$\mathcal{D}(B^3_\varepsilon \setminus T_\varepsilon)$: the disk complex of $B^3_\varepsilon \setminus T_\varepsilon$ ($\varepsilon = +$ or $-$)

$\left(\alpha \in \mathcal{D}^0(B^3_\varepsilon \setminus T_\varepsilon) \iff \alpha \text{ bounds a disk in } B^3_\varepsilon \setminus T_\varepsilon \right)$

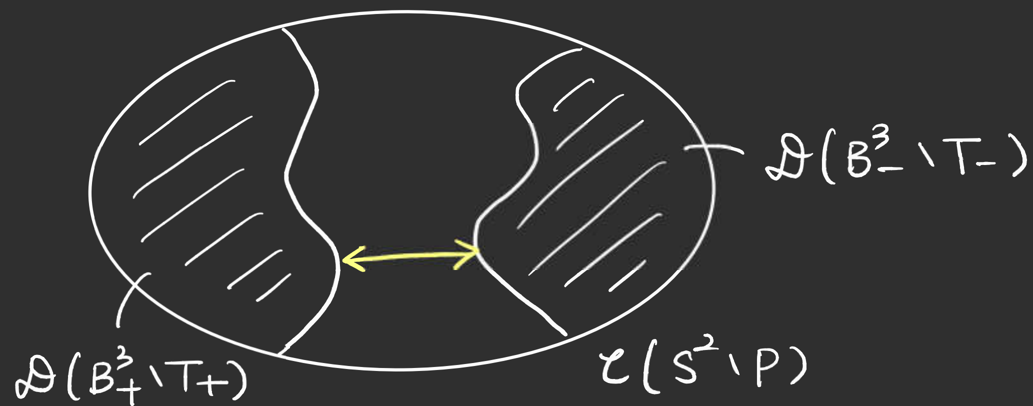
$\downarrow$

$d((B^3_+, T_+) \cup_{(S^2, P)} (B^3_-, T_-)) = d_{S^2 \setminus P}(\mathcal{D}(B^3_+ \setminus T_+), \mathcal{D}(B^3_-, T_-)) \in \mathbb{Z}_{\geq 0}$

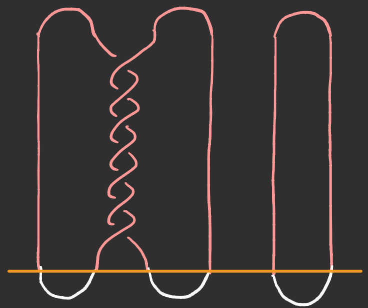
: the Hempel distance of $(B^3_+, T_+) \cup_{(S^2, P)} (B^3_-, T_-)$

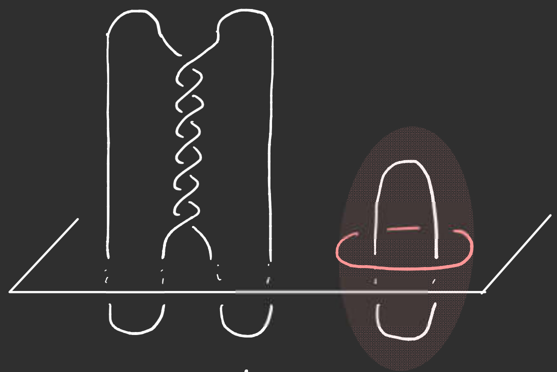


$\rightsquigarrow$

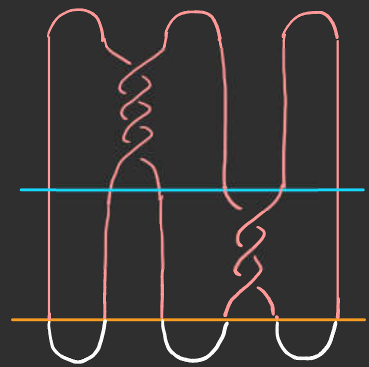


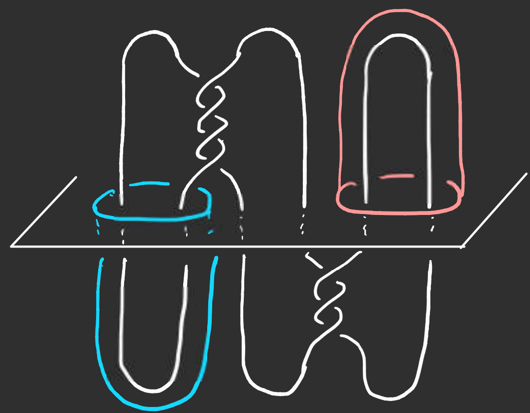
ex)





d = 0

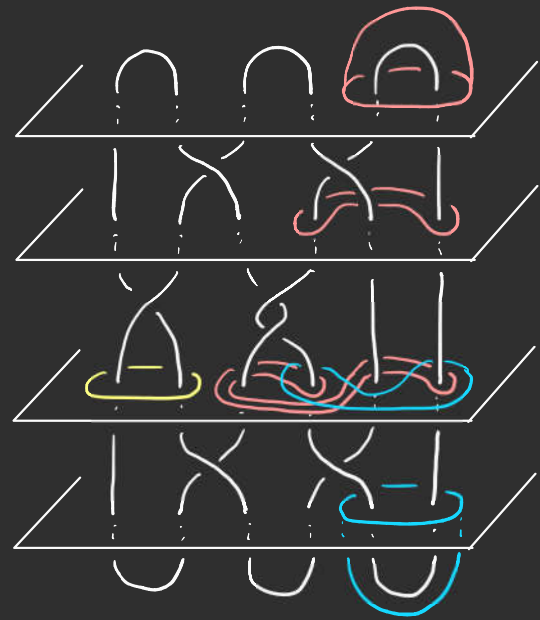




d ≤ 1

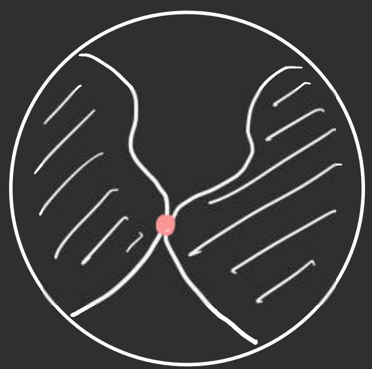
(d=1 in fact)

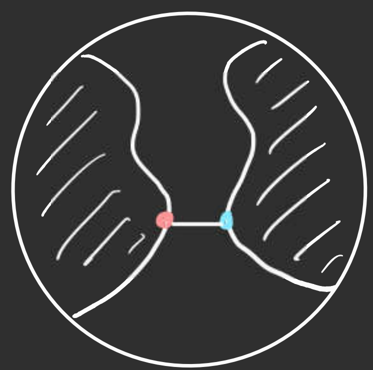
*



d ≤ 2

(d=2 in fact)



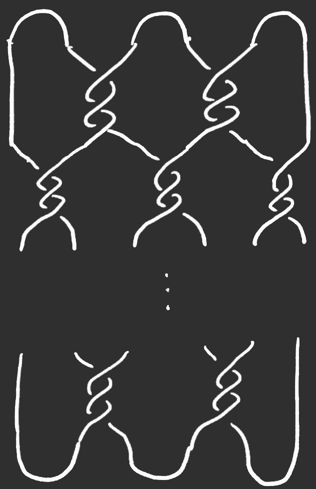


Rmk It is difficult to decide the Hempel distance for given bridge splittings in general.
($\because$ $\mathcal{C}(\Sigma)$: complicated , $\mathcal{D}$: infinite diameter)

Facts $\exists$ bridge splittings with high Hempel distance.

i.e., $\forall m \in \mathbb{Z}_{\geq 0}$, $\exists$ bridge splitting whose distance

- $> m$ [Saito '04], [Campisi-Rathbun '12]
  [Blair-Tomova-Yoshizawa '13]
  [Ichihara-Saito '13]
- $= m$ [Johnson-Moriah '16]
  [Ido-J.-Kobayashi ('15), '25]



1

2

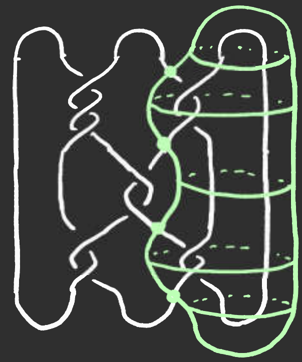
⋮

$2m-1$

$d = m$ [Johnson-Moriah '16]

Facts

- Hempel distance is bounded above in terms of χ(essential surface).
  [Bachman–Schleimer '05], [J. '14]



$\chi = -2$

<u>Cor.</u> $d(\text{some b.s. of } (S^3, K)) \geq n$ $\quad \geq 3$
$\Rightarrow \nexists$ essential surface with $-\chi < n-2$
(i.e., $\chi > 2-n$) $\quad \chi \geq 0$

- Hempel distance is bounded above in terms of χ(alternative bridge sphere). [Tomova '07], [Ido '15]

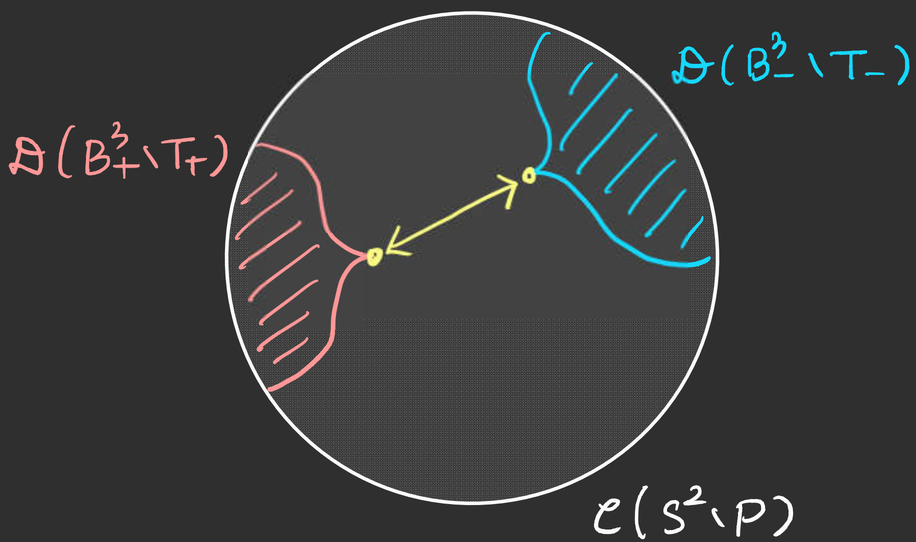
<u>Cor.</u> K : n-bridge knot & $d(\underbrace{n\text{-b. sp.}}) > 2n$
$\Rightarrow$ $\sim\sim$ is the unique n-bridge splitting.

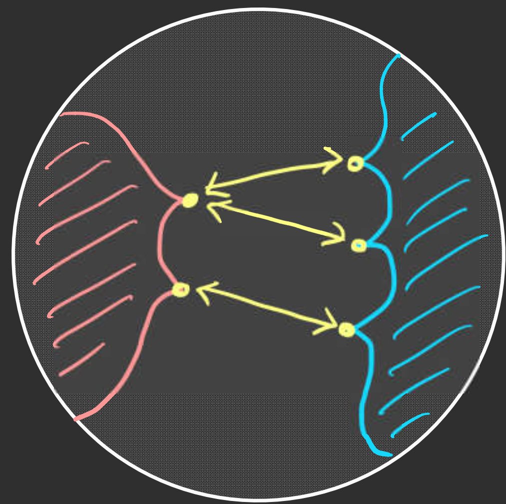
Keen bridge splittings

◎ $(B^3_+, T_+) \cup_{(S^2,P)} (B^3_-, T_-)$ is

keen $\overset{\text{def}}{\iff}$ $d((B^3_+, T_+) \cup_{(S^2,P)} (B^3_-, T_-))$ is realized by a unique pair of elements in $\mathcal{D}(B^3_+ \setminus T_+)$ & $\mathcal{D}(B^3_- \setminus T_-)$.

weakly keen $\overset{\text{def}}{\iff}$ $d((B^3_+, T_+) \cup_{(S^2,P)} (B^3_-, T_-))$ is realized by only finitely many pairs of elements in $\mathcal{D}(B^3_+ \setminus T_+)$ & $\mathcal{D}(B^3_- \setminus T_-)$.





Thm 1 [Ido-J.-Kobayashi '25]

(1) $\nexists$ keen 3-bridge splitting with Hempel distance 1.

(2) $\forall n \geq 3$, $\forall m \geq 1$ with $(n, m) \neq (3, 1)$,

$\exists$ keen n-bridge splittings with Hempel distance m.

Thm 2 [Ido-J.-Kobayashi]

$\forall n \geq 4$, $\forall m \geq 2$,

$\exists$ weakly keen, not keen

n-bridge splittings with Hempel distance m.

Cor $\exists$ bridge splittings with finite Goeritz group.

$$\mathcal{G} = MCG_+(S^3, K, B^3_\pm)$$

* $d \leq 1 \Rightarrow |\mathcal{G}| = \infty$.

$d \geq 3$ & (weakly) keen $\Rightarrow |\mathcal{G}| < \infty$.

(cf. [Iguch-Koda '20])

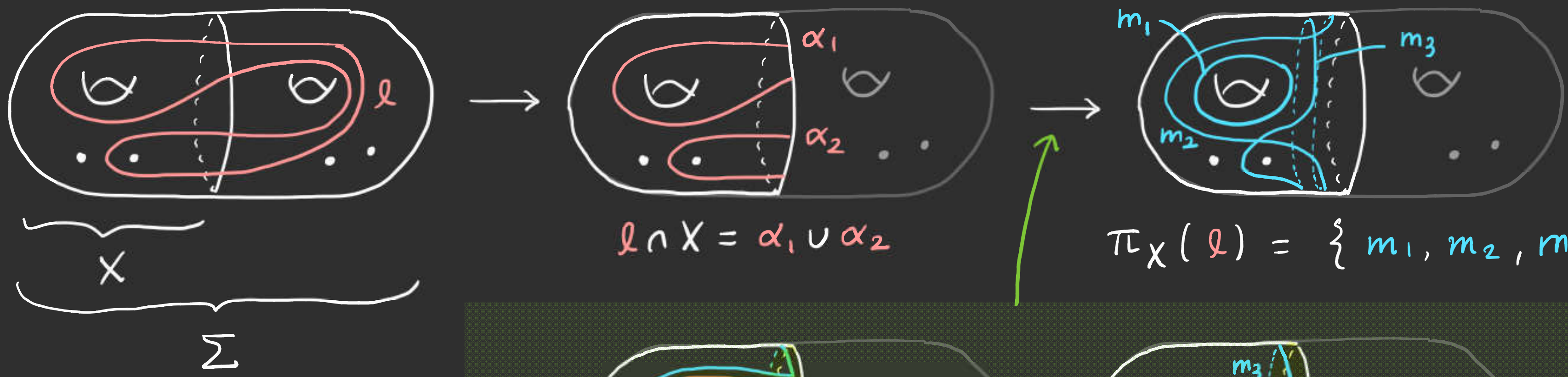
Idea of Proof for Thm 2

Key tool : "subsurface projection" introduced by Masur-Minsky.

- X : essential non-simple subsurface of Σ

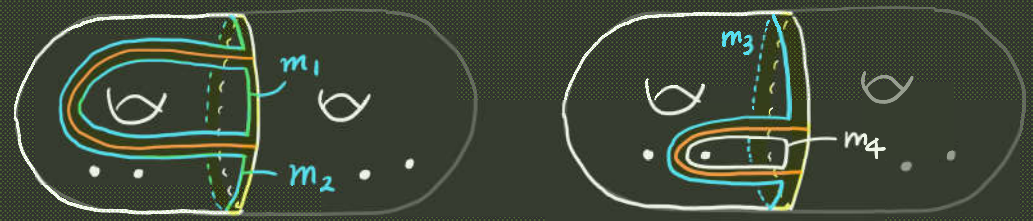
  The subsurface projection $\pi_X : \mathcal{C}(\Sigma)^{(0)} \longrightarrow \mathcal{P}(\mathcal{C}(X)^{(0)})$
  is defined as follows :

  ↑ the power set



$\ell \cap X = \alpha_1 \cup \alpha_2$

$\pi_X(\ell) = \{ m_1, m_2, m_3 \}$



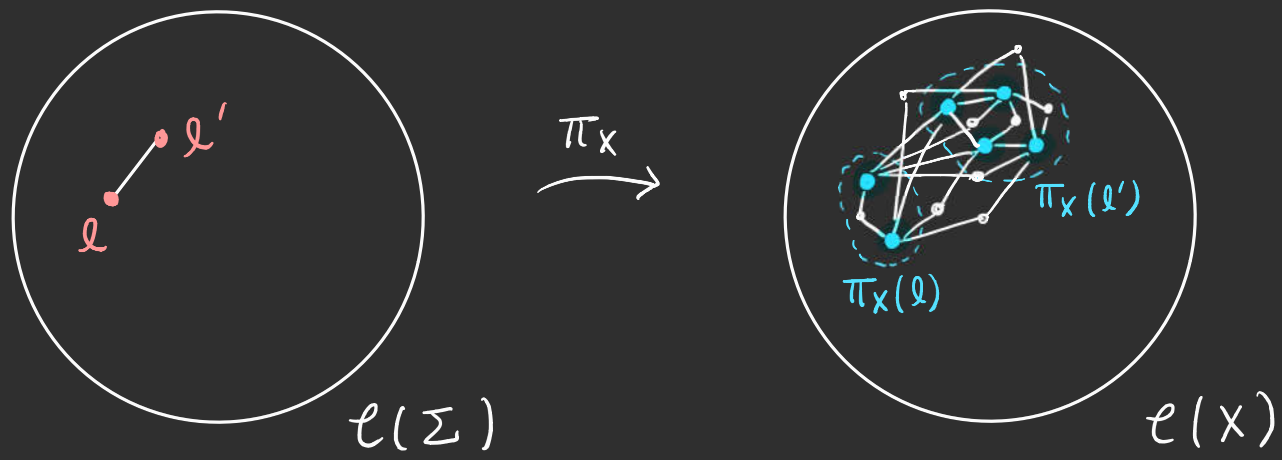
$\partial N_X(\alpha_1 \cup \partial X) = m_1 \cup m_2 \cup \partial X$ $\quad$ $\partial N_X(\alpha_2 \cup \partial X) = m_3 \cup m_4 \cup \partial X$

$\pi_X(\ell)$ = the union of
the set of the isotopy classes of the components
of $\partial N_X(\alpha \cup \partial X)$ which are essential in X
for every component α of $\ell \cap X$.

Lemma 1 (Masur-Minsky '00)

$$d_{\Sigma}(\ell, \ell') \leq 1 \Rightarrow \mathrm{diam}_X(\pi_X(\ell) \cup \pi_X(\ell')) \leq 2.$$

$(\ell \cap X \neq \phi,\ \ell' \cap X \neq \phi)$



Lemma 1'

$[\ell_0, \ell_1, \cdots, \ell_m]$: path in $\mathcal{C}(\Sigma)$ s.t. $\ell_i \cap X \neq \phi$ $(\forall i)$.

$$\Rightarrow \mathrm{diam}_X(\pi_X(\ell_0) \cup \cdots \cup \pi_X(\ell_m)) \leq 2m.$$

" If ℓ_0 & ℓ_m intersect complicatedly in X, then any geodesic connecting ℓ_0 & ℓ_m go throurgh a loop in $\Sigma \setminus X$. "

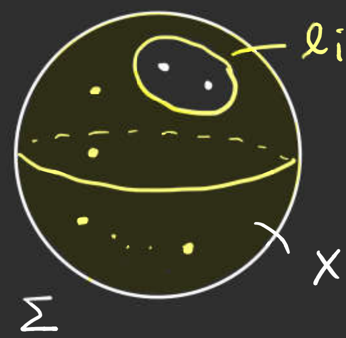
Lemma 1'

$[l_0, l_1, \cdots, l_m]$: path in $\mathcal{C}(\Sigma)$ s.t. $l_i \cap X \neq \phi \ (\forall i)$.

$\Rightarrow \ \mathrm{diam}_X(\pi_X(l_0) \cup \cdots \cup \pi_X(l_m)) \leq 2m$.

" If l_0 & l_m intersect complicatedly in X, then any geodesic connecting l_0 & l_m go through a loop in $\Sigma \setminus X$. "

ex)



If l_i cuts off a twice-punctured disk Δ & l_0 and l_m intersect complicatedly in $X = \overline{\Sigma \setminus \Delta}$,

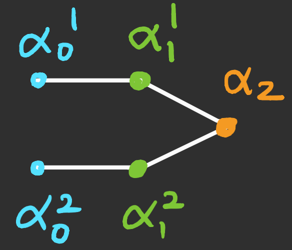
then any geodesic connecting l_0 & l_m goes through l_i.

Thm 2 [Ido-J.-Kobayashi]
$\forall n \geq 4$, $\forall m \geq 2$,
$\exists$ weakly keen, not keen
n-bridge splittings with Hempel distance m.

Idea of Proof for Thm 2

S : 2-sphere
$P\ (\subset S)$: the union of $2n$ points

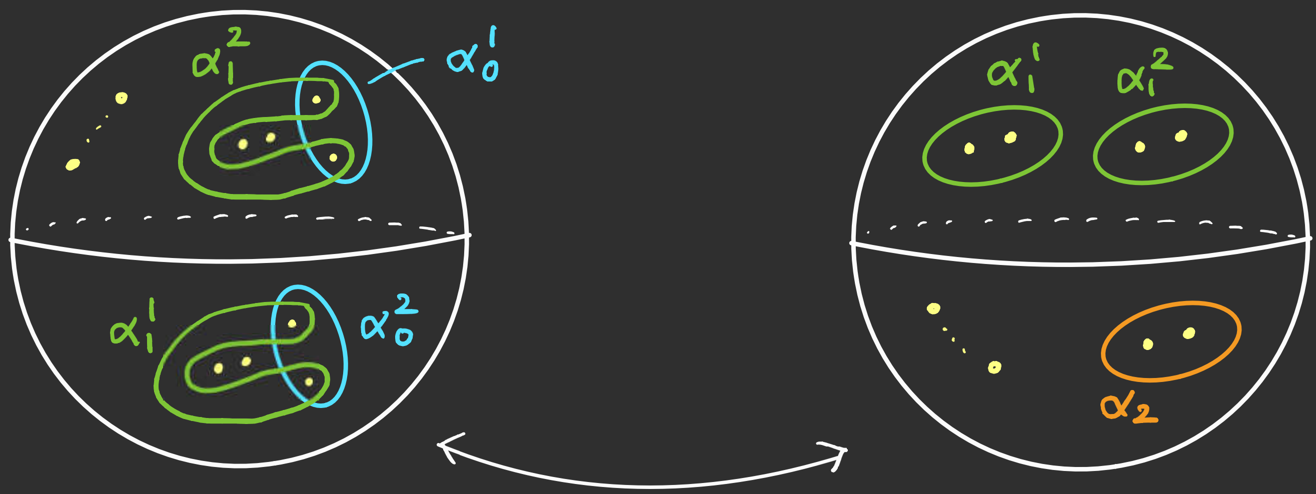
Step 1 construction of geodesics



$\subset \mathcal{C}(S \setminus P)$

s.t.

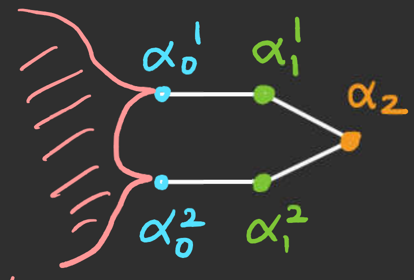
$[\alpha_0^i, \alpha_1^i, \alpha_2]$: unique geodesic connecting α_0^i & α_2 $(i=1,2)$.



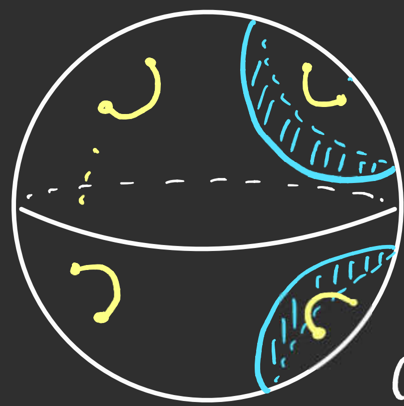
identify so that
$(\alpha_0^1 \cup \alpha_0^2)$ & α_2 intersect complicatedly
in the exterior of $(\alpha_1^1 \cup \alpha_1^2)$

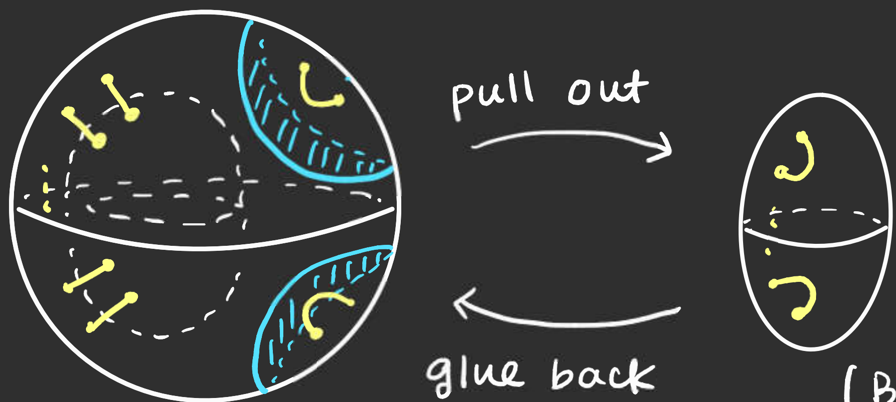
(B_+, T_+) : trivial n-tangle

Step 2 identifying $(\partial B_+, \partial T_+)$ with (S, P) so that $[\alpha_0^1, \alpha_1^1, \alpha_2]$, $[\alpha_0^2, \alpha_1^2, \alpha_2]$: the only two geodesics realizing $d_{F(P)}(\mathcal{D}(V_1 \cup T_1), \alpha_2) = 2$.



- Identify $(\partial B_+, \partial T_+)$ with (S, P) so that $\partial D_1^1 = \alpha_0^1$ & $\partial D_1^2 = \alpha_0^2$.
- Modify (B_+, T_+) as follows :

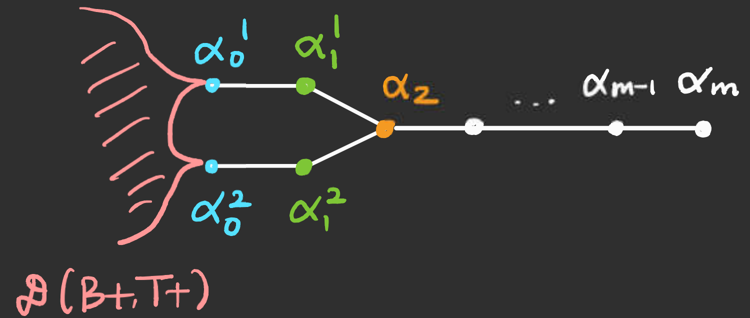




" $\mathcal{D}(B'_+, T'_+)$ intersects $\alpha_1^1 \cup \alpha_1^2$ complicatedly ".

Step 3 extending geodesics

so that

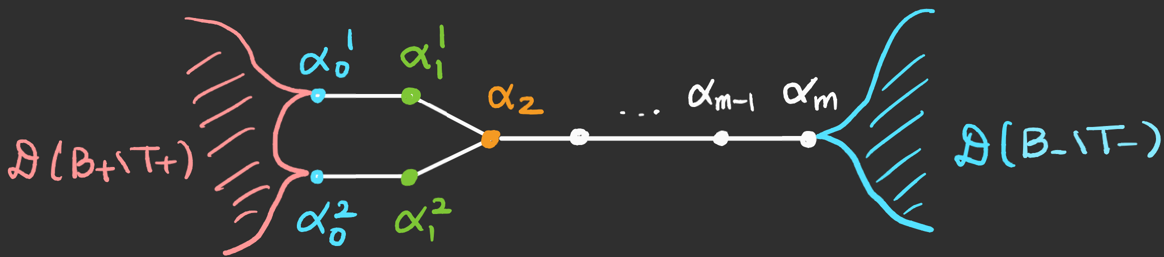


- $d_{s,p}(\mathcal{D}(B_+, T_+), \alpha_m) = m$.
- $\forall$ geodesic connecting $\mathcal{D}(B_+, T_+)$ & α_m goes through α_2 (& α_{m-1})

($\rightsquigarrow$ (α_0^1, α_m) & (α_0^2, α_m) : the only pairs realizing $d_{s,p}(\mathcal{D}(B_+, T_+), \alpha_m) = m$.)

Step 4 attaching (B_-, T_-) so that $\rightsquigarrow$ (B_-, T_-) : trivial n-tangle

(α_0^1, α_m) & (α_0^2, α_m) : the only pairs realizing $d_{s,p}(\mathcal{D}(B_+, T_+), \mathcal{D}(B_-, T_-)) = m$.



$\blacksquare$

Works in progress / future works

- on existence weakly keen (not keen) bridge splittings ($n=3$ or $m \le 1$).

  & application to Heegaard splittings of 3-manifolds.

- (strongly) keen bridge splittings vs essential surfaces

  - $\exists$ essential surface $F \subset \overline{S^3 \setminus N(K)}$
    $\Rightarrow d(\forall \text{ bridge splitting of } (S^3, K)) \le -\chi(F) + 2$.
    [Bachman - Schleimer '05]
  - $\exists$ essential meridional surface $F \subset \overline{S^3 \setminus N(K)}$
    $\Rightarrow d(\forall \text{ bridge splitting of } (S^3, K)) \le -\chi(F)$ [J. '14]