グラフ上の単純ランダムウォークの無限回衝突について

On infinite collisions of random walks on graphs

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講演記録 / Summary of the talk

Background

Why do we study random walk on graphs?

- Random walk and (discrete) heat equation

$u(n+1,x) - u(n,x) = \Delta u(n,x)$ (Δ : discrete Laplacian)

- f : initial condition.
- $X(n)$: n-th step of simple random walk started at x.
- $\mathbb{E}[\cdot]$: expectation

$\rightarrow u(n,x) = \mathbb{E}[f(X(n))]$ (continuous setting : Brownian motion)

- Random walk behavior $\longleftrightarrow$ Characterization of the graph
- Natural phenomena → **disordered media**

$\cdots$ Analysis on fractals and random media

Random walk behavior

Let us observe a random walker

e.g. Does the random walk **return to the starting point** within finite steps?

Definition 1

We say that the random walk is

recurrent if it returns with probability 1,

transient otherwise.

Example : SRW on $\mathbb{Z}$ and $\mathbb{Z}^2$ $\cdots$ recurrent,

SRW on $\mathbb{Z}^d$ $(d \geq 3)$ $\cdots$ transient (Pólya)

"A drunk man will eventually find his way home, but a drunk bird may get lost forever."

Collisions of random walks

- Pólya : How often do two walkers meet in the woods?

$X = \{X_n\}$, $Y = \{Y_n\}$: independent SRWs on G.

$$Z := \sum_{n=0}^{\infty} 1\{X_n = Y_n\}$$: the number of collisions between X and Y.

Definition 2 (Infinite / finite collision property)

We say G has the **infinite collision property** if $Z = \infty$ a.s. and it has the **finite collision property** if $Z < \infty$ a.s.

Fact : Either of these holds (0-1 law).

On transitive graphs

i.e. the graph looks the same from every vertex (*e.g.* $\mathbb{Z}^d$)

→ reduces to recurrence / transience

Finite collisions on a recurrent graph

Example

- If the graph is not transitive
  e.g. Comb($\mathbb{Z}$)
  : **recurrent** & **finite collisions**
  (Krishnapur-Peres, 2004)

Two walkers on different "teeth" are unlikely to meet

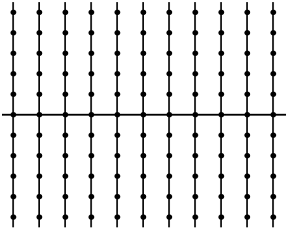

Figure: Comb($\mathbb{Z}$) from Chen-Wei-Zhang

Remark: Collision property is not monotone
(*e.g.* $\mathbb{Z} \subset \text{Comb}(\mathbb{Z}) \subset \mathbb{Z}^{\mathbf{2}}$)

Phase transition on comb graphs

Comb with shorter "teeth" ?

→ *e.g.* Comb($\mathbb{Z}, f$) : truncate at height $f(n)$

Number of collisions

- $f(x) = |x|^{\alpha}$

  → $\alpha \leq 1$: infinite / $\alpha > 1$: finite a.s.

  (Barlow-Peres-Sousi, 2012)

- $f(x) = |x| \log^{\beta}(|x| \vee 1)$

  → $\beta \leq 1$: infinite / $\beta > 2$: finite

  (Chen-Chen, 2011)

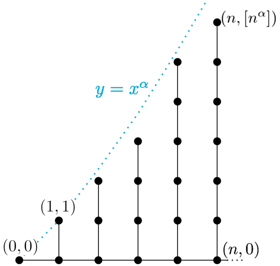

Figure: Comb($\mathbb{Z}, f$) with $f(x) = |x|^{\alpha}$

Backbone + short teeth → infinite collisions

Other examples?

RW on random graphs

Background

Anomalous diffusion in **disordered** media

Polymar ⟷ self-avoiding walk

Porous media ⟷ percolation cluster etc.

- Alexander-Orbach conjecture (1982)
  : RW diffusion is essentially slower on critical percolation clusters
- Kesten (1986)
  : First rigorous result in mathematics

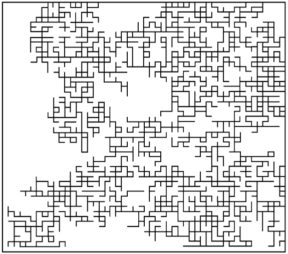

Figure: 2D percolation cluster (by Hunt-Ewing-Ghanbarian, 2014)

Aim : Typical behavior of RW on random graphs?

Collisions on random graphs

Example

- Critical Galton-Watson tree
- IIC of critical percolation on $\mathbb{Z}^d$
- Uniform spanning tree of Cayley graphs

Is the infinite collision property **typical** on these random graphs?

Figure: Garton-Watson tree (by Berglund)

Sufficient conditions : Barlow-Peres-Sousi (2012), Hutchcroft-Peres (2015)

Necessary condition is harder...

My work #1

1. Quantitative estimate?

→ Collisions on the **three-dimensional uniform spanning tree** (UST) **:** W. (2023)

Related models of UST

Loop-erased random walk (LERW),

random cluster model,

2D UST: Schramm-Loewner Evolution (SLE)

- scaling limit of planar random process

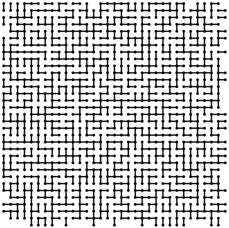

Figure: 2D UST (by Wilson)

Key points

- Effective resistance
- 3D UST is analizable via LERW

Idea of the proof

Inspired by Barlow-Peres-Sousi (2012)

X : nonnegative random variable on $(\Omega, \mathcal{F}, P)$
with $E[X^2] < \infty$, $\theta \in (0,1)$.

Then, $$P(X > \theta E[X]) \geq (1-\theta)^2 \frac{E[X^2]}{E[X]^2}.$$

(Cauchy-Schwarz inequality)

$E[Z_{B_r}]$ and $E[Z_{B_r}^2]$ can be written with the **effective resistance**

(regarding the graph as electrical network)

- $E[\#\text{collisions}] \asymp E[\#\text{visits}] =$ Green's function

  $+$ potential theory

- UST is analyzable **even in 3D** **(hardest in general)**

  thanks to the connection with **LERW**

My work #2

2. Triple collisions?

→ 4D SRW trace **"looks like" a comb with short teeth**

Previous results on triple collisions

- Comb graph with i.i.d. teeth of finite mean (Chen-Chen, 2011)
- $\mathbf{Comb}(\mathbb{Z}, 0 \vee (\log|x|)^{\alpha})$ → $0 < \alpha \leq 1$ or $\alpha > 1$

  (Croydon-De Ambroggio, 2024+)

Why do we expect infinite triple collisions on 4D SRW trace?

- 4D $=$ critical dimension of SRW intersection

  → Comb-like structure (long-range self intersection is rare)

- Volume growth is similar to $\mathbf{Comb}(\mathbb{Z}, (\log n)^{1/2})$

Main theorem : Almost sure infinite **triple** collisions

(Croydon-De Ambroggio-Shiraishi-W. in prep.)

Conclusion and comment

- Double collisions $\longleftrightarrow$ transitivity

  (Krishnapur-Peres, Barlow-Peres-Sousi, Hutchcroft-Peres)

- **Triple collisions $\longleftrightarrow$ more "one-dimensional" structure**

- **Future work** : criterion of in/finite collisions?

Open problems

- Quadruple collisions $\longleftrightarrow$ What kind of characterization?

  Bounded degree → finite collisions (Croydon-De Ambroggio)

- Stability of collision property

  - Does a small change of the graph affect collision property?

  Remark : Hard problem for 10+ years