多重ゼータ値と多重ポリログ関数の関係式

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Section 1： 多重ポリログ関数と 2 つの多重ゼータ関数

- $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index
- $s \in \mathbb{C}$

(1 変数)Euler–Zagier 型多重ゼータ関数

$$\zeta(\mathbf{k}; s) = \zeta(k_1, \ldots, k_r; s) = \sum_{0<m_1<\cdots<m_r<m_{r+1}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} m_{r+1}^{s}}$$

(1) $\mathrm{Re}(s) > 1 \Longrightarrow$ 絶対収束.

(2) 複素数平面全体に有理型関数として解析接続される.

(3) $r = 0$ ならば, $\zeta(\varnothing; s) = \zeta(s)$ とする.

多重ゼータ値

$k_r \geq 2$ に対して,

$$\zeta(\mathbf{k}) = \zeta(k_1, \ldots, k_r) = \sum_{0<m_1<m_2<\cdots<m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \cdots m_r^{k_r}} \qquad \in \mathbb{R}$$

- $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index
- $s, z \in \mathbb{C}$

荒川–金子型多重ゼータ関数

$$\xi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}(\mathbf{k}; 1 - e^{-t})}{e^t - 1} t^{s-1} dt$$

(1) $\mathrm{Re}(s) > 0 \Longrightarrow$ 絶対収束.

(2) 複素数平面全体に正則関数として解析接続される.

(3) $\xi(\mathbf{k}; -m) = (-1)^m C_m^{(\mathbf{k})} \quad (m \in \mathbb{Z}_{\geq 0})$

(4) $\xi(1; s) = s\zeta(s+1)$

多重ポリログ関数

$$\mathrm{Li}(\mathbf{k}; z) := \sum_{0<m_1<\cdots<m_r} \frac{z^{m_r}}{m_1^{k_1} \cdots m_r^{k_r}} \quad (|z| < 1)$$

(1 変数)Euler–Zagier 型多重ゼータ関数 [Review]

$$\zeta(\mathbf{k}; s) = \sum_{0<m_1<\cdots<m_r<m_{r+1}} \frac{1}{m_1^{k_1} \cdots m_r^{k_r} m_{r+1}^s}$$

$$= \frac{1}{\Gamma(s)} \int_0^\infty \frac{\mathrm{Li}(\mathbf{k}; e^{-t})}{e^t - 1} t^{s-1} dt$$

先行研究

- $a \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 0}$, $s, z \in \mathbb{C}$
- $\{1\}^a = \underbrace{1, \dots, 1}_{a}$

Theorem 1 (T. Arakawa–M. Kaneko'99)

$$\xi(\{1\}^{a-1}, b+1; s) = \sum_{j=0}^{b-1} (-1)^j \zeta(\{1\}^{a-1}, b+1-j) \zeta(\{1\}^j; s)$$

$$+ (-1)^b \sum_{\substack{e_1, \dots, e_{k-1}, d \geq 0 \\ e_1 + \cdots + e_b + d = a}} \binom{s+d-1}{d} \zeta(e_1+1, \dots, e_{k-1}+1; s+d).$$

$\Uparrow$

Theorem 2 (C. Xu'21)

$$\mathrm{Li}(\{1\}^{a-1}, b+1; 1-z) = \sum_{j=0}^{b-1} (-1)^j \zeta(\{1\}^{a-1}, b+1-j) \mathrm{Li}(\{1\}^j; z)$$

$$+ (-1)^b \sum_{\substack{e_1, \dots, e_{k-1}, d \geq 0 \\ e_1 + \cdots + e_b + d = a}} \mathrm{Li}(\{1\}^d; 1-z) \mathrm{Li}(e_1+1, \dots, e_{k-1}+1; z).$$

- $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index
- $a, m \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 0}$, $s, z \in \mathbb{C}$
- $\{1\}^a = \underbrace{1, \ldots, 1}_{a}$

Problem (Arakawa–Kaneko'99)

① $\xi(\mathbf{k}; s)$ は Euler-Zagier 型で表せる？

② ∃?「$z \mapsto 1 - z$ に対する $\mathrm{Li}(\mathbf{k}; z)$ の関数関係式」 $\Longrightarrow$ ①: 肯定的に解決

③ $\xi(\mathbf{k}; m+1)$ には，反転公式はある？

Theorem 3 (Arakawa–Kaneko'99)

$$\xi(\{1\}^{a-1}, b+1; m+1) - (-1)^b \xi(\{1\}^{m-1}, b+1; a+1)$$
$$= \sum_{j=0}^{b-1} (-1)^j \zeta(\{1\}^{a-1}, b+1-j) \zeta(\{1\}^{m-1}, j+2).$$

- $\mathbf{e} = (e_1, \ldots, e_r) \in (\mathbb{Z}_{\geq 0})^r$
- $\mathrm{wt}(\mathbf{e}) = e_1 + \cdots + e_r$: weight , $\mathrm{dep}(\mathbf{e}) = r$: depth
- $s, z \in \mathbb{C}$

Theorem 4 (Kaneko–H. Tsumura'18)

$$\xi(\mathbf{k}; s) = \sum_{\substack{\mathbf{k}':\text{index},\ d \geq 0 \\ \mathrm{wt}(\mathbf{k}')+d \leq \mathrm{wt}(\mathbf{k})}} c_{\mathbf{k}}(\mathbf{k}'; d) \binom{s+d-1}{d} \zeta(\mathbf{k}'; s+d).$$

Lemma 5 (Kaneko–Tsumura'18)

$$\mathrm{Li}(\mathbf{k}; 1-z) = \sum_{\substack{\mathbf{k}':\text{index},\ d \geq 0 \\ \mathrm{wt}(\mathbf{k}')+d \leq \mathrm{wt}(\mathbf{k})}} c_{\mathbf{k}}(\mathbf{k}'; d) \mathrm{Li}(\{1\}^d; 1-z) \mathrm{Li}(\mathbf{k}'; z).$$

$c_{\mathbf{k}}(\mathbf{k}'; d)$ は weight が $\mathrm{wt}(\mathbf{k}) - \mathrm{wt}(\mathbf{k}') - d$ を満たす多重ゼータ値の $\mathbb{Q}$-線形結合とする.

Today's talk (Probrem①②③を肯定的に完全解決！！)

① Theorem 4 の明示式 (Theorem 8)

② Lemma 5 の明示式 (Theorem 9)

③ Theorem 3 の一般化 (Corollary 10)

Notation

- $\mathbf{k} = (k_1, \ldots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index, $\mathbf{e} = (e_1, \ldots, e_r) \in (\mathbb{Z}_{\geq 0})^r$
  - $\mathbf{e}_+ := (e_1, \ldots, e_{r-1}, e_r + 1)$,
  - $\mathbf{k} + \mathbf{e} := (k_1 + e_1, \ldots, k_r + e_r)$,
  - $b(\mathbf{k}; \mathbf{e}) := \prod_{i=1}^{r} \binom{k_i + e_i - 1}{e_i}$.
  - $\mathbf{k}^\vee = (\underbrace{1, \ldots, 1}_{k_1} + \underbrace{1, \ldots, 1}_{k_2} + \cdots + \underbrace{1, \ldots, 1}_{k_r})$: $\mathbf{k}$ の Hoffman 双対.

- $\mathbf{k} = (\{1\}^{a_1 - 1}, b_1 + 1, \ldots, \{1\}^{a_n - 1}, b_n + 1)$ $(a_1, \ldots, a_n, b_1, \ldots, b_{n-1} \in \mathbb{Z}_{\geq 1}, b_n \in \mathbb{Z}_{\geq 0})$
  - $\mathbf{k}^i = (\{1\}^{a_{i+1} - 1}, b_{i+1} + 1, \ldots, \{1\}^{a_n - 1}, b_n + 1)$ $(0 \leq i \leq n)$,
  - $\mathbf{k}_j = (\{1\}^{a_1 - 1}, b_1 + 1, \ldots, \{1\}^{a_j - 1}, b_j + 1)$ $(0 \leq j \leq n)$,
  - $b_n \geq 1 \Longrightarrow \mathbf{k}^\dagger = (\{1\}^{b_n - 1}, a_n + 1, \ldots, \{1\}^{b_1 - 1}, a_1 + 1)$: $\mathbf{k}$ の双対.

  Remark. $\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\dagger)$: 双対公式

定理 4 (Kaneko-Tsumura'18) の明示式

- $a_1, \ldots, a_n, b_1, \ldots, b_{n-1} \in \mathbb{Z}_{\geq 1}$, $b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \ldots, \{1\}^{a_n-1}, b_n+1)$: index

Theorem 6 (K.)

空でない $\mathbf{k}$ に対して,

$$
\begin{aligned}
&\xi(\mathbf{k}; s) \\
&= (1-\delta_{0,b_n})\zeta(\mathbf{k})\zeta(s) \\
&\quad - \sum_{l=1}^{n} \sum_{j=0}^{b_l-2} (-1)^{j+\mathrm{wt}(\mathbf{k}^l)} \zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l - j)\zeta((j+1, \mathbf{k}^l)^\vee; s) \\
&\quad + \sum_{l=1}^{n} (-1)^{b_l+\mathrm{wt}(\mathbf{k}^l)} \sum_{d=0}^{a_l} \sum_{\substack{\mathrm{wt}(\mathbf{e}_1)+\mathrm{wt}(\mathbf{e}_2)+d=a_l \\ \mathrm{dep}(\mathbf{e}_1)=n_1,\ \mathrm{dep}(\mathbf{e}_2)=n_2}} (-1)^{\mathrm{wt}(\mathbf{e}_1)} b\left((\mathbf{k}_{l-1})^\dagger; \mathbf{e}_1\right) \zeta((\mathbf{k}_{l-1})^\dagger + \mathbf{e}_1) \\
&\qquad \times b\left((b_l, \mathbf{k}^l)^\vee; \mathbf{e}_2\right) \binom{s+d-1}{d} \zeta((b_l, \mathbf{k}^l)^\vee + \mathbf{e}_2; s+d)
\end{aligned}
$$

ただし, $\mathbf{e}_1 \in (\mathbb{Z}_{\geq 0})^{n_1}$, $\mathbf{e}_2 \in (\mathbb{Z}_{\geq 0})^{n_2}$, $n_1 = \mathrm{dep}((\mathbf{k}_{l-1})^\dagger)$, $n_2 = \mathrm{dep}((b_l, \mathbf{k}^l)^\vee)$.

Remark. $n = 1 \Longrightarrow$ [Arakawa-Kaneko'99]

補題 5 (Kaneko-Tsumura'18) の明示式

- $a_1, \ldots, a_n, b_1, \ldots, b_{n-1} \in \mathbb{Z}_{\geq 1}$, $b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \ldots, \{1\}^{a_n-1}, b_n+1)$: index

Theorem 7 (K.)

空でない $\mathbf{k}$ に対して,

$$\begin{aligned}
&\mathrm{Li}(\mathbf{k}; 1-z) \\
&= (1-\delta_{0,b_n})\zeta(\mathbf{k}) \\
&\quad - \sum_{l=1}^{n} \sum_{j=0}^{b_l-2} (-1)^{j+\mathrm{wt}(\mathbf{k}^l)} \zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l - j) \mathrm{Li}((j+1, \mathbf{k}^l)^\vee; z) \\
&\quad + \sum_{l=1}^{n} (-1)^{b_l+\mathrm{wt}(\mathbf{k}^l)} \sum_{d=0}^{a_l} \sum_{\substack{\mathrm{wt}(\mathbf{e}_1)+\mathrm{wt}(\mathbf{e}_2)+d=a_l \\ \mathrm{dep}(\mathbf{e}_1)=n_1,\ \mathrm{dep}(\mathbf{e}_2)=n_2}} (-1)^{\mathrm{wt}(\mathbf{e}_1)} b\left((\mathbf{k}_{l-1})^\dagger; \mathbf{e}_1\right) \zeta((\mathbf{k}_{l-1})^\dagger + \mathbf{e}_1) \\
&\qquad \times b\left((b_l, \mathbf{k}^l)^\vee; \mathbf{e}_2\right) \mathrm{Li}(\{1\}^d; 1-z) \mathrm{Li}((b_l, \mathbf{k}^l)^\vee + \mathbf{e}_2; z).
\end{aligned}$$

ただし, $\mathbf{e}_1 \in (\mathbb{Z}_{\geq 0})^{n_1}$, $\mathbf{e}_2 \in (\mathbb{Z}_{\geq 0})^{n_2}$, $n_1 = \mathrm{dep}((\mathbf{k}_{l-1})^\dagger)$, $n_2 = \mathrm{dep}((b_l, \mathbf{k}^l)^\vee)$.

Remark. $n = 1 \Longrightarrow$ [C. Xu'21]

定理 6 の系

- $a_1, \dots, a_n, b_1, \dots, b_{n-1} \in \mathbb{Z}_{\geq 1}$, $b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \dots, \{1\}^{a_n-1}, b_n+1)$: index
- $\overleftarrow{\mathbf{k}} = (b_n+1, \{1\}^{a_n-1}, \dots, b_1+1, \{1\}^{a_1-1})$

Corollary 8 (K.)

空でない $\mathbf{k}$ と正の整数 m に対して,

$$
\begin{aligned}
&\xi(\mathbf{k}; m+1) - (-1)^{\mathrm{wt}(\mathbf{k})-a_1}\xi(\{1\}^{m-1}, \overleftarrow{(b_1, \mathbf{k}^1)_+}; a_1+1) \\
&= (1-\delta_{0,b_n})\zeta(\mathbf{k})\zeta(m+1) \\
&\quad - \sum_{l=1}^{n}\sum_{j=0}^{b_l-2}(-1)^{j+\mathrm{wt}(\mathbf{k}^l)}\zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l-j)\zeta(\{1\}^{m-1}, \overleftarrow{(j+2, \mathbf{k}^l)_+}) \\
&\quad + \sum_{l=2}^{n}\sum_{d=0}^{a_l}(-1)^{b_l+\mathrm{wt}(\mathbf{k}^l)+d}\xi((\mathbf{k}_{l-1})_-; d+1)\xi(\{1\}^{m-1}, \overleftarrow{(b_l, \mathbf{k}^l)_+}; a_l-d+1).
\end{aligned}
$$

Remark. $a_2, \dots, a_n = 1 \Longrightarrow$ [C. Xu'21]

Remark. $n = 1 \Longrightarrow$ [Arakawa-Kaneko'99]

Step of proofs

① $z \mapsto 1-z$ に関する $\mathrm{Li}(\mathbf{k};z)$ の関係式		② ξ と ζ の関係式		③ ξ の反転公式
Theorem 2	$\Longrightarrow$	Theorem 1		Theorem 3
Lemma 5	$\Longrightarrow$	Theorem 4		
|	|			
$\mathrm{wt}(\mathbf{k})$ に関する帰納法	同様	– point: ξ と ζ の積分表示		
|	|			
Theorem 7	$\Longrightarrow$	Theorem 6	$\Longrightarrow$	Corollary 8
			|	
			point: $s \mapsto m+1$	

Idea of proofs

① $z \mapsto 1-z$ に関する $\mathrm{Li}(\mathbf{k};z)$ の関係式		② ξ と ζ の関係式		③ ξ の反転公式
Theorem 2		Theorem 1		Theorem 3
Lemma 5	$\Longrightarrow$	Theorem 4		
Theorem 7	$\longleftarrow$	Theorem 6	$\longleftarrow$	Corollary 8
				point: 山本積分表示

Section 3： 山本積分表示

多重ゼータ値の反復積分表示

$$\zeta(1,2) = \sum_{0<m<n} \frac{1}{mn^2} = \int_{0<t_1<t_2<t_3<1} \frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}$$

実際,

$$\int_0^{t_2} \frac{dt_1}{1-t_1} = \sum_{m=1}^{\infty} \int_0^{t_2} t_1^{m-1} dt = \sum_{m=1}^{\infty} \frac{t_2^m}{m},$$

$$\sum_{m=1}^{\infty} \frac{1}{m} \int_0^{t_3} \frac{t_2^m}{1-t_2} dt_2 = \sum_{0<m<n} \frac{1}{m} \int_0^{t_3} t_2^{n-1} dt_2 = \sum_{0<m<n} \frac{t_3^n}{mn},$$

$$\sum_{0<m<n} \frac{1}{mn} \int_0^1 t_3^{n-1} dt_3 = \sum_{0<m<n} \frac{1}{mn^2} = \zeta(1,2).$$

Remark. **多重ポリログ関数の反復積分表示**

$$\mathrm{Li}(1,2;z) = \sum_{0<m<n} \frac{z^n}{mn^2} = \int_{0<t_1<t_2<t_3<z} \frac{dt_1}{1-t_1}\frac{dt_2}{1-t_2}\frac{dt_3}{t_3}$$

Definition 9 (Yamamoto'17)

- $(X, \preceq)$: **有限半順序集合**, $\delta_X : X \to \{0,1\}$: labeling map
- $X := ((X, \preceq), \delta_X)$: 2 **色半順序集合** (2-poset)
- 2-poset X : 'admissible' $\overset{\text{def}}{\iff} \begin{cases} \delta_X(x) = 0 & (x \in X : \textbf{極大元}), \\ \delta_X(x) = 1 & (x \in X : \textbf{極小元}). \end{cases}$
- X: admissible 2-poset

$$I(X) := \int_{D(X)} \prod_{x \in X} \omega_{\delta_X(x)}(t_x),$$

ただし,

$$D(X) := \left\{ (t_x)_{x \in X} \in (0,1)^X \mid t_x < t_y \text{ if } x \prec y \right\},$$

$$\omega_0(t) := \frac{dt}{t} \quad \omega_1(t) := \frac{dt}{1-t}.$$

Notation

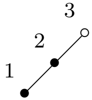
2-poset X		Hasse diagram
$x \in X$	$\longleftrightarrow$	a vertex
$\delta_X(x) = 0$	$\longleftrightarrow$	○
$\delta_X(x) = 1$	$\longleftrightarrow$	●
$x \prec y$	$\longleftrightarrow$	

Example 2-poset $X = (\{1 \prec 2 \prec 3\}, \delta_X)$ が

$(\delta_X(1), \delta_X(2), \delta_X(3)) = (1, 1, 0)$ を満たすとする.

$\Longrightarrow X$ は admissible であり,

$$I(X) = \int_{0<t_1<t_2<t_3<1} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} = I\left(\quad \right)$$

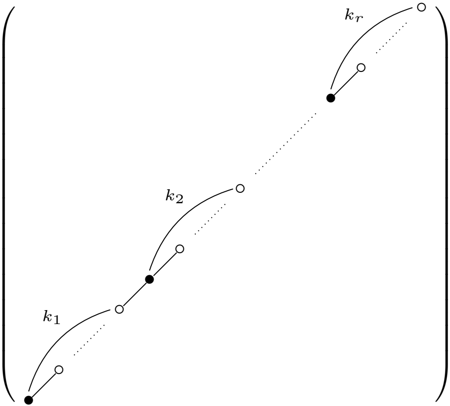


$$= \sum_{0<m<n} \frac{1}{mn^2} = \zeta(1,2).$$

多重ゼータ値の山本積分表示

$k_r \geq 2$ を満たす index $(k_1, \ldots, k_r)$ に対して,

$$\zeta\,(k_1, k_2, \ldots, k_r) = I$$



Remark admissible 2-poset X が全順序ならば,

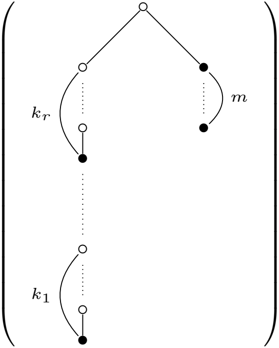
$$I(X) = \text{‘多重ゼータ値’}.$$

荒川–金子型多重ゼータ関数の特殊値の山本積分表示

index $(k_1, \ldots, k_r)$ と非負整数 m に対して，$1 - e^{-t} \mapsto x$ とすると，

$$
\begin{aligned}
\xi(k_1, \ldots, k_r; m+1) &= \frac{1}{\Gamma(m+1)} \int_0^\infty \frac{\mathrm{Li}(\mathbf{k}; 1 - e^{-t})}{e^t - 1} t^m dt \\
&= \frac{1}{m!} \int_0^1 \mathrm{Li}(\mathbf{k}; x)(-\log(1-x))^m \frac{dx}{x}
\end{aligned}
$$

$= I$



Proposition 10 (Yamamoto'17)

$X = ((X, \preceq), \delta_X)$, $Y = ((Y, \preceq), \delta_Y)$: 2-posets

(1) 2-poset $X \amalg Y = ((X \amalg Y, \preceq'), \delta_{X \amalg Y})$ を次で定義する.

$$x \preceq' y \iff x, y \in X \text{ and } x \preceq y \text{ in } X \text{ または } x, y \in Y \text{ and } x \preceq y \text{ in } Y$$

とし, $\delta_{X \amalg Y} : X \amalg Y \to \{0, 1\}$ を δ_X と δ_Y の直和とする.

$\Longrightarrow X, Y$ が admissible ならば $X \amalg Y$ も admissible であり,

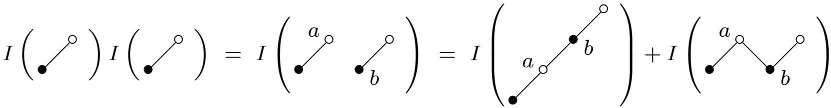
$$I(X)\, I(Y) = I(X \amalg Y).$$

(2) $a, b \in X$: 比較不可能, X_a^b: X に関係 $a \prec b$ を追加した 2-poset

$\Longrightarrow X$ が admissible ならば, X_a^b, X_b^a も admissible であり,

$$I(X) = I(X_a^b) + I(X_b^a).$$

example



Section 4： Corollary の組合せ的証明

- $a_1, \ldots, a_n, b_1, \ldots, b_{n-1} \in \mathbb{Z}_{\geq 1}$, $b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1+1, \ldots, \{1\}^{a_n-1}, b_n+1)$: index
- $\overleftarrow{\mathbf{k}} = (b_n+1, \{1\}^{a_n-1}, \ldots, b_1+1, \{1\}^{a_1-1})$

Corollary 8 (K.) [Review]

空でない $\mathbf{k}$ と正の整数 m に対して,

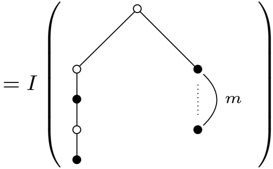
$$
\begin{aligned}
&\xi(\mathbf{k}; m+1) - (-1)^{\mathrm{wt}(\mathbf{k})-a_1}\xi(\{1\}^{m-1}, \overleftarrow{(b_1, \mathbf{k}^1)_+}; a_1+1) \\
&= (1-\delta_{0,b_n})\zeta(\mathbf{k})\zeta(m+1) \\
&\quad - \sum_{l=1}^{n}\sum_{j=0}^{b_l-2}(-1)^{j+\mathrm{wt}(\mathbf{k}^l)}\zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l-j)\zeta(\{1\}^{m-1}, \overleftarrow{(j+2, \mathbf{k}^l)_+}) \\
&\quad + \sum_{l=2}^{n}\sum_{d=0}^{a_l}(-1)^{b_l+\mathrm{wt}(\mathbf{k}^l)+d}\xi((\mathbf{k}_{l-1})_-; d+1)\xi(\{1\}^{m-1}, \overleftarrow{(b_l, \mathbf{k}^l)_+}; a_l-d+1).
\end{aligned}
$$

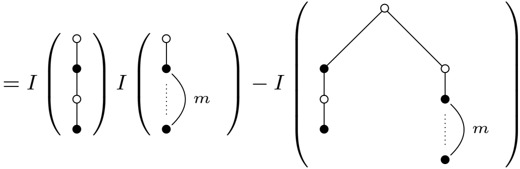
$\mathbf{k} = (2,2)$ の場合

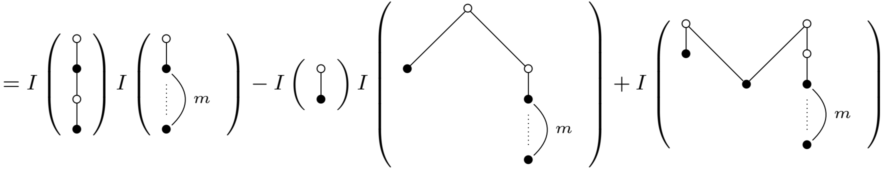
$$
\begin{aligned}
&\xi(2,2; m+1) + \xi(\{1\}^{m-1}, 3, 1; 2) \\
&= \zeta(2,2)\zeta(m+1) - \xi(1;1)\xi(\{1\}^{m-1}, 2; 2) + \xi(1;2)\xi(\{1\}^{m-1}, 2; 1)
\end{aligned}
$$

Corollary 8 の組合せ論的証明 ($\mathbf{k} = (2,2)$ の場合)

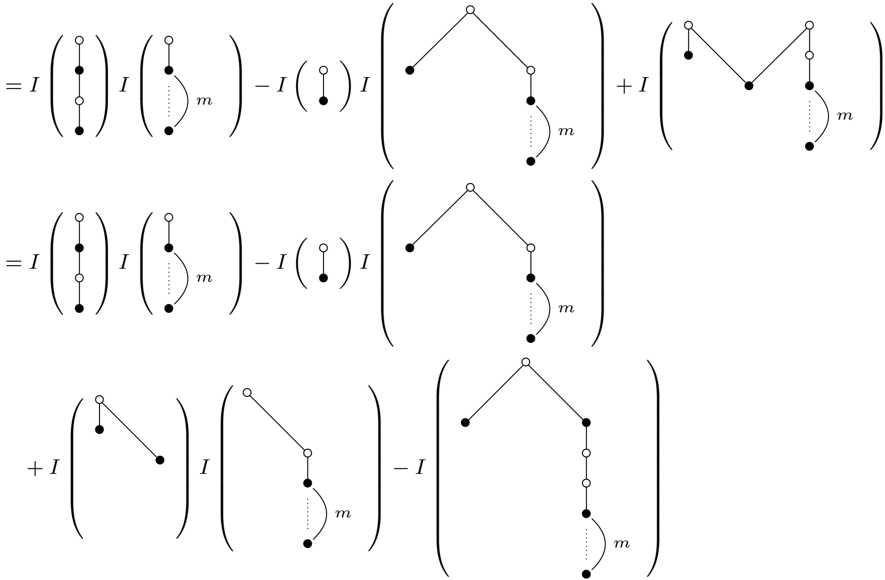
$\xi\,(2,2;m+1)$



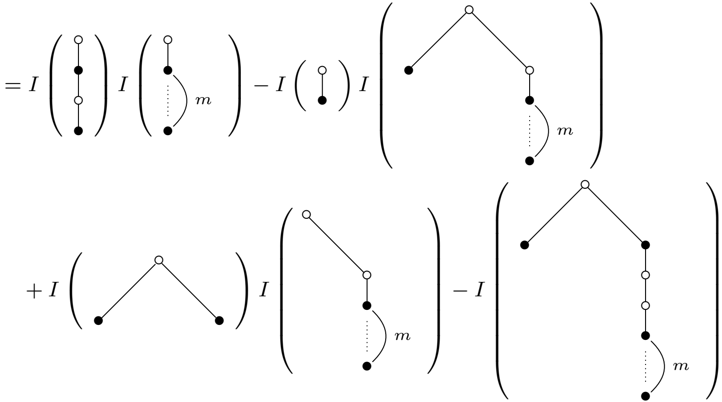




$\xi(2,2;m+1)$



$\xi\,(2,2;m+1)$



$$= \zeta(2,2)\zeta(\{1\}^{m-1},2) - \xi(1;1)\xi(\{1\}^{m-1},2;2) + \xi(1;2)\xi(\{1\}^{m-1},2;1) - \xi(\{1\}^{m-1},3,1;2)$$

$$\Longrightarrow \quad \xi\,(2,2;m+1) + \xi(\{1\}^{m-1},3,1;2)$$
$$= \zeta(2,2)\zeta(\{1\}^{m-1},2) - \xi(1;1)\xi(\{1\}^{m-1},2;2) + \xi(1;2)\xi(\{1\}^{m-1},2;1) \qquad \square$$

Remark

次の level 2 類似たちに対する Theorem 6, 7, Corollary 8 も得られた;

$\xi(\mathbf{k};s)$ の level 2 類似

$$\psi(k_1,\cdots,k_r;s)=\frac{1}{\Gamma(s)}\int_0^\infty \frac{\mathrm{A}(k_1,\ldots,k_r;\tanh(t/2))}{\sinh(t)}t^{s-1}dt$$

$\zeta(\mathbf{k};s)$ の level 2 類似

$$T(k_1,\ldots,k_r;s)=2^r\sum_{\substack{0<m_1<\cdots<m_{r+1}\\ m_i\equiv i \bmod 2}}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}m_{r+1}^s}$$

$\mathrm{Li}(\mathbf{k};z)$ の level 2 類似

$$\mathrm{A}(k_1,\ldots,k_r;z)=2^r\sum_{\substack{0<m_1<\cdots<m_r\\ m_i\equiv i \bmod 2}}\frac{z^{m_r}}{m_1^{k_1}\cdots m_r^{k_r}}$$

$\zeta(\mathbf{k})$ の level 2 類似

$$T(k_1,\ldots,k_r)=2^r\sum_{\substack{0<m_1<\cdots<m_r\\ m_i\equiv i \bmod 2}}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$