

多重ゼータ値と多重ポリログ関数の関係式

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- Section 1** : 多重ポリログ関数と2つの多重ゼータ関数
- Section 2** : Main Theorems
- Section 3** : 山本積分表示
- Section 4** : Corollary の組合せ論的証明

Section 1 : 多重ポリログ関数と2つの多重ゼータ関数

- $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index
- $s \in \mathbb{C}$

(1変数)Euler-Zagier型多重ゼータ関数

$$\zeta(\mathbf{k}; s) = \zeta(k_1, \dots, k_r; s) = \sum_{0 < m_1 < \dots < m_r < m_{r+1}} \frac{1}{m_1^{k_1} \dots m_r^{k_r} m_{r+1}^s}$$

- (1) $\operatorname{Re}(s) > 1 \implies$ 絶対収束.
- (2) 複素数平面全体に有理型関数として解析接続される.
- (3) $r = 0$ ならば, $\zeta(\emptyset; s) = \zeta(s)$ とする.

多重ゼータ値

$k_r \geq 2$ に対して,

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{0 < m_1 < m_2 < \dots < m_r} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_r^{k_r}} \in \mathbb{R}$$

- $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index
- $s, z \in \mathbb{C}$

荒川-金子型多重ゼータ関数

$$\xi(\mathbf{k}; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}(\mathbf{k}; 1 - e^{-t})}{e^t - 1} t^{s-1} dt$$

- (1) $\text{Re}(s) > 0 \implies$ 絶対収束.
- (2) 複素数平面全体に正則関数として解析接続される.
- (3) $\xi(\mathbf{k}; -m) = (-1)^m C_m^{(\mathbf{k})}$ ($m \in \mathbb{Z}_{\geq 0}$)
- (4) $\xi(1; s) = s\zeta(s+1)$

多重ポリログ関数

$$\text{Li}(\mathbf{k}; z) := \sum_{0 < m_1 < \dots < m_r} \frac{z^{m_r}}{m_1^{k_1} \dots m_r^{k_r}} \quad (|z| < 1)$$

(1変数)Euler-Zagier型多重ゼータ関数 [Review]

$$\begin{aligned} \zeta(\mathbf{k}; s) &= \sum_{0 < m_1 < \dots < m_r < m_{r+1}} \frac{1}{m_1^{k_1} \dots m_r^{k_r} m_{r+1}^s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{\text{Li}(\mathbf{k}; e^{-t})}{e^t - 1} t^{s-1} dt \end{aligned}$$

先行研究

- $a \in \mathbb{Z}_{\geq 1}, b \in \mathbb{Z}_{\geq 0}, s, z \in \mathbb{C}$
- $\{1\}^a = \underbrace{1, \dots, 1}_a$

Theorem 1 (T. Arakawa–M. Kaneko'99)

$$\begin{aligned} \xi(\{1\}^{a-1}, b+1; s) &= \sum_{j=0}^{b-1} (-1)^j \zeta(\{1\}^{a-1}, b+1-j) \zeta(\{1\}^j; s) \\ &+ (-1)^b \sum_{\substack{e_1, \dots, e_{k-1}, d \geq 0 \\ e_1 + \dots + e_b + d = a}} \binom{s+d-1}{d} \zeta(e_1+1, \dots, e_{k-1}+1; s+d). \end{aligned}$$



Theorem 2 (C. Xu'21)

$$\begin{aligned} \text{Li}(\{1\}^{a-1}, b+1; 1-z) &= \sum_{j=0}^{b-1} (-1)^j \zeta(\{1\}^{a-1}, b+1-j) \text{Li}(\{1\}^j; z) \\ &+ (-1)^b \sum_{\substack{e_1, \dots, e_{k-1}, d \geq 0 \\ e_1 + \dots + e_b + d = a}} \text{Li}(\{1\}^d; 1-z) \text{Li}(e_1+1, \dots, e_{k-1}+1; z). \end{aligned}$$

- $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index
- $a, m \in \mathbb{Z}_{\geq 1}$, $b \in \mathbb{Z}_{\geq 0}$, $s, z \in \mathbb{C}$
- $\{1\}^a = \underbrace{1, \dots, 1}_a$

Problem (Arakawa–Kaneko'99)

- ① $\xi(\mathbf{k}; s)$ は Euler-Zagier 型で表せる?
- ② $\exists?$ 「 $z \mapsto 1 - z$ に対する $\text{Li}(\mathbf{k}; z)$ の関数関係式」 \implies ①: 肯定的に解決
- ③ $\xi(\mathbf{k}; m + 1)$ には, 反転公式はある?

Theorem 3 (Arakawa–Kaneko'99)

$$\begin{aligned} & \xi(\{1\}^{a-1}, b+1; m+1) - (-1)^b \xi(\{1\}^{m-1}, b+1; a+1) \\ &= \sum_{j=0}^{b-1} (-1)^j \zeta(\{1\}^{a-1}, b+1-j) \zeta(\{1\}^{m-1}, j+2). \end{aligned}$$

- $\mathbf{e} = (e_1, \dots, e_r) \in (\mathbb{Z}_{\geq 0})^r$
- $\text{wt}(\mathbf{e}) = e_1 + \dots + e_r$: weight , $\text{dep}(\mathbf{e}) = r$: depth
- $s, z \in \mathbb{C}$

Theorem 4 (Kaneko–H. Tsumura'18)

$$\xi(\mathbf{k}; s) = \sum_{\substack{\mathbf{k}': \text{index}, d \geq 0 \\ \text{wt}(\mathbf{k}') + d \leq \text{wt}(\mathbf{k})}} c_{\mathbf{k}}(\mathbf{k}'; d) \binom{s + d - 1}{d} \zeta(\mathbf{k}'; s + d).$$

Lemma 5 (Kaneko–Tsumura'18)

$$\text{Li}(\mathbf{k}; 1 - z) = \sum_{\substack{\mathbf{k}': \text{index}, d \geq 0 \\ \text{wt}(\mathbf{k}') + d \leq \text{wt}(\mathbf{k})}} c_{\mathbf{k}}(\mathbf{k}'; d) \text{Li}(\{1\}^d; 1 - z) \text{Li}(\mathbf{k}'; z).$$

$c_{\mathbf{k}}(\mathbf{k}'; d)$ は weight が $\text{wt}(\mathbf{k}) - \text{wt}(\mathbf{k}') - d$ を満たす多重ゼータ値の \mathbb{Q} -線形結合とする.

Today's talk (Problem①②③を肯定的に完全解決！！)

- ① Theorem 4 の明示式 (Theorem 8)
- ② Lemma 5 の明示式 (Theorem 9)
- ③ Theorem 3 の一般化 (Corollary 10)

Notation

- $\mathbf{k} = (k_1, \dots, k_r) \in (\mathbb{Z}_{\geq 1})^r$: index, $\mathbf{e} = (e_1, \dots, e_r) \in (\mathbb{Z}_{\geq 0})^r$
 - ▶ $\mathbf{e}_+ := (e_1, \dots, e_{r-1}, e_r + 1)$,
 - ▶ $\mathbf{k} + \mathbf{e} := (k_1 + e_1, \dots, k_r + e_r)$,
 - ▶ $b(\mathbf{k}; \mathbf{e}) := \prod_{i=1}^r \binom{k_i + e_i - 1}{e_i}$.
 - ▶ $\mathbf{k}^\vee = (\underbrace{1, \dots, 1}_{k_1} + \underbrace{1, \dots, 1}_{k_2} + \dots + \underbrace{1, \dots, 1}_{k_r})$: \mathbf{k} の Hoffman 双対.

- $\mathbf{k} = (\{1\}^{a_1-1}, b_1 + 1, \dots, \{1\}^{a_n-1}, b_n + 1)$ ($a_1, \dots, a_n, b_1, \dots, b_{n-1} \in \mathbb{Z}_{\geq 1}, b_n \in \mathbb{Z}_{\geq 0}$)
 - ▶ $\mathbf{k}^i = (\{1\}^{a_{i+1}-1}, b_{i+1} + 1, \dots, \{1\}^{a_n-1}, b_n + 1)$ ($0 \leq i \leq n$),
 - ▶ $\mathbf{k}^j = (\{1\}^{a_1-1}, b_1 + 1, \dots, \{1\}^{a_j-1}, b_j + 1)$ ($0 \leq j \leq n$),
 - ▶ $b_n \geq 1 \implies \mathbf{k}^\dagger = (\{1\}^{b_n-1}, a_n + 1, \dots, \{1\}^{b_1-1}, a_1 + 1)$: \mathbf{k} の双対.

Remark. $\zeta(\mathbf{k}) = \zeta(\mathbf{k}^\dagger)$: 双対公式

定理 4 (Kaneko-Tsumura'18) の明示式

- $a_1, \dots, a_n, b_1, \dots, b_{n-1} \in \mathbb{Z}_{\geq 1}, b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1 + 1, \dots, \{1\}^{a_n-1}, b_n + 1)$: index

Theorem 6 (K.)

空でない \mathbf{k} に対して,

$$\begin{aligned}
 & \xi(\mathbf{k}; s) \\
 &= (1 - \delta_{0, b_n}) \zeta(\mathbf{k}) \zeta(s) \\
 & \quad - \sum_{l=1}^n \sum_{j=0}^{b_l-2} (-1)^{j+\text{wt}(\mathbf{k}^l)} \zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l - j) \zeta((j+1, \mathbf{k}^l)^\vee; s) \\
 & \quad + \sum_{l=1}^n (-1)^{b_l+\text{wt}(\mathbf{k}^l)} \sum_{d=0}^{a_l} \sum_{\substack{\text{wt}(\mathbf{e}_1)+\text{wt}(\mathbf{e}_2)+d=a_l \\ \text{dep}(\mathbf{e}_1)=n_1, \text{dep}(\mathbf{e}_2)=n_2}} (-1)^{\text{wt}(\mathbf{e}_1)} b \left((\mathbf{k}_{l-1})^\dagger; \mathbf{e}_1 \right) \zeta((\mathbf{k}_{l-1})^\dagger + \mathbf{e}_1) \\
 & \quad \times b \left((b_l, \mathbf{k}^l)^\vee; \mathbf{e}_2 \right) \binom{s+d-1}{d} \zeta((b_l, \mathbf{k}^l)^\vee + \mathbf{e}_2; s+d)
 \end{aligned}$$

ただし, $\mathbf{e}_1 \in (\mathbb{Z}_{\geq 0})^{n_1}, \mathbf{e}_2 \in (\mathbb{Z}_{\geq 0})^{n_2}, n_1 = \text{dep}((\mathbf{k}_{l-1})^\dagger), n_2 = \text{dep}((b_l, \mathbf{k}^l)^\vee)$.

Remark. $n = 1 \implies$ [Arakawa-Kaneko'99]

補題 5 (Kaneko-Tsumura'18) の明示式

- $a_1, \dots, a_n, b_1, \dots, b_{n-1} \in \mathbb{Z}_{\geq 1}, b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1 + 1, \dots, \{1\}^{a_n-1}, b_n + 1)$: index

Theorem 7 (K.)

空でない \mathbf{k} に対して,

$$\begin{aligned}
 & \text{Li}(\mathbf{k}; 1 - z) \\
 &= (1 - \delta_{0, b_n}) \zeta(\mathbf{k}) \\
 & \quad - \sum_{l=1}^n \sum_{j=0}^{b_l-2} (-1)^{j+\text{wt}(\mathbf{k}^l)} \zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l - j) \text{Li}((j+1, \mathbf{k}^l)^\vee; z) \\
 & \quad + \sum_{l=1}^n (-1)^{b_l+\text{wt}(\mathbf{k}^l)} \sum_{\substack{d=0 \\ \text{wt}(\mathbf{e}_1)+\text{wt}(\mathbf{e}_2)+d=a_l \\ \text{dep}(\mathbf{e}_1)=n_1, \text{dep}(\mathbf{e}_2)=n_2}} \sum (-1)^{\text{wt}(\mathbf{e}_1)} b \left((\mathbf{k}_{l-1})^\dagger; \mathbf{e}_1 \right) \zeta((\mathbf{k}_{l-1})^\dagger + \mathbf{e}_1) \\
 & \quad \times b \left((b_l, \mathbf{k}^l)^\vee; \mathbf{e}_2 \right) \text{Li}(\{1\}^d; 1 - z) \text{Li}((b_l, \mathbf{k}^l)^\vee + \mathbf{e}_2; z).
 \end{aligned}$$

ただし, $\mathbf{e}_1 \in (\mathbb{Z}_{\geq 0})^{n_1}, \mathbf{e}_2 \in (\mathbb{Z}_{\geq 0})^{n_2}, n_1 = \text{dep}((\mathbf{k}_{l-1})^\dagger), n_2 = \text{dep}((b_l, \mathbf{k}^l)^\vee)$.

Remark. $n = 1 \implies$ [C. Xu'21]

定理 6 の系

- $a_1, \dots, a_n, b_1, \dots, b_{n-1} \in \mathbb{Z}_{\geq 1}, b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1 + 1, \dots, \{1\}^{a_n-1}, b_n + 1)$: index
- $\overleftarrow{\mathbf{k}} = (b_n + 1, \{1\}^{a_n-1}, \dots, b_1 + 1, \{1\}^{a_1-1})$

Corollary 8 (K.)

空でない \mathbf{k} と正の整数 m に対して,

$$\begin{aligned} & \xi(\mathbf{k}; m+1) - (-1)^{\text{wt}(\mathbf{k})-a_1} \xi(\{1\}^{m-1}, \overleftarrow{(b_1, \mathbf{k}^1)_+}; a_1 + 1) \\ &= (1 - \delta_{0, b_n}) \zeta(\mathbf{k}) \zeta(m+1) \\ & \quad - \sum_{l=1}^n \sum_{j=0}^{b_l-2} (-1)^{j+\text{wt}(\mathbf{k}^l)} \zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l - j) \zeta(\{1\}^{m-1}, \overleftarrow{(j+2, \mathbf{k}^l)_+}) \\ & \quad + \sum_{l=2}^n \sum_{d=0}^{a_l} (-1)^{b_l+\text{wt}(\mathbf{k}^l)+d} \xi((\mathbf{k}_{l-1})_-; d+1) \xi(\{1\}^{m-1}, \overleftarrow{(b_l, \mathbf{k}^l)_+}; a_l - d + 1). \end{aligned}$$

Remark. $a_2, \dots, a_n = 1 \implies$ [C. Xu'21]

Remark. $n = 1 \implies$ [Arakawa-Kaneko'99]

Step of proofs

① $z \mapsto 1 - z$ に関する
 $\text{Li}(\mathbf{k}; z)$ の関係式

Theorem 2

\implies

Theorem 1

Theorem 3

Lemma 5

\implies

Theorem 4

wt(\mathbf{k}) に関する帰納法

同様

- point: ξ と ζ の積分表示

Theorem 7

\implies

Theorem 6

\implies

Corollary 8

point: $s \mapsto m + 1$

Idea of proofs

① $z \mapsto 1 - z$ に関する
 $\text{Li}(k; z)$ の関係式

Theorem 2

Lemma 5

Theorem 7

② ξ と ζ の関係式

Theorem 1

Theorem 4

Theorem 6

③ ξ の反転公式

Theorem 3

Corollary 8

point: 山本積分表示

\implies

\longleftarrow

\longleftarrow

|

Section 3 : 山本積分表示

多重ゼータ値の反復積分表示

$$\begin{aligned}\zeta(1, 2) &= \sum_{0 < m < n} \frac{1}{mn^2} \\ &= \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3}\end{aligned}$$

実際,

$$\begin{aligned}\int_0^{t_2} \frac{dt_1}{1-t_1} &= \sum_{m=1}^{\infty} \int_0^{t_2} t_1^{m-1} dt_1 = \sum_{m=1}^{\infty} \frac{t_2^m}{m}, \\ \sum_{m=1}^{\infty} \frac{1}{m} \int_0^{t_3} \frac{t_2^m}{1-t_2} dt_2 &= \sum_{0 < m < n} \frac{1}{m} \int_0^{t_3} t_2^{n-1} dt_2 = \sum_{0 < m < n} \frac{t_3^n}{mn}, \\ \sum_{0 < m < n} \frac{1}{mn} \int_0^1 t_3^{n-1} dt_3 &= \sum_{0 < m < n} \frac{1}{mn^2} = \zeta(1, 2).\end{aligned}$$

Remark. 多重ポリログ関数の反復積分表示

$$\text{Li}(1, 2; z) = \sum_{0 < m < n} \frac{z^n}{mn^2} = \int_{0 < t_1 < t_2 < t_3 < z} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3}$$

Definition 9 (Yamamoto'17)

- (X, \preceq) : 有限半順序集合, $\delta_X : X \rightarrow \{0, 1\}$: labeling map
- $X := ((X, \preceq), \delta_X)$: 2色半順序集合 (2-poset)
- 2-poset X : 'admissible' $\stackrel{\text{def}}{\iff} \begin{cases} \delta_X(x) = 0 & (x \in X : \text{極大元}), \\ \delta_X(x) = 1 & (x \in X : \text{極小元}). \end{cases}$
- X : admissible 2-poset

$$I(X) := \int_{D(X)} \prod_{x \in X} \omega_{\delta_X(x)}(t_x),$$

ただし,

$$D(X) := \left\{ (t_x)_{x \in X} \in (0, 1)^X \mid t_x < t_y \text{ if } x \prec y \right\},$$
$$\omega_0(t) := \frac{dt}{t} \quad \omega_1(t) := \frac{dt}{1-t}.$$

Notation

2-poset X		Hasse diagram
$x \in X$	\longleftrightarrow	a vertex
$\delta_X(x) = 0$	\longleftrightarrow	\circ
$\delta_X(x) = 1$	\longleftrightarrow	\bullet
$x \prec y$	\longleftrightarrow	

Example 2-poset $X = (\{1 \prec 2 \prec 3\}, \delta_X)$ が

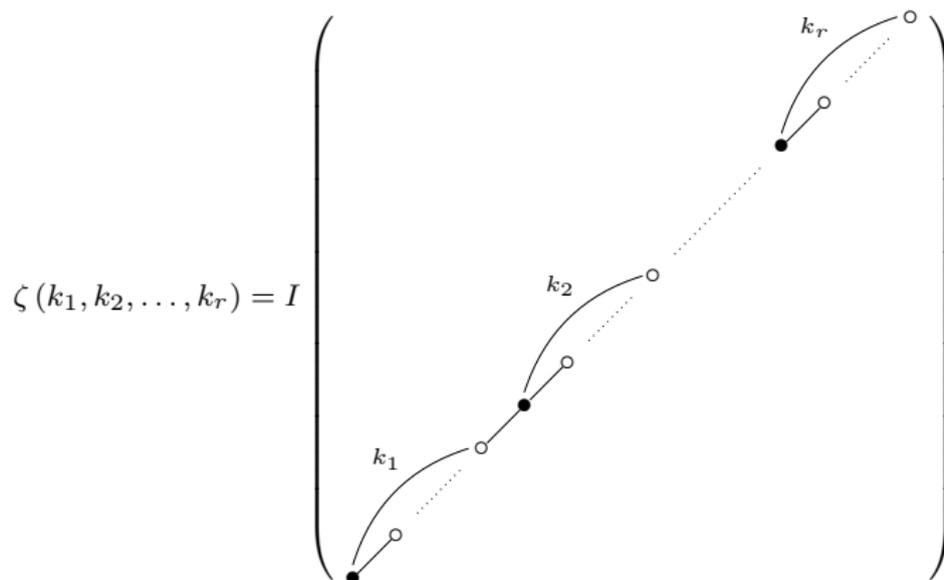
$(\delta_X(1), \delta_X(2), \delta_X(3)) = (1, 1, 0)$ を満たすとする.

$\implies X$ は admissible であり,

$$I(X) = \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{1-t_1} \frac{dt_2}{1-t_2} \frac{dt_3}{t_3} = I \left(\begin{array}{c} 3 \\ \diagup \circ \\ 2 \bullet \\ \diagdown \bullet \\ 1 \end{array} \right)$$
$$= \sum_{0 < m < n} \frac{1}{mn^2} = \zeta(1, 2).$$

多重ゼータ値の山本積分表示

$k_r \geq 2$ を満たす index (k_1, \dots, k_r) に対して,



Remark admissible 2-poset X が全順序ならば,

$$I(X) = \text{'多重ゼータ値'}.$$

荒川-金子型多重ゼータ関数の特殊値の山本積分表示

index (k_1, \dots, k_r) と非負整数 m に対して, $1 - e^{-t} \mapsto x$ とすると,

$$\begin{aligned} \xi(k_1, \dots, k_r; m+1) &= \frac{1}{\Gamma(m+1)} \int_0^\infty \frac{\text{Li}(\mathbf{k}; 1 - e^{-t})}{e^t - 1} t^m dt \\ &= \frac{1}{m!} \int_0^1 \text{Li}(\mathbf{k}; x) (-\log(1-x))^m \frac{dx}{x} \end{aligned}$$

$$= I \left(\begin{array}{c} \text{Diagram} \end{array} \right)$$

The diagram is enclosed in large parentheses and represents a tree structure. At the top is a root node (open circle). Two edges descend from the root to two intermediate nodes (open circles). From the left intermediate node, a vertical chain of nodes descends: an open circle, a dotted line, another open circle, and finally a solid black circle. A curved line labeled k_r connects the left intermediate node to the solid black circle. From the right intermediate node, a vertical chain of nodes descends: a solid black circle, a dotted line, another solid black circle, and finally a solid black circle. A curved line labeled m connects the right intermediate node to the second solid black circle from the top. From the bottom-most solid black circle, a vertical chain of nodes descends: a dotted line, an open circle, another dotted line, another open circle, and finally a solid black circle. A curved line labeled k_1 connects the bottom-most solid black circle to the second open circle from the bottom.

Section 4 : Corollary の組合せ的証明

- $a_1, \dots, a_n, b_1, \dots, b_{n-1} \in \mathbb{Z}_{\geq 1}, b_n \in \mathbb{Z}_{\geq 0}$
- $\mathbf{k} = (\{1\}^{a_1-1}, b_1 + 1, \dots, \{1\}^{a_n-1}, b_n + 1)$: index
- $\overleftarrow{\mathbf{k}} = (b_n + 1, \{1\}^{a_n-1}, \dots, b_1 + 1, \{1\}^{a_1-1})$

Corollary 8 (K.) [Review]

空でない \mathbf{k} と正の整数 m に対して,

$$\begin{aligned}
 & \xi(\mathbf{k}; m+1) - (-1)^{\text{wt}(\mathbf{k})-a_1} \xi(\{1\}^{m-1}, \overleftarrow{(b_1, \mathbf{k}^1)}_+; a_1 + 1) \\
 &= (1 - \delta_{0, b_n}) \zeta(\mathbf{k}) \zeta(m+1) \\
 & \quad - \sum_{l=1}^n \sum_{j=0}^{b_l-2} (-1)^{j+\text{wt}(\mathbf{k}^l)} \zeta(\mathbf{k}_{l-1}, \{1\}^{a_l-1}, b_l - j) \zeta(\{1\}^{m-1}, \overleftarrow{(j+2, \mathbf{k}^l)}_+) \\
 & \quad + \sum_{l=2}^n \sum_{d=0}^{a_l} (-1)^{b_l+\text{wt}(\mathbf{k}^l)+d} \xi((\mathbf{k}_{l-1})_-; d+1) \xi(\{1\}^{m-1}, \overleftarrow{(b_l, \mathbf{k}^l)}_+; a_l - d + 1).
 \end{aligned}$$

$\mathbf{k} = (2, 2)$ の場合

$$\begin{aligned}
 & \xi(2, 2; m+1) + \xi(\{1\}^{m-1}, 3, 1; 2) \\
 &= \zeta(2, 2) \zeta(m+1) - \xi(1; 1) \xi(\{1\}^{m-1}, 2; 2) + \xi(1; 2) \xi(\{1\}^{m-1}, 2; 1)
 \end{aligned}$$

Corollary 8 の組合せ論的証明 ($k = (2, 2)$ の場合)

$\xi(2, 2; m + 1)$

$$= I \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)$$

Diagram 1: A tree with root node (white) and two children (white and black). The white child has two children (black and white). The black child has a vertical chain of nodes (black, white, black) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 2: A tree with root node (white) and two children (white and black). The white child has a vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

$$= I \left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right) - I \left(\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right)$$

Diagram 3: A vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 4: A vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 5: A tree with root node (white) and two children (black and white). The black child has a vertical chain of nodes (black, white, black) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 6: A tree with root node (white) and two children (white and black). The white child has a vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

$$= I \left(\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right) - I \left(\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right) + I \left(\begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right)$$

Diagram 7: A vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 8: A vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 9: A vertical chain of nodes (white, black) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 10: A tree with root node (white) and two children (black and white). The black child has a vertical chain of nodes (black, white, black) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 11: A tree with root node (white) and two children (black and white). The black child has a vertical chain of nodes (black, white, black) with a dashed line and a bracket labeled m indicating m nodes.

Diagram 12: A tree with root node (white) and two children (white and black). The white child has a vertical chain of nodes (white, black, white) with a dashed line and a bracket labeled m indicating m nodes.

$\xi(2, 2; m + 1)$

$$\begin{aligned}
 &= I \left(\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m - I \left(\begin{array}{c} \circ \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m + I \left(\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m \\
 &= I \left(\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m - I \left(\begin{array}{c} \circ \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m \\
 &\quad + I \left(\begin{array}{c} \circ \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m - I \left(\begin{array}{c} \circ \\ \bullet \\ \circ \\ \bullet \\ \vdots \\ \bullet \end{array} \right) \Bigg|_m
 \end{aligned}$$

$$\xi(2, 2; m + 1)$$

$$\begin{aligned}
 &= I \left(\begin{array}{c} \circ \\ | \\ \bullet \\ | \\ \circ \\ | \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ | \\ \bullet \\ \vdots \\ \bullet \end{array} \right)_m - I \left(\begin{array}{c} \circ \\ | \\ \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ / \quad \backslash \\ \bullet \quad \circ \\ | \quad | \\ \bullet \quad \bullet \\ \vdots \\ \bullet \end{array} \right)_m \\
 &+ I \left(\begin{array}{c} \circ \\ / \quad \backslash \\ \bullet \quad \bullet \end{array} \right) I \left(\begin{array}{c} \circ \\ | \\ \circ \\ | \\ \bullet \\ \vdots \\ \bullet \end{array} \right)_m - I \left(\begin{array}{c} \circ \\ / \quad \backslash \\ \bullet \quad \bullet \\ | \quad | \\ \circ \quad \circ \\ | \quad | \\ \bullet \quad \bullet \\ \vdots \\ \bullet \end{array} \right)_m \\
 &= \zeta(2, 2)\zeta(\{1\}^{m-1}, 2) - \xi(1; 1)\xi(\{1\}^{m-1}, 2; 2) + \xi(1; 2)\xi(\{1\}^{m-1}, 2; 1) - \xi(\{1\}^{m-1}, 3, 1; 2) \\
 &\implies \xi(2, 2; m + 1) + \xi(\{1\}^{m-1}, 3, 1; 2) \\
 &= \zeta(2, 2)\zeta(\{1\}^{m-1}, 2) - \xi(1; 1)\xi(\{1\}^{m-1}, 2; 2) + \xi(1; 2)\xi(\{1\}^{m-1}, 2; 1) \quad \square
 \end{aligned}$$

Remark

次の level 2 類似たちに対する Theorem 6, 7, Corollary 8 も得られた;

$\xi(\mathbf{k}; s)$ の level 2 類似

$$\psi(k_1, \dots, k_r; s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{A(k_1, \dots, k_r; \tanh(t/2))}{\sinh(t)} t^{s-1} dt$$

$\zeta(\mathbf{k}; s)$ の level 2 類似

$$T(k_1, \dots, k_r; s) = 2^r \sum_{\substack{0 < m_1 < \dots < m_{r+1} \\ m_i \equiv i \pmod{2}}} \frac{1}{m_1^{k_1} \dots m_r^{k_r} m_{r+1}^s}$$

$\text{Li}(\mathbf{k}; z)$ の level 2 類似

$$A(k_1, \dots, k_r; z) = 2^r \sum_{\substack{0 < m_1 < \dots < m_r \\ m_i \equiv i \pmod{2}}} \frac{z^{m_r}}{m_1^{k_1} \dots m_r^{k_r}}$$

$\zeta(\mathbf{k})$ の level 2 類似

$$T(k_1, \dots, k_r) = 2^r \sum_{\substack{0 < m_1 < \dots < m_r \\ m_i \equiv i \pmod{2}}} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}$$