

開リーマン面の変動に関する2階変分公式と擬凸領域

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Motivation: Simultaneous uniformization problem

Let $\pi : \mathcal{R} \rightarrow \Delta$ be a holomorphic submersion of a 2-dimensional complex manifold $\mathcal{R}$ to a disk $\Delta = \{t \in \mathbb{C} \mid |t| < \rho\}$. Assume that each fiber $R(t) = \pi^{-1}(t)$, $t \in \Delta$, is non-compact and irreducible. Assume that

- ① $\mathcal{R}$ is a 2-dim Stein manifold,
- ② each $R(t)$, $t \in \Delta$, is planar.

Fact 1 (Nishino (1969, Kyoto), cf: Chirka2012, Ohsawa2020)

$R(t)$, $t \in \Delta$, is conformal equivalent to $\mathbb{C} \implies \mathcal{R} \cong \Delta \times \mathbb{C}$.

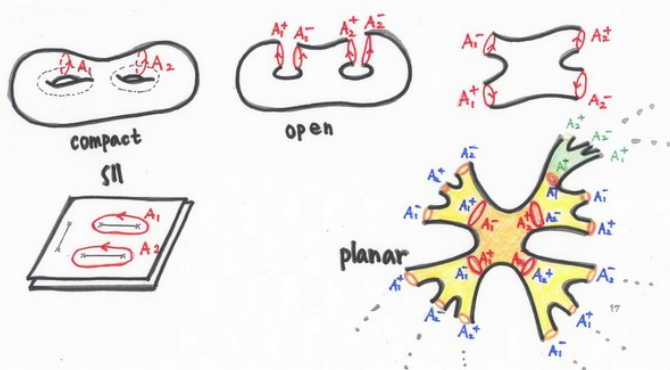
Fact 2 (Yamaguchi (1976, Kyoto), Maitani-Yamaguchi (2004, Math. Ann.))

$R(t)$, $t \in \Delta$, is of class $O_G \implies \mathcal{R} \cong$ a univalent domain in $\Delta \times \mathbb{P}^1$.

Theorem 3 (H; 2014, Ann. Polonici Math)

$R(t)$, $t \in \Delta$ is of class O_{AD} and (C1) type $\implies \mathcal{R} \cong$ a univalent domain in $\Delta \times \mathbb{P}^1$.

- Schottky covering $\tilde{S}$ of a compact Riemann surface S ($g \geq 2$)



Theorem 4 (H; 2011, Michigan Math)

Let Δ be a disk in $\mathbb{C}_t$. Let $\pi : \mathcal{S} \rightarrow \Delta$ be a holomorphic family of compact Riemann surfaces $S(t) = \pi^{-1}(t)$, $t \in \Delta$, of genus $g \geq 2$. Let $\tilde{S}(t)$ denote the **Schottky covering of $S(t)$** for each $t \in \Delta$. Then the total space $\tilde{\mathcal{S}} := \bigcup_{t \in \Delta} (t, \tilde{S}(t))$ is biholomorphic to a univalent domain in $\Delta \times \mathbb{P}^1$.

Smooth variation of domains $D(t)$, $t \in \mathbb{C}$

For $t \in \Delta = \{|t| < \rho\} \subset \mathbb{C}_t$, we consider the variation of domains $D(t)$:

$$\mathcal{D} : t \in \Delta \rightarrow D(t) \subset \mathbb{C}_z.$$

We identify $\mathcal{D}$ with a subset $\bigcup_{t \in \Delta} (t, D(t))$ of $\Delta \times \mathbb{C}_z$, and write

$$\mathcal{D} = \bigcup_{t \in \Delta} (t, D(t)) \quad \text{and} \quad \partial \mathcal{D} = \bigcup_{t \in \Delta} (t, \partial D(t)).$$

We say that $\mathcal{D}$ is a smooth variation if there exists a C^ω -defining function $\varphi(t, z)$ of $\partial \mathcal{D}$ in $\Delta \times \mathbb{C}_z$, and $\frac{\partial \varphi(t, z)}{\partial z} \neq 0$ at any $(t, z) \in \partial \mathcal{D}$.

- A Stein manifold has a C^ω strictly plurisubharmonic exhaustion function.

We discuss here some analytic invariants associated with a holomorphic family of open Riemann surfaces.

Smooth variation of open Riemann surfaces $(R(t), \chi(t))$

Let $\pi : \tilde{\mathcal{R}} \rightarrow \Delta$ be a holomorphic submersion of a 2-dimensional complex manifold $\tilde{\mathcal{R}}$ to a disk $\Delta = \{t \in \mathbb{C} \mid |t| < r\}$. Assume that $\tilde{R}(t) := \pi^{-1}(t)$, $t \in \Delta$, is **non-compact** and irreducible.

Let $\pi|_{\mathcal{R}} : \mathcal{R} \rightarrow \Delta$ be a sub-holomorphic family of $\tilde{\mathcal{R}}$ s.t. each fiber $(\pi|_{\mathcal{R}})^{-1}(t) =: R(t) (\subseteq \tilde{R}(t))$ is **an open Riemann surface of genus g ($1 \leq g < \infty$)** with **C^ω smooth boundary $\partial R(t) = \sum_{j=1}^{\nu} C_j(t)$** in $\tilde{R}(t)$.

We identify $\mathcal{R}$ with the **smooth variation of $R(t)$** in $\tilde{\mathcal{R}}$.

For $t \in \Delta$, we may take a canonical homology basis

$\chi(t) = \{A_k(t), B_k(t)\}_{k=1}^g$ of $R(t)$ mod dividing cycles $C_j(t)$ ($j = 1, \dots, \nu$) s.t. all $A_k(t)$ and $B_k(t)$ move continuously in $\mathcal{R}$ with $t \in \Delta$.

Variational formula for the Robin constant

Let D be a domain in $\mathbb{C}$ s.t. $\partial D = \sum_{j=1}^{\nu} C_j$ is C^ω smooth, and fix $\zeta \in D$.

Definition 5 (the Green function $g(z, \zeta)$ for (D, ζ))

$g(z, \zeta)$ is harmonic on $D \setminus \{\zeta\}$ and continuous on $\overline{D}$ s.t.

① $g(z, \zeta)$ has a pole of $\log \frac{1}{|z-\zeta|}$,

② $g(z, \zeta) = 0$ on C_j ($j = 1, \dots, \nu$).

$\lambda(\zeta) := \lim_{z \rightarrow \zeta} (g(z, \zeta) - \log \frac{1}{|z-\zeta|})$ is called the Robin constant for (D, ζ) .

Fact 6 (Schiffer 1946, Suita 1972)

Let $K(z, \zeta)$ be the Bergman kernel function on D , and $K(\zeta, \zeta)$ be the Bergman kernel on D . Then

$$K(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \bar{\zeta}} \text{ (by Schiffer),} \quad K(\zeta, \zeta) = -\frac{1}{\pi} \frac{\partial^2 \lambda(\zeta)}{\partial \zeta \partial \bar{\zeta}} \text{ (by Suita).}$$

Assume that a smooth variation $\mathcal{D}$ contains $\Delta \times \{0\}$. Each $D(t)$ carries the Green function $g(t, z)$ and the Robin constant $\lambda(t)$ for $(D(t), 0)$. Then,

Fact 7 (Hadamard; 1907, Mém.l'Acadé.de Sci.)

$$\frac{\partial \lambda(t)}{\partial t} = -\frac{1}{\pi} \int_{\partial D(t)} \kappa_1(t, z) \left| \frac{\partial g(t, z)}{\partial z} \right|^2 ds_z.$$

Fact 8 (Levenberg-Yamaguchi; 1991, Mem.AMS.)

$$\frac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial D(t)} \kappa_2(t, z) \left| \frac{\partial g(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{D(t)} \left| \frac{\partial^2 g(t, z)}{\partial t \partial \bar{z}} \right|^2 dx dy.$$

Here, for the defining function $\varphi(t, z)$ of $\partial \mathcal{D}$ in $\Delta \times \mathbb{C}$, we set

$$\kappa_1(t, z) = \frac{\partial \varphi}{\partial t} \left| \frac{\partial \varphi}{\partial z} \right|^{-1},$$

$$\kappa_2(t, z) = \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2 \operatorname{Re} \left\{ \frac{\partial^2 \varphi}{\partial t \partial \bar{z}} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right) \left| \frac{\partial \varphi}{\partial z} \right|^{-3} \text{ on } \partial \mathcal{D}.$$

The total space $\mathcal{D}$ is a 2-dim pseudoconvex domain in $\Delta \times \mathbb{C}_z$.

$\implies \kappa_2(t, z) \geq 0 \implies \lambda(t)$ is superharmonic on Δ .

Fact 9 (Maitani-Yamaguchi, 2004)

Let $K(t, z, \zeta)$ be the Bergman kernel function on $R(t)$, $t \in \Delta$.
 If $\mathcal{R} = \cup_{t \in \Delta} (t, R(t))$ is a 2-dim pseudoconvex domain in $\Delta \times \tilde{R}$,
 then $\log K(t, \zeta, \zeta)$ is plurisubharmonic on $\mathcal{R}$.

Fact 10 (Berndtsson, 2006)

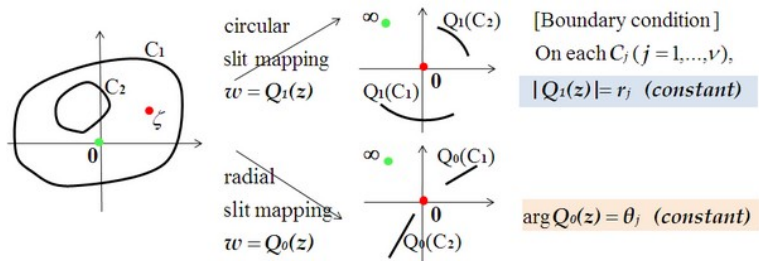
Let $\mathcal{D}$ be a pseudoconvex domain in $\mathbb{C}_t^k \times \mathbb{C}_z^n$, and ϕ be a plurisubharmonic function on $\mathcal{D}$. For each t , let ϕ^t be the restriction of ϕ to the n -dim slice $\mathcal{D}_t = \{z \mid (t, z) \in \mathcal{D}\}$ of $\mathcal{D}$, and denote by $K_t(z, \zeta)$ the Bergman kernel function of $\mathcal{D}_t$ with the weight function ϕ^t .
 $\implies \log K_t(\zeta, \zeta)$ is plurisubharmonic on $\mathcal{D}$.

Variational formulas for principal functions

$\mathbb{C}_z \supset D \supset \{0, \zeta\}$, $\partial D = \sum_{j=1}^{\nu} C_j : C^\omega$ smooth.

Let $\mathcal{Q}(D)$ be the set of all univalent functions $w = Q(z)$ on D s.t.

$$Q(z) = \begin{cases} \frac{1}{z} + b_0 + b_1 z + \cdots & \text{at } z = 0, \\ B_1(z - \zeta) + B_2(z - \zeta)^2 + \cdots & \text{at } z = \zeta. \end{cases}$$



- $q_i(z) := \log |Q_i(z)|$ ($i = 1, 0$) : L_i -principal function for $(D, 0, \zeta)$.
- $\beta_i := \log \left| \frac{dQ_i}{dz}(\zeta) \right|$ ($i = 1, 0$) : L_i -constant for $(D, 0, \zeta)$.
- $h := \beta_1 - \beta_0$: harmonic span for $(D, 0, \zeta)$.

For $t \in \Delta = \{|t| < \rho\} \subset \mathbb{C}$, let $R(t)$ be a bordered Riemann surface of genus g (≥ 0). Let $\{A_k(t), B_k(t)\}_{k=1}^g$ be a canonical homology basis of $R(t)$ modulo dividing cycles $\partial R(t)$. Assume that $R(t) \ni \{0, \zeta(t)\}$ and $z = \zeta(t) (\neq 0)$ is holomorphic on Δ .

Lemma 11 (H; 2011, Michigan)

$$\frac{\partial^2 \beta_1(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} \kappa_2(t, z) \left| \frac{\partial q_1(t, z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 q_1(t, z)}{\partial \bar{t} \partial z} \right|^2 dx dy.$$

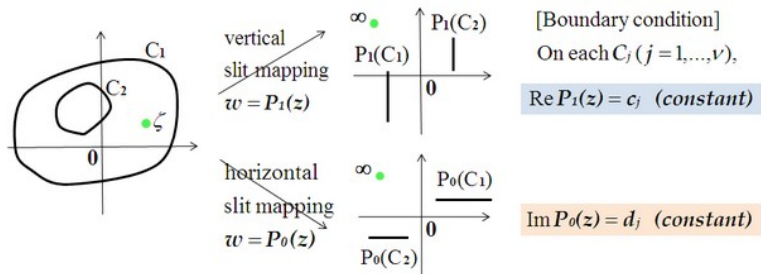
→ The simultaneous uniformization of the Schottky covering of cpt R.S.

Lemma 12 (H-Maitani-Yamaguchi; 2011, Nagoya)

$$\begin{aligned} \frac{\partial^2 \beta_0(t)}{\partial t \partial \bar{t}} = & -\frac{1}{\pi} \int_{\partial R(t)} \kappa_2(t, z) \left| \frac{\partial q_0(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 q_0(t, z)}{\partial \bar{t} \partial z} \right|^2 dx dy \\ & - \frac{2}{\pi} \operatorname{Im} \left\{ \sum_{k=1}^g \frac{\partial}{\partial t} \left(\int_{A_k(t)} *dq_0(t, z) \right) \frac{\partial}{\partial \bar{t}} \left(\int_{B_k(t)} *dq_0(t, z) \right) \right\}. \end{aligned}$$

If $\mathcal{R}$ is pseudoconvex in $\tilde{\mathcal{R}}$ and each $R(t)$ is planar, then the harmonic span $h(t) := \beta_1(t) - \beta_0(t)$ for $(R(t), 0, \zeta(t))$ is subharmonic on Δ .

$$P(z) = \frac{1}{z - \zeta} + 0 + A_1(z - \zeta) + A_2(z - \zeta)^2 + \cdots \quad \text{at } z = \zeta.$$



M.Schiffer (1943) showed that the maximizing function of the Schiffer span s for (D, ζ) is $\frac{P_1(z) + P_0(z)}{2} \in \mathcal{P}(D)$.

- $p_i(z) := \text{Re } P_i(z)$ ($i = 1, 0$) : L_i -principal function for (D, ζ) .
- $\alpha_i := \text{Re } A_{i,1}$ ($i = 1, 0$) : L_i -constant for (D, ζ) .
- $s := \alpha_0 - \alpha_1$: Schiffer span for (D, ζ) .

Assume that the total space $\pi : \mathcal{D} \rightarrow \Delta$ contains $\Delta \times \{\zeta\}$, and each $D(t) := \pi^{-1}(t)$ is **planar**.

Lemma 13 (H; 2014, Polon.Math.)

$$\begin{aligned} \frac{\partial^2 s(t)}{\partial t \partial \bar{t}} = & \frac{1}{\pi} \int_{\partial D(t)} \kappa_2(t, z) \left(\left| \frac{\partial p_0(t, z)}{\partial z} \right|^2 + \left| \frac{\partial p_1(t, z)}{\partial z} \right|^2 \right) ds_z \\ & + \frac{4}{\pi} \iint_{D(t)} \left(\left| \frac{\partial^2 p_0(t, z)}{\partial \bar{t} \partial z} \right|^2 + \left| \frac{\partial^2 p_1(t, z)}{\partial \bar{t} \partial z} \right|^2 \right) dx dy. \end{aligned}$$

Theorem 14 (H; 2014, Polon.Math.)

$\mathcal{D}$ is a **2-dim pseudoconvex domain** in $\Delta \times \mathbb{C}_z$

$\implies \boxed{s(t) := \alpha_0(t) - \alpha_1(t)}$ is **subharmonic** on Δ .

$\implies$ Schiffer span $s(t)$ for $(D(t), \zeta)$ is **logarithmic subharmonic** on Δ .

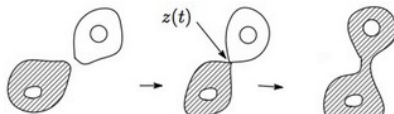
$\longrightarrow$ The simultaneous uniformization of **planar** $R(t)$, $t \in \Delta$, **of class O_{AD}** .

Non-smooth variations under pseudoconvexity

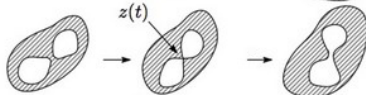
$\mathcal{D} = \cup_{t \in \Delta} (t, D(t))$ is
pseudoconvex in $\Delta \times \mathbb{C}_z$



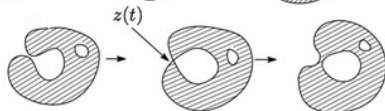
(FI)



(FII)



(FIII)



$D(t'), t' \in \Delta'$

$D(t), t \in \ell$

$D(t''), t'' \in \Delta''$

Lemma 15 (H; 2013, JMSJ)

- ① For $\mathcal{D}$ of type (FI), the harmonic span $h(t)$ for $(D(t), 0, \zeta(t))$ is C^1 subharmonic on Δ (but not of class C^2 , in general).
- ② For $\mathcal{D}$ of type (FII) or (FIII), there exists a counterexample s.t. $h(t)$ is neither of class C^1 nor subharmonic on Δ .

How are they related?

Theorem 16 (H; 2016, Math Z.)

Let R be a finite bordered **planar** Riemann surface. Then

- ① The metrics $s(\zeta)|d\zeta|^2$ and $\frac{1}{\pi} \frac{\partial^2 h(\zeta)}{\partial \zeta \partial \bar{\zeta}} |d\zeta|^2$ are identical on R ;
- ② $s(\zeta)|d\zeta|^2$ is of negative curvature at every point $\zeta \in R$ and complete on R .

Theorem 17 (H; 2016, Math Z.)

Let $R(t)$ be a bordered **planar** Riemann surface in $\tilde{R}$. Assume that total space $\mathcal{R}$ is **2-dimensional pseudoconvex** in $\Delta \times \tilde{R}$, and $\mathcal{R}$ contains two holomorphic sections σ_0 and σ_ζ over Δ with $\sigma_0 \cap \sigma_\zeta = \emptyset$. Then $\log s(t, \zeta)$ and $\log \frac{\partial^2 h(t, \zeta)}{\partial \zeta \partial \bar{\zeta}} |d\zeta|^2$ are plurisubharmonic in two variables (t, ζ) on $\mathcal{R}$.

Remark. This is the same phenomenon as the Bergman metric $K(t, \zeta, \bar{\zeta})|d\zeta|^2$ is in Fact 9.

Reproducing kernel $\tilde{K}(\zeta, \zeta)$ for $S(R)$

Let R be a bordered Riemann surface of genus $g(\geq 0)$ with a finite number of C^ω -smooth contours C_j ($j = 1, \dots, \nu$) in a larger Riemann surface $\tilde{R}$.

Let $A(R)$ be the space of L^2 -analytic differentials on R .

Let $\tilde{S}(R)$ be the space of **semi-exact** L^2 -analytic differentials on R .

Let $\tilde{K}(z, \zeta)$ be the reproducing kernel function for $S(R)$.

We have two metrics $ds^2 := K(\zeta, \zeta)|d\zeta|^2$ and $\tilde{ds}^2 := \tilde{K}(\zeta, \zeta)|d\zeta|^2$.

In view of the orthogonal relations $A(R) = S(R) \oplus S(R)^\perp$,

we see that $\tilde{ds}^2 - ds^2 \leq 0$. For example, **for the annulus**

$A := \{\sqrt{r} < |z| < \frac{1}{\sqrt{r}}\}$ ($0 < r < 1$),

$$\tilde{K}(\zeta, \zeta) - K(\zeta, \zeta) = \frac{1}{2\pi|\zeta|^2 \log r} < 0.$$

Lemma 18 (H; 2016, MZ)

$$\tilde{K}(z, \zeta) = \frac{2}{\pi} \frac{\partial^2 q_1(z, \zeta)}{\partial z \partial \bar{\zeta}} = -\frac{2}{\pi} \frac{\partial^2 q_0(z, \zeta)}{\partial z \partial \bar{\zeta}}, \quad \tilde{K}(\zeta, \zeta) = \frac{1}{2\pi} \frac{\partial^2 h(\zeta)}{\partial \zeta \partial \bar{\zeta}}.$$

Closings of an open Riemann surface

For a given open Riemann surface R of genus g ($1 \leq g < \infty$), we fix a canonical homology basis $\chi^R = \{A_j^R, B_j^R\}_{j=1}^g$ of R modulo dividing cycles.

Definition 19 (closings)

Let S be a closed Riemann surface of the same genus g , $\chi^S = \{A_j^S, B_j^S\}_{j=1}^g$ be a canonical homology basis of S , and ι be a conformal embedding of R into S such that

$$\begin{aligned}\iota(A_j^R) &\text{ is } \text{homologous} \text{ to } A_j^S \text{ and} \\ \iota(B_j^R) &\text{ is } \text{homologous} \text{ to } B_j^S \quad (j = 1, 2, \dots, g).\end{aligned}$$

Two triplets (S, χ^S, ι) and $(S', \chi^{S'}, \iota')$ are called *equivalent* if there exists a conformal bijection f of S onto S' such that $f \circ \iota = \iota'$.

Each equivalence class is referred to as a *closing of (R, χ^R)* . We denote by $\mathcal{C}(R, \chi^R)$ the set of all closings of (R, χ^R) .

Our aim is to study the set $\mathfrak{M}(R, \chi^R)$ of the period matrices of $\mathcal{C}(R, \chi^R)$.

- Oikawa (1957): $\mathfrak{M}(R, \chi^R)$ is compact and connected in $\mathfrak{T}_g$.
- Shiba (1987): $g = 1$, the set $\mathfrak{M}(R, \chi^R)$ of moduli is a closed disk in $\mathbb{H}$.
- Schmieder-Shiba (1998) and Masumoto (1999): each diagonal element τ_{jj} of $T[S, \chi^S, \iota]$ is a closed disk $\mathfrak{M}_j(R, \chi^R)$ in $\mathbb{H}$ for $2 \leq j \leq g (< \infty)$.

Remark

In the **higher genus cases**, even at a boundary point of $\mathfrak{M}_j(R, \chi^R)$, the closing of (R, χ) is **not always uniquely determined**.

Theorem 20 (H-Shiba-Yamaguchi (2017, Kyoto); $g = 1$)

Let $\pi : \mathcal{R} \rightarrow \Delta$ be a smooth variation of open tori $(R(t), \chi(t))$, $t \in \Delta$. Assume that $\mathcal{R}$ is **pseudoconvex** in $\tilde{\mathcal{R}}$. Then the following holds:

- ① the **hyperbolic diameter** $\sigma(t)$ of $\mathfrak{M}(R(t), \chi^R(t))$ is **subharmonic** on Δ .
- ② $\sigma(t)$ is harmonic on Δ if and only if $\mathcal{R}$ is a trivial family $\Delta \times R(0)$.

Theorem 21 (H (2022, Math.Z.); $1 \leq g < \infty$)

For a marked open Riemann surface R of genus g ($1 \leq g < \infty$) and a real g -vector $\mathbf{a} = (a_1, \dots, a_g) \neq \mathbf{0}$, the diameter of the **$\mathbf{a}$ -directional moduli disk** $\mathfrak{M}_{\mathbf{a}}(R(t))$ for higher genera is **subharmonic** when $\mathcal{R}$ is **pseudoconvex**.

Variational formula for hydrodynamic differentials

Definition 22 (hydrodynamic differential $\phi^\theta \in \Sigma^\theta(R)$)

Let $\theta \in \mathbb{R}$. A holomorphic differential ϕ^θ on an open Riemann surface R is called a *hydrodynamic differential with parameter value θ* if

- ① outside a compact set on R , exact $\phi = d\Phi$ and Dirichlet finite,
- ② on each component of ∂R , $\text{Im}[e^{-\pi i \theta / 2} \Phi] = \text{const. along } \partial R$.

Lemma 23 (H-Shiba-Yamaguchi (2017, Kyoto); $g = 1$)

Lemma 24 (H (2022, Math.Z.); $1 \leq g < \infty$, $\phi_{\mathbf{a}}^\theta \in \Sigma^\theta(R)$ for $\theta = 1, 0$)

Let $\phi_{\mathbf{a}}^\theta(t, z) = f_{\mathbf{a}}^\theta(t, z)dz$ ($\theta = 1, 0$) by use the local parameter z of $\overline{R(t)}$.
 Let $\tau_{\mathbf{a}}^\theta(t) = \sum_{k=1}^g \int_{A_k} \phi_{\mathbf{a}}^\theta(t, z) \int_{B_k} \phi_{\mathbf{a}}^\theta(t, z)$. Then, for $t \in \Delta$,

$$\frac{\partial^2 \text{Im} \tau_{\mathbf{a}}^1(t)}{\partial t \partial \bar{t}} = \frac{1}{2} \int_{\partial R(t)} \kappa(t, z) |f_{\mathbf{a}}^1(t, z)|^2 |dz| + \left\| \frac{\partial \phi_{\mathbf{a}}^1(t, z)}{\partial \bar{t}} \right\|_{R(t)}^2,$$

$$\frac{\partial^2 \text{Im} \tau_{\mathbf{a}}^0(t)}{\partial t \partial \bar{t}} = - \left(\frac{1}{2} \int_{\partial R(t)} \kappa(t, z) |f_{\mathbf{a}}^0(t, z)|^2 |dz| + \left\| \frac{\partial \phi_{\mathbf{a}}^0(t, z)}{\partial \bar{t}} \right\|_{R(t)}^2 \right).$$

Main Theorem 1 (H-Shiba, submitted)

Let (R, χ^R) be a marked open Riemann surface of finite genus g .

Then, the period matrices T of the closings of (R, χ^R) satisfy a system of inequalities in $\mathfrak{S}_g$:

$$|\mathbf{c} (T - \textcolor{red}{T}^*) {}^t \mathbf{c}| \leq \bar{\mathbf{c}} \textcolor{blue}{Q} {}^t \mathbf{c}$$

for every $\mathbf{c} \in \mathbb{C}^g$. Here,

$$\exists 1 \textcolor{red}{T}^* = \frac{\textcolor{red}{T}^\theta + \textcolor{red}{T}^{\theta+1}}{2} \in \mathfrak{S}_g \quad \text{and} \quad \frac{\textcolor{blue}{T}^\theta - \textcolor{blue}{T}^{\theta+1}}{2} = e^{\pi i(\theta - \frac{1}{2})} \textcolor{blue}{Q} \notin \mathfrak{S}_g,$$

where the $g \times g$ matrix Q is a positive semi-definite Hermitian matrix. Equality holds for some if and only if T is the period matrix of a *hydrodynamic closing* of (R, χ^R) .

Main Theorem 2 (H, submitted)

If $\mathcal{R}$ is a two-dimensional pseudoconvex domain fibered by open Riemann surfaces $R(t)$ of the same topological type, then $\rho_{\mathbf{c}}(t)$ is subharmonic for t .

Thank you so much for your attention.

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