COLLECTIVE COHN-VOSSEN PROBLEM, PARABOLIC LOCALIZATION PRINCIPLE AND PERIOD CONDITION OF ALGEBRAIC MINIMAL SURFACES

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Abstract. According to the interpretation proposed in [KKM] of Osserman's theory of algebraic minimal surfaces [O1,2], this theory studies the value distribution of the Gauss map of algebraic minimal surfaces by estimating the basic ratio R, which is the ratio of the Fubini-Study area against the hyperbolic area, of the algebraic minimal surface under question, by using the Riemann-Roch (or Gauss-Bonnet) formula under the presence of the period condition. In this paper, we further develop the approach of [KKR] :

(i) We propose a "partition function" Z_{1-r} involving a parameter $1-r$ ($0<r<1$ being the radius of the disk $|z|<r$) which formally yields the basic ratio R if we take the limit $1-r\to 0$ before performing the (potentially infinite) sum which counts states with certain weights. We reveal what kind of mathematics emerges if we interchange the order of taking the (potentially infinite) sum and taking the limit $1-r\to 0$ into the standard order in the semi-classical limit. We show that this change of limits, i.e., the study of the asymptotic behavior of the partition function Z_{1-r} when $1-r\to 0$ ($r\to 1$) reduces to the Nevanlinna Theoretical study of the Weierstrass data.

(ii) We propose a version of the Nevanlinna theory coupled with the $\pi_1(M)$-action on the unit disk $\mathbb{D}=\{|z|<1\}$ where $\pi_1(M)$ is the fundamental group of the algebraic minimal surface under question. This theory, which we call the Nevanlinna-Galois theory, fits to the purpose of investigating the asymptotic behavior of the partition function Z_{1-r}. We reveal the prominent role of the action of $\pi_1(M)$ (a free Fuchsian group) on $\mathbb{D}$ in the classical minimal surface theory whose effect on the Nevanlinna-Galois theory culminates in the "collective Cohn-Vossen inequality". As an application, we establish an effective version of the Lemma on Logarithmic Derivative (LLD) valid for the lifted Gauss map of algebraic minimal surfaces and more generally for meromorphic functions on $\mathbb{D}$ whose height transform has comparable growth as $\log\frac{1}{1-r}$ when $r\to 1$.

(iii) We propose a Nevanlinna theory interpretation of the period condition of algebraic minimal surfaces. More precisely, we propose a canonical construction, from the Weierstrass data, of a pair $(e^H,\mathcal{D})$ of a holomorphic function e^H on $\mathbb{D}$ and a divisor $\mathcal{D}$ (potentially of infinite degree) on $\mathbb{P}^1$ so that e^H maximally approximates $\mathcal{D}$ and the pair $(e^H,\mathcal{D})$ fully encodes the period condition. We show that we can apply our effective LLD to the pair $(e^H,\mathcal{D})$. Combining these two results, we establish Nevanlinna's Second Main Theorem for the Gauss map and give the optimal upper bound for the totally ramified value number of the lifted Gauss map of algebraic minimal surfaces. In particular, we prove that the Gauss map of any algebraic minimal surface can omit at most two values.

§1. Introduction.

This note is an extended version of my lecture delivered at 2019 Oka Symposium. My lecture was based on the preprints [KM1,2]. These papers are very much involved. Therefore, the purpose of this note is to introduce main ideas in [KM1,2] with geometric intuitions behind results in [KM1,2] rather than proofs. In the course of preparing this note, I discovered the reason why the Cohn-Vossen ratio R (for definition, see below) being 2 is so special in the theory of algebraic minimal surfaces. In this note, I will put emphasis on this discovery.

We study the period condition of algebraic minimal surfaces from the view point of the action of a free Fuchsian group action on $\mathbb{D}$ and Nevanlinna theory. An open Riemann surface M on which the Weierstrass data (g,ω) is defined is called the **basic domain**. We write $(M,(g,\omega))$ for an algebraic minimal surface, i.e., a complete minimal surface in $\mathbb{R}^3$ with finite total curvature, where M is an open Riemann surface, (g,ω) the Weierstrass data, i.e., $g: M\to\mathbb{P}^1$ the Gauss map and ω a holomorphic 1-form on M satisfying the regularity condition and the period condition. The algebraic minimal surface

M is immersed in $\mathbb{R}^3$ by the Ennneper-Weierstrass representation formula

$$M \ni z \longmapsto \int_{z_0}^{z} \frac{1}{2}\Re \begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} \omega \in \mathbb{R}^3 \ .$$

The regularity condition means that the induced metric $ds^2 = \frac{1}{4}(1+|g|^2)^2)|\omega|^2$ is positive, i.e., g has a pole of order m at $P \in M$ if and only if ω has zero of order $2m$. The period condition means that the Enneper-Weierstrass representation is single-valued, i.e., we have

$$\frac{1}{2}\int_{\gamma} \begin{pmatrix} 1-g^2 \\ i(1+g^2) \\ 2g \end{pmatrix} \omega \in i\,\mathbb{R}^3 \ , \quad \forall \gamma \in H_1(M,\mathbb{Z}) \ .$$

We denote ω_{FS} (resp. ω_{hyp}) for the Fubini-Study metric on $\mathbb{P}^1$ (resp. the Poincaré metric on $\mathbb{D}$) normalized so that the absolute value of the Gaussian curvature is 1. In [KKM], the authors have shown that the Cohn-Vossen ratio

$$R = \frac{-\int_M g^*\omega_{\mathrm{FS}}}{\int_M \omega_{\mathrm{hyp}}}$$

plays the essential role in Osserman's theory of algebraic minimal surfaces [O1,2]. Namely, Osserman's main results are expressed in terms of the Cohn-Vossen ratio by

$$R > 1 \quad (\,\text{Riemann-Roch} + \text{period condition}\,)$$

and

$$\nu_g \le 2 + \frac{2}{R} \quad (\,\text{Riemann-Hurwitz}\,)$$

for any algebraic minimal surface. Here ν_g is the totally ramified value number of the Gauss map (cf. [KKM]). In particular we have $\nu_g < 4$ and therefore the Gauss map can omit at most three values. It is a folklore conjecture that the Gauss map of an algebraic minimal surface can omit at most two values. This conjecture is called Osserman's problem. If $R > 2$ then $\nu_g < 3$ holds. In particular, the Gauss map g can omit at most two values. Therefore, to settle Osserman's problem, it suffices to show that $\nu_g < 3$ holds under the assumption $R \le 2$. In [KM2, Part II], the authors obtained the Second Main Theorem for the Gauss map of an algebraic minimal surface $(M,(g,\omega))$ under the assumption $R \le 2$. The estimate $\nu_g < 3$ follows from the Second Main Theorem.

The purpose of the present note is to prove the Second Main Theorem for the (lifted) Gauss map of an algebraic minimal surface without assuming anything on the Cohn-Vossen ratio. Our strategy in this note is essentially the same as in [KM2, Part II]. We will follow the argument in [KM2, Part II]. At every place where we use inequalities proved in [KM2, Part II] under the assumption $R \le 2$, we replace them with more general (but weaker) ones which hold without assuming anything on R. The main point is Theorem 2.8.3.1 which states that $N_{g,\infty}(r)$ (and therefore $m_{g,\infty}(r)$ also) behaves in a special way relative to $T_g(r)$, where ∞ (resp. 0) $\notin \{g(P)\,|\,P \in \overline{M}\setminus M \text{ or } dg(P)=0\}$ is the repelling (resp. absorbing) fixed point of the hyperbolic translation T and the metric of $\mathbb{P}^1$ on which $T_g(r)$ is defined is the pillow-case metric $\omega_{\mathrm{FS}}^{\mathrm{modified}}$, where T and $\omega_{\mathrm{FS}}^{\mathrm{modified}}$ appeared in the proof of Metrized Riemann-Hurwitz Theorem [Part I, Theorem 4.2] (see **2.4** of the present note).

Theorem 1.1 (Theorem 2.8.3.1) (parabolic localization in terms of group theoretic approximation). *Suppose that $(M,(g,\omega))$ be a pseudo-algebraic minimal surface. For $\infty \notin \{g(P)\,|\,P \in \overline{M}\setminus M$ or $dg(P)=0\}$ which appears in the proof of Metrized Riemann-Hurwitz Theorem [Part I, Theorem 4.2], we have*

$$N_{g,\infty}(r) \sim O(n_{\mathrm{th}}^{-1})\,T_g(r) \ ,$$
$$m_{g,\infty}(r) \sim (1-O(n_{\mathrm{th}}^{-1}))\,T_g(r) \ .$$

Here the metric on $\mathbb{P}^1$ with which $T_g(r)$ is defined is the pillow-case metric $\omega_{\mathrm{FS}}^{\mathrm{modified}}$ in Metrizd Riemann-Hurwitz Theorem [Part I, Theorem 4.2].

Here $n_{\rm th}$ is the *threshold of the Euclidean distortion* introduced in [KM1] (for definition, see **2.3** of the present note), which can be taken arbitrarily large positive number. Theorem 1.1 is a Nevanlinna Theoretic interpretation of the parabolic localization principle introduced in [KM1,2] (see **2.3** of the present note). The "parabolic localization" was introduced in [KM1] which is a sort of localization phenomenon arising from the action of the free Fuchsian group $\pi_1(M)$ on $\mathbb{D}$ (for a brief discussion, see **2.3** of the present note). For the meaning of the "group theoretic approximation", see **2.8.3** of the present note.

The Main Theorem of the present note (in fact, the same as that of [KM2]) is the following :

Theorem 1.2 (Theorem 2.10.6) (Second Main Theorem for the Gauss map of an algebraic minimal surface). *Let $D = \{a_1, \dots, a_q\}$ be a set of distinct points of $\mathbb{P}^1$. The lifted Gauss map $g : \mathbb{D} \to \mathbb{P}^1$ of an algebraic minimal surface $(M, (g, \omega))$ satisfies the Second Main Theorem*

$$m_{g,D}(r) + N_{g,\mathrm{Ram}}(r) \le \{4(e-2) + \varepsilon)\}\, T_g(r) \ ,$$

where ε is any small positive number.

In particular, we have $\nu_g \le 4(e-2) \le 2.88$ and therefore the Gauss map g can omit at most two values.

§2. Proof of Main Theorem.

2.1. Partition Function.

Let M be an open Riemann surface and (g, ω) a Weierstrass data on M, where $g : M \to \mathbb{P}^1$ is the (stereographically projected) Gauss map and ω is a holomorphic 1-form satisfying the regularity condition, i.e., g has a pole of order m at $P \in M$ if and only if ω has zero of order $2m$ at the same point. The Enneper-Weierstrass representation

$$M \ni z \longmapsto X(z) := \int_{z_0}^{z} \frac{1}{2} \Re \begin{pmatrix} 1 - g^2 \\ i(1 + g^2) \\ 2g \end{pmatrix} \omega \in \mathbb{R}^3$$

defines a holomorphic map from the universal covering surface of M into $\mathbb{R}^3$. We say that $(M, (g, \omega))$ is a **pseudo-algebraic minimal surface** if M is a finite Riemann surface, i.e., $M = \overline{M} \setminus \{P_1, \dots, P_{n(M)}\}$ where $\overline{M}$ is a compact Riemann surface, and the Weierstrass data (g, ω) extends meromorphically across the punctured points $\{P_1, \dots, P_{n(M)}\}$. A pseudo-algebraic minimal surface is an **algebraic minimal surface** if in addition the period condition is satisfied :

$$\frac{1}{2} \int_\gamma \begin{pmatrix} 1 - g^2 \\ i(1 + g^2) \\ 2g \end{pmatrix} \omega \in i\,\mathbb{R}^3 \ , \quad \forall \gamma \in H_1(M, \mathbb{Z}) \ ,$$

i.e., the Enneper-Weierstrass representation $X(z)$ is single-valued (by Huber [Hu] and Osserman [O1,2], this definition of algebraic minimal surface coincides with the classical one). Throughout this paper we assume that the universal covering surface $\widetilde{M} = \mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$ is the unit disk in $\mathbb{C}$.

The first step toward the Second Main Theorem for the lifted Gauss map $g : \mathbb{D} \to \mathbb{P}^1$ is to replace the Cohn-Vossen ratio R by the partition function Z defined by

$$Z(r) = \sum_{\alpha \in \pi_1(M)} \frac{\mathrm{Area}_{\mathrm{FS}}(F_\alpha \cap \mathbb{D}(r))}{\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(r))} \ .$$

As soon as we fix a reference fundamental domain F, we have a one to one correspondence between $\alpha \in \pi_1(M)$ and a fundamental domain $F_\alpha = \alpha(F)$. We then count the number of elements α of $\pi_1(M)$ with the Cohn-Vossen type weight

$$\frac{\mathrm{Area}_{\mathrm{FS}}(F_\alpha \cap \mathbb{D}(r))}{\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}(r))} \ , \ \mathbb{D}(r) = \{z \in \mathbb{D} \,|\, |z| < r\}$$

to define the partition function $Z(r)$. In particular, for a fixed $0 < r < 1$, the sum in $Z(r)$ is finite. The idea behind the replacement $R \mapsto Z(r)$ is to embed Osserman's problem into the realm of the free Fuchsian group $(\pi_1(M))$-action and Nevanlinna Theory on $\mathbb{D}$ by lifting the Weierstrass data (g, ω) to the universal covering $\mathbb{D}$ without looking at individual algebraic minimal surface $M = \pi_1(M)\backslash\mathbb{D}$ with unlifted Weierstrass data (g, ω). The number $1 - r$ plays the role of the Planck constant. The study of the semi-classical limit $\lim_{r\to 1} Z(r)$ reduces to the comparison of following two Nevanlinna Theoretic functions $T_g(r)$ and $T_{\rm hyp}(r)$, i.e., the height transform of $\frac{1}{4\pi}\omega_{\rm FS}$ and $\frac{1}{4\pi}\omega_{\rm hyp}$, in the presence of the $\pi_1(M)$-action on $\mathbb{D}$. Here,

$$\omega_{\rm FS} = 2\sqrt{-1}\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \quad \text{on } \mathbb{P}^1 \ ,$$
$$\omega_{\rm hyp} = 2\sqrt{-1}\frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2} \quad \text{on } \mathbb{D}^1 \ .$$

are the Fubini-Study metric and the Poinbcaré metric normalized so that the absolute value of the Gaussian curvature is 1. Explicitly $T_g(r)$ and $T_{\rm hyp}(r)$ are defined as

$$T_g(r) = \int_0^r \frac{dt}{t} \int_{\mathbb{D}(t)} \frac{1}{4\pi} g^*\omega_{\rm FS} \ ,$$
$$T_{\rm hyp}(r) = \int_0^r \frac{dt}{t} \int_{\mathbb{D}(t)} \frac{1}{4\pi}\omega_{\rm hyp} \overset{\text{asymptotically as } r\to 1}{=} \frac{1}{2}\log\frac{1}{1-r} \ .$$

2.2. Period Condition.

Every minimal surface is locally represented by the Enneper-Weierstrass representation formula. The freedom of rotating a minimal surface in $\mathbb{R}^3$ corresponds to the freedom in the choice of a unit tangent vector from the tangent bundle of

$$\mathbb{S} := \left\{ a\frac{1}{2}(1 - g^2) + b\frac{i}{2}(1 + g^2) + cg \,\middle|\, (a, b, c) \in \mathbb{R}^3 \ , \ a^2 + b^2 + c^2 = 1 \right\} ,$$

where $\mathbb{S}$ is the unit sphere of the 3-dimensional $\mathbb{R}$-vector space V spanned by the ON basis $\{\frac{1}{2}(1 - g^2), \frac{i}{2}(1 + g^2), g\}$. Therefore, instead of working with three components in the Enneper-Weierstrass representation formula, we choose to work with a random polynomial $p(g)$ in g chosen from $\mathbb{S}$. Given $p(g) \in \mathbb{S}$ we define

$$H(z) := \int_{z_0}^{z} p(g)\omega \ ,$$

which defines a holomorphic function on $\mathbb{D}$ by the regularity condition. If z_0 is a reference point in $\mathbb{D}$ and $\alpha \in \pi_1(M)$, then the period condition implies that there exists a $\gamma \in H_1(M, \mathbb{Z})$ such that

$$H(\alpha z) - H(z) = \int_\gamma p(g)\omega \in i\,\mathbb{R}^3 \ .$$

Therefore

$$|e^H(z)| = |e^H(\alpha z)| \quad \forall z \in \mathbb{D} \text{ and } \alpha \in \pi_1(M) \ .$$

Thus we have

Lemma 2.2.1. *The period condition is equivalent to saying that $|e^H|$ is invariant under the action of $\pi_1(M)$.*

2.3. Parabolic Localization.

In **2.3**, we do not need the period condition. Hence we work on pseudo-algebraic minimal surfaces. In order to study the semi-classical limit $\lim_{r\to 1} Z(r)$ in terms of the Nevanlinna Theory, we need to analyze the deviation from the action of $\pi_1(M)$ to $\mathbb{D}$ from the Euclidean similarity. This is because the concentric disks $|z| < r$ and their boundaries $|z| = r$ are involved in every Nevanlinna theoretic function so that every such function is a function in r. As $\mathbb{D}$ and $\mathbb{H} = \{z \in \mathbb{C} \,|\, \Im(z) > 0\}$ are conformally equivalent via the Cayley transform $\mathbb{H} \ni w \mapsto z = \frac{w-i}{w+i} \in D$, we may work on $\mathbb{H}$ and translate obtained results into $\mathbb{D}$.

In order to analyze the Euclidean distortion of fundamental domains, we consider

$$T(x) = \frac{x}{x+1}$$

which is a parabolic translation with a unique fixed point $0 \in \partial\mathbb{H}$ ($\partial\mathbb{H}$ contains ∞). By iteration we have

$$T^n(x) = \frac{x}{nx+1} \ .$$

We call the iteration of a parabolic translation a parabolic sequence. Let $I = [a, b] \subset \mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$ be a closed interval in $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$. We imagine a fundamental domain F having 0 and end points $\{a, b\}$ of I as three vertices and all other vertices locate in the interior (a, b) of I. We set $I_n := T^n(I)$ and call it the **cluster part**. A simple computation shows that

$$\mathrm{dist}_{\,\mathrm{euc}}(0, I_n) = O(n^{-1})$$

and

$$\mathrm{diam}_{\,\mathrm{euc}}(I_n) = O(n^{-2}) \ .$$

Visually this is interpreted as follows. Look at the image of $\partial\mathbb{D}(r) = \{z \in \mathbb{D} \,|\, |z| = r\}$ in M for $r < 1$ close to 1. Then as soon as the image path γ approaches to a punctured point P (corresponding to a parabolic fixed point), the image curve γ rotates many times around P and stays longer. This is the localization induced from the parabolic sequence. We call this localization phenomenon as the **parabolic localization principle**. One of the basic fact is the non-occurrence of the multi-cluster part.

Lemma 2.3.1 [KM1, Part I, Lemma 3.4.1.4] (non-existence of multi-cluster part). *There is at most one cluster part in every fundamental domain.*

Lemma 2.3.1 implies that the following definition makes sense. We call the ratio

$$\frac{\mathrm{dist}_{\,\mathrm{euc}}(0, I_n)}{\mathrm{diam}_{\,\mathrm{euc}}(I_n)}$$

the **Euclidean distortion**. We fix a large number n_{th} and say that a fundamental domain is Euclidean distorted if the Euclidean distortion is larger than n_{th} (n_{th} is the **threshold of Euclidean distortion**). The parabolic localization means the distortion in the Euclidean sense induced by the iteration of a parabolic translation and the truncation of the Euclidean distorted fundamental domains by $\mathbb{D}(r)$, r being comparable to the Euclidean height of the cluster part, has the effect of isolating the neighborhood of (more precisely, the annular domain around) the parabolic fixed point under question. This is the origin of the name "parabolic localization". In order to get useful information from the parabolic localization principle, knowing the contribution from parabolic sequences in any measure theoretic quantity associated to $\mathbb{D}(r)$ as well as the asymptotic behavior of the number N_1 (resp. N_2) of the Euclidean distorted (resp. non-distorted) fundamental domains is essential[1]. For this, we have the following basic results :

Lemma 2.3.2 [KM1, Part I, Lemma 3.4.3.7] (dominating parabolic sequences covering $\mathbb{D}(r)$). *Set*

$$\alpha(r) := 1 - \frac{2 \log n_{\mathrm{th}}}{\log \frac{1}{1-r}} \ .$$

Then the iteration images of fundamental domains of parabolic translations contained in the minimal covering of $\mathbb{D}(r)$ is dominated by those parabolic sequences starting at fundamental domains of Euclidean diameter $(1-r)^{\alpha(r)}$.

Lemma 2.3.2 indicates that we can count any measure theoretic quantity associated to $\mathbb{D}(r)$ in the following way.

[1] We count the number N_1 (resp. N_2) of Euclidean distorted (resp. non-distorted) fundamental domains in the minimal covering by fundamental domains of $\mathbb{D}(r)$ and therefore N_i should be interpreted as $N_i(r)$ $(i = 1, 2)$.

Lemma 2.3.3 [KM1, Part I, Lemma 3.4.3.8] (counting rule of quantities associated to $\mathbb{D}(r)$). *In order to count any measure theoretic quantity associated to $\mathbb{D}(r)$, we first classify all parabolic sequences according to the Euclidean diameter of the initial Euclidean non-distorted fundamental domain. Then we count the contribution from individual parabolic sequence starting at a Euclidean non-distorted fundamental domain of Euclidean diameter $O(1-t)$. Then we multiply the number $O((1-t)^{-1})$ of parabolic sequences starting at a Euclidean non distorted fundamental domain with Euclidean radius $O(1-t)$ and integrate the result against the measure*

$$d\mu(t) := \frac{dt}{1-t}$$

from, say, $t = 2^{-1}$ to $t = 1-(1-r)^{\alpha(r)}$.

For instance, we count the quantity in question in the case that the contribution from individual parabolic sequence is 1. In this case we have

$$\int_{\frac{1}{2}}^{1-(1-r)^{\alpha(r)}} \frac{1}{1-t} d\mu(t) = \int_{\frac{1}{2}}^{1-(1-r)^{\alpha(r)}} \frac{dt}{(1-t)^2} = \frac{1}{(1-r)^{\alpha(r)}} \ .$$

This counts the number of the fundamental domains of Euclidean radius $O(1-r)^{\alpha(r)}$.

Lemma 2.3.4 [KM1, Part I, Lemma 3.4.3.9-10] (comparison of numbers of Euclidean distorted / non-distorted fundamental domains covering $\mathbb{D}(r)$). *Fix any large number n_{th} as the threshold of the Euclidean distortion. Then we have*

$$\frac{N_1}{N_2} = O(n_{\mathrm{th}}(\log n_{\mathrm{th}})^{\frac{1}{2}})$$

as $r \to 1$. This means that the contribution from Euclidean distorted fundamental domains asymptotically dominates as $r \to 1$ the significant portion of the minimal covering of $\mathbb{D}(r)$ by truncated fundamental domains (the intersection of fundamental domains and $\mathbb{D}(r)$). In particular, the ratio of the contribution from parabolic sequences against the Fubini-Study area $\int_{\mathbb{D}(r)} g^*\omega_{\mathrm{FS}}$ of $\mathbb{D}(r)$ is bounded below by $1-\varepsilon$ for any $\varepsilon > 0$ uniformly when $r \to 1$.

As for the uniformity of the cluster part, we have

Lemma 2.3.5 [KM1, Part I, Lemma 3.4.1.5] (uniformity of cluster part). *The variation of the distribution of finitely many parabolic fixed points in the cluster part is uniform.*

Lemma 2.3.5 implies that hyperbolic translations in $\pi_1(M)$ do not contribute to the formation of cluster parts [KM1, Part I, Lemma 3.4.2.1]. Using Lemma 2.3.5, we can introduce a dynamical system on the boundary $S^1 = \partial\mathbb{D}$ defined by the evolution of orbits of fundamental domains and the concept of the scale. The existence of such a dynamical system is the background geometry developed in later sections.

Definition 2.3.6 [KM1, Part I, Definition 3.4.2.2](A dynamical system on $\mathbb{S}^1$ induced from the action of the free Fuchsian group $\pi(M)$ and the associated scale on $\mathbb{S}^1$). We fix a reference fundamental domain F_0 wrt. the action of $\pi_1(M)$ where M is the basic domain of a pseudo-algebraic minimal surface. Then F_0 is realized as an ideal geodesic $(4G + 2(n(M)-1))$-gon with vertices on $\mathbb{S}^1 = \partial\mathbb{D}$. We can then introduce the dynamical system on $S^1 = \partial\mathbb{D}$ arising from the vertices of fundamental domains obtained by applying words of $\pi_1(M)$, i.e., the evolving orbit of vertices of fundamental domains. The uniformity of the cluster part enables us to introduce the concept of the **scale** on $\mathbb{S}^1$ by saying that this dynamical system is in the scale $\rho > 0$ if the mesh of the distributed vertices is comparable to ρ.

2.4. Metrized Riemann-Hurwitz Theorem.

We assume that the Gauss map $g : M \to \mathbb{P}^1$ has no poles at the punctured points and all poles are simple, i.e., $0, \infty \notin \{g(P) \mid P \in \overline{M} \setminus M \text{ and } dg(P) \neq 0\}$. The parabolic localization principle truncated at $\mathbb{D}(r)$ isolates neighborhoods of the punctured points. Therefore, we need to establish a tool which allows us to get useful information from local computation around punctured points. The tool we need is the Metrized Riemann-Hurwitz Theorem. To formulate it, we need to introduce a certain **cell decomposition** of the basic domain of a pseudo-algebraic minimal surface. This is outlined as follows. The 1-form part $\omega = hdz$ of the Weierstrass data should be understood as a holomorphic map sending the coordinate vector $\frac{\partial}{\partial z}$ to the tangent vector $h(z)\frac{\partial}{\partial w}$ in $T_{g(z)}\mathbb{P}^1$. So, we can compare dg and h as linear maps T_zM to $T_{g(z)}\mathbb{P}^1$ at each $z \in M$.

Let T be a hyperbolic translation of $\mathbb{P}^1$ having ∞ (resp. 0) as a repelling (resp. absorbing) fixed point. We replace the stereo-graphically projected Gauss map $g : M \to \mathbb{P}^1$ with the composition

$$T(g) \, : \, M \xrightarrow{g} \mathbb{P}^1 \xrightarrow{T} \mathbb{P}^1 \ .$$

Then we interpret the composition $T(g)$ as a new g. Under this interpretation, the union $\{Q_j\}$ of the values $\{g(P_1), \ldots, g(P_N)\}$ of g (here g means $T(g)$) at the punctured points and the values of g at places where g ramifies are located near 0, the absorbing fixed point of T.

Of course the whole Weierstrass data (g, ω) should be replaced by this composition, i.e., we should think that the original Weierstrass data should be replaced by the composed pair

$$(\widetilde{g}, \widetilde{\omega}) := (T(g), T_*(\omega)) \ .$$

Here, the 1-form part ω of the Weierstrass data is regarded as the holomorphic map $T_zM \to T_{g(z)}\mathbb{P}^1$ and T_* means to take the image by the induced map T_* acting on tangent vectors of $\mathbb{P}^1$. Then $p(\widetilde{g})$ means the operator acting on tangent vectors in $T_{T(g)}\mathbb{P}^1$ as the multiplication by $p(\widetilde{g})$.

Remark 2.4.1 (gauge invariance of the period condition). The effect of this replacement to the period condition should be examined. Let $p(g)$ represent any linear combination of $1 - g^2$, $i(1 + g^2)$ and $2g$ (in the Enneper-Weierstrass representation) with $\mathbb{R}$-coefficients a, b, c with $a^2 + b^2 + c^2 = 1$. The integrand $p(g)\omega$ is understood as follows. Let z be any local coordinate of M and w any affine coordinate of $\mathbb{P}^1$. Let $\gamma(t)$ $(t \in [0, 1])$ be a parameterization of $\gamma \in H_1(M, \mathbb{Z})$. Then we have

$$\int_\gamma p(g)\omega = \int_0^1 dw(p(g)\omega(\dot{\gamma}(t)))dt \ .$$

Here the expression of ω depends on the choice of local coordinate functions z in M and w in $\mathbb{P}^1$. This convention understood, we have

$$\int_\gamma p(g)\omega = \int_\gamma p(\widetilde{g})\widetilde{\omega} \ , \quad \forall \gamma \in H_1(M, \mathbb{Z}) \ .$$

The period condition implies

$$\Re \int_\gamma p(g)\omega = 0 \ , \quad \forall \gamma \in H_1(M, \mathbb{Z}) \ .$$

Therefore we have

$$0 = \Re \int_\gamma p(g)\omega = \Re \int_\gamma p(\widetilde{g})\widetilde{\omega} \ , \quad \forall \gamma \in H_1(M, \mathbb{Z}) \ .$$

in general. We interpret this as the gauge invariance of the period integral. This means that the period condition is preserved in the operation $(g, \omega) \mapsto (\widetilde{g}, \widetilde{\omega})$. We say that T is a strong hyperbolic translation if a small circle centered at the repelling fixed point ∞ is mapped to a small circle centered at the absorbing fixed point 0. By the effect of composing a strong T, the g-image of all punctured points and critical points locate near the absorbing fixed point 0 where the distance to 0 is measured by the Fubini-Study metric ω_{FS} of $\mathbb{P}^1$.

From here on we work on the new $(\widetilde{g}, \widetilde{\omega})$ and therefore we mean by (g, ω) as the new $(\widetilde{g}, \widetilde{\omega}) = (T(g), T_*\omega)$ obtained from the original (g, ω) by the application of the strong parabolic translation T.

We return to the discussion on Metrized Riemann-Hurwitz Theorem. Let $\{Q_j\}$ be the collection of values of g at the punctured points and critical points, i.e., those points P satisfying $dg(P) = 0$. We proceed to the construction of a good cell decomposition of the basic domain M of a pseudo-algebraic minimal surface. We pick $\{R_k\}$ ($\sharp\{R_k\} = \sharp\{Q_j\} - 1$) near ∞, the repelling fixed point of T. We then conformally deform $\omega_{\rm FS}$ to the **pillow case metric** (i.e., two euclidean discs of radius slightly smaller than $\sqrt{2}$ bridged by highly curved equator) which we regard as a modified Fubini-Study metric $\omega_{\rm FS}^{\rm modified}$ on $\mathbb{P}^1$. We make a slit consisting of broken geodesics connecting $Q_1, R_1, Q_2, \ldots, Q_l, R_l, Q_{l+1}$ ($l + 1 = \sharp\{Q_j\}$) in this order to make a polygon. Then we develop the polygon onto M via g^{-1} (the inverse map of g) to decompose M into cells. We then take its dual decomposition. After this procedure is done, we can make a "microscope" which works well in studying the semi-classical analysis $\lim_{r\to 1} Z(r)$.

Lemma 2.4.2 [KM1, Part I, Theorem 4.2, Lemma 5.1, Corollary 5.2] (Metrized Riemann-Hurwitz Theorem). *(1) There exists a local parameter ζ_i centered at each punctured point $P_i \in \{P_1, \ldots, P_{n(M)}\}$ with the following properties :*

(1-i) The globally defined Poincaré metric $\omega_{\rm hyp}$ on M is locally expressed as

$$\frac{4|d\zeta_i|^2}{|\zeta_i|^2(\log(c^{-2}|\zeta_i|^{-2}))^2} \ , \ c^{-2} \geq 2 \ .$$

(1-ii) The pillow case metric interpreted as a modified Fubini-Study metric is $\geq 4|d\zeta_i|^2$.

(2) The ratio of the Fubini-Study area against the hyperbolic area of the disk $|\zeta_i| \leq r$ ($r^2 < 2$) is bounded below by the quantity

$$r^2 \log(2/r^2) \ .$$

(3) The stationary value of the local area ratio $\mathrm{Area}_{\rm FS}/\mathrm{Area}_{\rm hyp}$ on truncated fundamental domain $F \cap \mathbb{D}(r)$ for Euclidean distorted fundamental domain whose cluster part has Euclidean height $\leq 1 - r$ is at most $2e^{-1}$.

The stationary value calculus based on Lemma 3.4.1 (3) implies the following :

Lemma 2.4.3 [KM1, Part I, Lemma 6.1]. *The portion of the annular domain $1 - (1 - r)^{\frac{\alpha+1}{2}} < |z| < r$ ($\alpha < 1$) covered by the parabolic sequences has area ratio*

$$\mathrm{Area}_{\rm FS}/\mathrm{Area}_{\rm hyp} \geq 2e^{-1}$$

asymptotically as $r \to 1$.

2.5. Collective Cohn-Vossen Inequality.

To interpret Lemma 2.4.3 in terms of the Nevanlinna Theory, we introduce the invariant κ_g in the following way.

Definition 2.5.1 [KM1, Part I, Definition 2.2.4]. *Let $(M, (g, \omega))$ be a pseudo-algebraic minimal surface. The invariant κ_g is defined as*

$$\kappa_g = \inf\left\{\kappa \,\middle|\, \lim_{r\to 1} \int_0^r \exp(\kappa\, T_g(t))dt = \infty\right\} \ .$$

If we define $\kappa_{\rm hyp}$ similarly by using $T_{\rm hyp}(r)$, we have $\kappa_{\rm hyp}(r) = 2$. Clearly we have $\kappa_g > 0$. The invariant κ_g is characterized by the asymptotic property

$$\kappa_g T_g(r) = \log \frac{1}{1-r} \quad \text{as } r \to 1$$

in the sense that

$$\kappa_g = \lim_{r\to 1} \frac{\log \frac{1}{1-r}}{T_g(r)}$$

holds.

Theorem 2.5.2 (Collective Cohn-Vossen Inequality) [KM1, Part I, Theorem 7.1]. *For every pseudo-algebraic minimal surface in* $\mathbb{R}^3$*, we have*

$$\kappa_g \leq e\ .$$

Equivalently, this means

$$T_g(r) \geq 2e^{-1}\, T_{\rm hyp}(r) \quad \Leftrightarrow \quad T_g(r) \geq e^{-1}\,\log\frac{1}{1-r}\ .$$

2.6. Effective LLD (Lemma on Logarithmic Derivative).

Our approach is based on the effective version of Lemma on Logarithmic Derivative which holds for meromorphic functions on $\mathbb{D}$ with small growth. For instance, Theorem 2.5.2 (Collective Cohn-Vossen inequality) implies that the lifted Gauss map $g : \mathbb{D} \to \mathbb{P}^1$ has small growth.

Using the fact that the correspondence $[0,1) \ni r \mapsto \rho(r) := \log\frac{1}{1-r} \in [0,\infty)$ is one to one, we introduce the following symbol : The symbol $\leq_\delta$ means that the inequality with this symbol holds outside of a Borel set E_δ in $(\rho \in)\,[0,\infty)$ of finite Lebesgue measure, where the Lebesgue measure of $[0,\infty)$ is understood as $d\rho$.

Let $f : \mathbb{D} \to \mathbb{P}^1$ be a meromorphic function. We define the invariant κ_f by

$$\kappa_f = \inf\left\{\kappa \,\middle|\, \lim_{r\to 1}\int_0^r \exp(\kappa T_f(t))dt = \infty\right\}\ .$$

We interpret $df = f^{(1)}$ as a holomorphic map $df : T'\mathbb{D} \to T'\mathbb{P}^1$ defined by $(df)_z(\frac{\partial}{\partial z}) = f'(z)\frac{\partial}{\partial w}$ where w is the standard affine coordinate of $\mathbb{P}^1$. For a divisor D on $\mathbb{P}^1$ we define its first jet space as $D^{(1)} \subset T'\mathbb{P}^1$ which consists of points $(z,0) \in T'_z\mathbb{P}^1$ where $z \in D$. We can define the proximity function $m_{f^{(1)},D^{(1)}}(r)$ in the usual way. Moreover, we denote the divisor at infinity which appears in the projective completion $\overline{T'\mathbb{P}^1}$ of $T'\mathbb{P}^1$ and define the proximity function $m_{f^{(1)},S_\infty}(r)$ in the usual way. Then we have

Theorem 2.6.1 [KM2, Part II, Corollary 4.2, Corollary 4.2′] (Effective geometric LLD). *For any meromorphic function* $f : \mathbb{D} \to \mathbb{P}^1$ *satisfying the condition* $0 < \kappa_f < \infty$ *and for any small* $\delta > 0$*, we have*

$$\begin{cases} m_{f,D}(r) - m_{f^{(1)},D^{(1)}}(r) \leq_\delta \alpha_{\rm LLD}(r)(\kappa_f+\delta)\,T_f(r) \\ m_{f^{(1)},S_\infty}(r) \leq_\delta \beta_{\rm LLD}(r)(\kappa_f+\delta)\,T_f(r) \end{cases}$$

where $\alpha_{\rm LLD}(r) + \beta_{\rm LLD}(r) = 1$*.*

The definition of κ_f implies that the RHS is written in terms of κ_g and $T_g(r)$:

$$\begin{cases} m_{f,D}(r) - m_{f^{(1)},D^{(1)}}(r) \leq_\delta \alpha_{\rm LLD}(r)(1+\delta)\log\dfrac{1}{1-r} = \alpha_{\rm LLD}(r)(\kappa_g+\delta)\,T_g(r)\ , \\ m_{f^{(1)},S_\infty}(r) \leq_\delta \beta_{\rm LLD}(r)(1+\delta)\log\dfrac{1}{1-r} = \beta_{\rm LLD}(r)(\kappa_g+\delta)\,T_g(r)\ . \end{cases}$$

2.7. Period Condition Encoding Pair $(e^H, \mathcal{D})$ and Characterization of Algebraic Minimal Surfaces.

The period condition is described in terms of e^H, where e^H was introduced in **2.2**. Lemma 2.2.1 says that the period condition is equivalent to the invariance of $|e^H|$ under the action of $\pi_1(M)^2$. To continue the study of the period condition in terms of e^H, we introduce the $\mathcal{D}$ with potentially infinite degree by setting

$$\mathcal{D} = \{e^H(z) \in \mathbb{P}^1 \,|\, g(z) = \infty\}\ ,$$

[2] Note that the original (g,ω) is replaced by the new $(\widetilde{g},\widetilde{\omega}) = (T(g), T_*\omega)$ and we write it as (g,ω). Therefore it is essential to notice that the period condition is preserved in the application of T (cf. Remark 2.4.1).

i.e., $\mathcal{D}$ is a collection of values of e^H at places where g has poles. The regularity condition implies that the value of e^H at any place where g has a pole is well-defined. Restricting the domain of z in $\mathbb{D}(r)$ $(r < 1)$, we get a usual divisor of finite degree. As r approaches to 1, the degree becomes indefinitely large. So we say that $\mathcal{D}$ has potentially infinite degree. We introduce the pair

$$(e^H, \mathcal{D})$$

and call it the **period condition encoding pair**. The main idea of this paper is to apply Theorem 2.6.1 (Effective geometric LLD for meromorphic functions on $\mathbb{D}$ with small growth) to $(e^H, \mathcal{D})$. In order to do so, we must show that e^H has small growth.

Theorem 2.7.1 [KM2, Part II, Theorem 5.3.3] (growth of $T_{e^H}(r)$ under the period condition). *Let $(M, (g, \omega))$ be an algebraic minimal surface, i.e., the Weierstrass data satisfies the period coindition. Then, the growth of the order function $T_{e^H}(r)$ is comparable to that of the height transform $T_{\mathrm{hyp}}(r)$ of the hyperbolic area function. It therefore follows that $0 < \kappa_{e^H} < \infty$, i.e.,*

$$\exists\, C > 0 \quad \text{s.t.} \quad C \log \frac{1}{1-r} < T_{e^H}(r) < C^{-1} \log \frac{1}{1-r} \ .$$

Therefore Theorem 2.6.1 applies to the holomorphic function $e^H : \mathbb{D} \to \mathbb{P}^1$ with approximation target $\mathcal{D}$, i.e., effective geometric LLD applies to the period condition encoding pair $(e^H, \mathcal{D})$.

The idea behind the proof of Theorem 2.7.1 is the counting rule of measure theoretic quantities associated to $\mathbb{D}(r)$ and therefore essentially relies on Lemmas 2.3.2 and 2.3.3 :

Step 1. We decompose

$$T_{e^H}(r) = \frac{1}{2} \int_0^{2\pi} \log(1 + |e^H|^2) \frac{d\theta}{2\pi}$$

into contribution from parabolic sequences.

Step 2. We sort the minimum set of fundamental domains covering $\mathbb{D}(r)$ into parabolic sequences starting at fundamental domain with euclidean distortion n_{th}.

Step 3. We integrate the result of Step 1 from, say, $\frac{1}{2}$ to $1 - (1-r)^{\alpha(r)}$ against the measure $d\mu(t) = \dfrac{dt}{1-t}$, where $\alpha(r) = 1 - \dfrac{2 \log n_{\mathrm{th}}}{\log \frac{1}{1-r}}$.

Theorem 2.7.2 [KM2, Part II, Lemma 5.3.4, Lemma 5.3.5, Corollary 5.3.6] (characterization of the period condition in terms of the growth of $T_{e^H}(r)$). *(1) For any pseudo-algebraic minimal surface we have*

$$T_{e^H}(r) = O\left(\left(\log \frac{1}{1-r}\right)^2\right) .$$

Suppose that a pseudo-algebraic minimal surface does not satisfy the period condition. Then $T_{e^H}(r)$ grows like $\left(\log \dfrac{1}{1-r}\right)^2$.

(2) A pseudo-algebraic minimal surface is algebraic, i.e., the period condition holds, if and only if $T_{e^H}(r)$ grows like $\log \dfrac{1}{1-r}$ when $r \to 1$, i.e.,

$$\exists\, C > 0 \quad \text{s.t.} \quad C \log \frac{1}{1-r} < T_{e^H}(r) < C^{-1} \log \frac{1}{1-r} \ .$$

2.8. Nevanlinna Theory Interpretation of Parabolic Localization Principle.

We want to decode the period condition encoding pair $(e^H, \mathcal{D})$ by applying Theorem 2.6.1 (effective geometric LLD) to extract geometric information from $(e^H, \mathcal{D})$. In the decoding procedure, we need to know how parabolic localization principle is interpreted in Nevanlinna Theory.

2.8.1. Parabolic Localization and Singular Coordinate Change $\zeta \mapsto z$.

Let $(M,(g,\omega))$ be a pseudo-algebraic minimal surface. Let z be the standard linear coordinate on $\mathbb{D}$ which is the restriction of the linear coordinate z of $\mathbb{C}$. As we are working on a finite Riemann surface, we have another natural coordinate defined by a local parameter ζ around a punctured point $P \in \overline{M}\backslash M$. Taking derivatives w.r.to z and ζ, we have two proximity functions $m_{g^{(1)},S_\infty}(r)$ and $m^{\zeta}_{g^{(1)},S_\infty}(r)$, where $m^{\zeta}_{g^{(1)},S_\infty}(r)$ means the proximity function of $g^{(1)}$ to S_∞ taking values in the projective completion $\overline{T'\mathbb{P}^1}$ where the derivative is taken w.r.to the local parameter ζ.

Lemma 2.8.1.1 [KM2, Part II, Lemma 5.2.1] (LLD type formula arising from singular coordinate change $\zeta \mapsto z$). *The effect of the singular coordinate transform from the local parameter ζ at $P \in \overline{M} \setminus M$ to the lineal coordinate z on $\mathbb{D}$ is Nevanlinna theoretically described as*

$$m_{g^{(1)},S_\infty}(r) - m^{\zeta}_{g^{(1)},S_\infty}(r) = \log \frac{1}{1-r} \quad \text{asymptotically as } r \to 1 \ .$$

2.8.2. Parabolic Localization and Completeness of Induced Metric.

Let $(M,(g,\omega))$ be a pseudo-algebraic minimal surface. The completeness of the metric ds^2 on M induced from the Euclidean metric of $\mathbb{R}^3$ by the Enneper-Weierstrass representation $X : \widetilde{M} \to \mathbb{R}^3$ is translated into a Nevanlinna theoretic statement. Let $\mathbb{S}$ be the unit sphere in the 3-dimensional $\mathbb{R}$-vector space V with ON basis $\{\frac{1}{2}(1-g^2), \frac{i}{2}(1+g^2), g\}$ (cf. **2,2**). Then, we have the following interpretation of the parabolic localization in terms of the completeness of ds^2, i.e., the estimate of the magnitude of $m_{h,S_\infty}(r)$:

Lemma 2.8.2.1 [KM2, Part II, Lemma 5.2.2] (Nevanlinna theory interpretation of the parabolic localization in terms of the completeness of ds^2). *Let $p(g)$ represent a generic element of $\mathbb{S}$. Then we have*

$$T_{H',S_\infty}(r) = 2T_g(r) + 2N_{g,\infty}(r) \quad \text{and} \quad m_{h,S_\infty}(r) = 2T_g(r) + 2N_{g,\infty}(r) \ .$$

The proof relies on Lemmas 2.3.2-3 [KM1, Part I, Lemmas 3.4.3.7-8].

2.8.3. Parabolic Localization and Group Theoretic Approximation.

Let $(M,(g,\omega))$ be a pseudo-algebraic minimal surface.

Theorem 2.8.3.1 [KN, Part II, Theorem 5.2.8] (Nevanlinna Theory interpretation of parabolic localization in terms of group theoretic approximation). *As soon as we fix the threshold $n_{\rm th}$ of the strength of Euclidean distortion, we have*

(1) Let $A(r) \sim B(r)$ mean that $A(r)$ and $B(r)$ are the same order as $r \to 1$. Then we have

$$N_{g,\infty}(r) \sim O(n_{\rm th}^{-1}) \log \frac{1}{1-r} \quad \text{and} \quad m_{g,\infty}(r) \sim (1 - O(n_{\rm th}^{-1})) \log \frac{1}{1-r} \ .$$

(2) The contribution to the hyperbolic area of the minimal covering of $\mathbb{D}(r)$ from the intervals $J \in \mathcal{J}_r$ over all $\mathcal{J}_r$'s is bounded above by

$$(1-r)^{-1} n_{\rm th}^{-1} \ .$$

Here we have defined the collection of intervals $\mathcal{J}_r$ in $\partial\mathbb{D}(r)$ in [KM2] :

Definition 2.8.3.2 [KM2, Part II, Definition 5.2.3-4]. (1) For each fixed r satisfying $0 < r < 1$ and close to 1, we define the subsequence $\mathcal{S}_r$ of a parabolic sequence in the following way. The subsequence $\mathcal{S}_r$ starts at the first fundamental domain in the parabolic sequence under question whose strength of Euclidean distortion first exceeds $n_{\rm th}$ and ends at the first fundamental domain the cluster part of which has Euclidean height less than $1 - r$.

(2) Let $0 < r < 1$. We decompose $\partial\mathbb{D}(r)$ into intervals whose end points are given by those of the cluster part. Let $\mathcal{J}_r$ be the collection of intervals corresponding to the subsequence $\mathcal{S}_r$ and $\mathcal{I}_r$ the rest (i.e., those intervals corresponding to the cluster part of the parabolic sequence under question whose cluster part has Euclidean height $< 1 - r$).

Geometry behind Theorem 2.8.3.1. The geometric origin of Theorem 2.8.3.1 is the **group theoretic approximation**, which is one of the Nevanlinna theoretic interpretation of the parabolic localization principle. To describe this phenomenon, we recall the structure of the First Main Theorem. The counting function $N_{g,w}(r)$ does not depend on the metric of $\mathcal{O}_{\mathbb{P}^1}(1)$. However, the proximity function $m_{g,w}(r)$ and the order function $T_f(r)$ do depend on the metric. How the asymptotic behavior of $T_g(r)$ as $r \to 1$ depends on the metric is described as follows. If we replace ω_{FS} by another metric $\omega_{\mathrm{FS}} + dd^c f$, then we have $T_{g,(1/4\pi)(\omega_{\mathrm{FS}}+dd^c f)}(r) - T_{g,(1/4\pi)\omega_{\mathrm{FS}}}(r) = (1/4\pi)\int_0^{2\pi} f(re^{i\theta})\frac{\pi}{2\pi}$. In the First Main Theorem $T_g(r) = m_{g,w}(r) + N_{g,w}(r) - m_{g,w}(0)$, the dependency on the metric cancels. In our setting, we operate a strong hyperbolic translation T with ∞ (resp. 0) the repelling fixed point (resp. absorbing fixed point) so that the g-image of the punctured points and critical points of g is contained in a small disk of 0. By the effect of the application of T, the behavior of $N_{g,\infty}(r)$ becomes special, while $T_{g,(1/4\pi)\omega_{\mathrm{FS}}^{\mathrm{modified}}}(r)$ differs from $T_{g,(1/4\pi)\omega_{\mathrm{FS}}}(r)$ by $O(1)$. This observation is important, because $N_{g,\infty}(r)$ does not depend on the metric of $\mathbb{P}^1$ involved in the definition of $T_g(r)$. Instead of changing the metric, we choose to work with a strong hyperbolic translation T in order to make the behavior of $N_{g,\infty}(r)$ special (the change of $T_g(r)$ being only by the quantity $O(1)$)[3]. In the classical Nevanlinna theory we have

$$\mathfrak{M}_{w\in\mathbb{P}^1} m_{f,w}(r) = O(1)$$

for any meromorphic function $f : \mathbb{C} \to \mathbb{P}^1$. In our setting, the effect of the Metrized Riemann-Hurwitz Theorem (Lemma 2.4.2) should be taken into axxount. We recall that in the proof of MRH in [Part I, Theorem 4.2], the Fubini-Study form was deformed by applying a "strong" hyperbolic translation T and further deformed into a pillow-case metric $\omega_{\mathrm{FS}}^{\mathrm{modified}}$. In this procedure, the effect that ∞ is the repelling fixed point of T plays an important role. We are ready to describe the phenomenon of the group theoretic approximation. We first subdivide the set-throretical image $f(\mathbb{D}(t)) \subset \mathbb{P}^1$ into the union of small cells so that any two cells does not share interior points. Then we collect the pre-image of these cells in $\mathbb{D}(t)$ and integrate the area w.r.to the measure $\frac{1}{4\pi}g^*\omega_{\mathrm{FS}}^{\mathrm{modified}}$ taking the multiplicities into account and then integrate against dt/t from 0 to r. By this procedure, we get the order function $T_{g,(1/4\pi)\omega_{\mathrm{FS}}^{\mathrm{modified}}}(r)$. The counting function does not depend on the metric of $\mathbb{P}^1$ and the same integration procedure is performed against the delta measure $g^*\infty$ instead of $\frac{1}{4\pi}\omega_{\mathrm{FS}}^{\mathrm{modified}}$. By the effect that ∞ is the repelling fixed point in the argument of Metrized Riemann-Hurwitz Theorem [Part I, Theorem 4.2], $\exists\varepsilon > 0$ s.th. any open set containing ∞ containing a ball of radius $\geq \varepsilon$ is expanded by g into an open set of S^2 with $\omega_{\mathrm{FS}}^{\mathrm{modified}}$ whose boundary is contained in a small disk, say, of $\omega_{\mathrm{FS}}^{\mathrm{modified}}$-radius $1/100$, centered at 0 (the absorbing fixed point of T). Therefore, the cells touching $\partial\mathbb{D}(r)$ is counted with high weight in the integration defining the order function $T_{f,(1/4\pi)\omega_{\mathrm{FS}}^{\mathrm{modified}}}(r)$. The weight becomes "heavier" as T becomes "stronger". On the other hand, in the integration defining the counting function $N_{g,\infty}(r)$, those cells which are counted in the order function with heavy weight are not counted because those cells locate near the boundary $\partial\mathbb{D}(r)$ because of the effect of the parabolic localization (see below). By the effect of the parabolic localization, even under the limit of subdivisions so that $\max\{\mathrm{diam}_{\omega_{\mathrm{FS}}}(\mathrm{cells})\} \to 0$, the ratio of the number (counted with multiplicities) of cells whose pre-image by g intersects the boundary of $\mathbb{D}(r)$ against that of all cells in the subdivision indefinitely increases as $r \to 1$. We note that

(1) the multiplicities of those cells whose pre-image under g intersects $\partial\mathbb{D}(r)$ diverges to ∞ with order $\frac{1}{1-r}$ as $r \to 1$,
and that

(2) the significant portion of the collection of those cells in $\mathbb{P}^1$ whose radius (w.r.to ω_{FS}) is of order $1 - r$ and whose pre-image under g intersects the boundary $\partial\mathbb{D}(r)$ is not counted in the counting function, while they are contained in the order function.

[3] This is the gauge transformation induced by the application of T by which the period condition is preserved, i.e., the new pair $(T(g), T_*\omega)$ satisfies the period condition just as (g, ω) does.

The above observation implies that the ratio of $N_{g,\infty}(r)$ against $T_g(r)$ (in fact $T_{g,(1/4\pi)\omega_{\rm FS}^{\rm modified}}(r)$) becomes significantly small when r approaches to 1. We call this phenomenon as the **group theoretic approximation** introduced by the parabolic localization and Metrized Riemann-Hurwitz Theorem (Lemma 2.4.2). This approximation is NOT an approximation for the original holomorphic map g to ∞ but is a RELATIVE approximation which means that $N_{g,\infty}(r)$ occupies only a small portion of $T_g(r)$ w.r. to the pillow-case metric $(1/4\pi)\omega_{\rm FS}^{\rm modified}$ of $\mathbb{P}^1$ *arising from the application of the strong hyperbolic translation* T. This phenomenon does not happen for entire holomorphic map $f : \mathbb{C} \to \mathbb{P}^1$. Indeed, in the limit of subdivisions so that $\max\{\mathrm{diam}_{\,\omega_{\rm FS}}(\text{cells})\} \to 0$, the ratio of the number of cells intersecting $\partial\mathbb{D}(r)$ against the that of all cells in subdivisions tends to 0 (counted with multiplicities).

Proof of Theorem 2.8.3.1. Before the proof, we recall that by rotation (if necessary), we may assume that the extended Gauss map g does not have poles at punctured points and in addition that g has only simple poles. We recall that in this setting the point ∞ is the repelling fixed point of a strong hyperbolic translation T used in Metrized Riemann-Hurwitz Theorem [Part I, Theorem 4.2]. The proof consists of the justification of the intuitive explanation in the above speculation based on the meaning of ∞, i.e., the strong hyperbolic translation T in the Riemann-Hurwitz Theorem has ∞ as the repelling fixed point. Then $N_{g,\infty}(r)$ has a canonical meaning. More generally, so does $N_{g,a}(r)$, where a is any value which does not belongs to the set $\{g(P)\,|\,P \in \overline{M} \setminus M\}$) *as soon as a is chosen as the repelling fixed point of the strong hyperbolic translation in the proof of Metrized Riemann-Hurwitz Theorem.* To explain the reason why $N_{g,\infty}(r)$ has a canonical meaning, we work on $\mathbb{H}$. Consider a sequence of fundamental domains having ∞ as a parabolic fixed point. Then the parabolic sequence consists of parallel translations of a reference fundamental domain. Suppose that the Euclidean height of the cluster part is 1. The boundary $\partial\mathbb{D}(r)$ corresponds to the circle

$$C_r \ : \ r = \frac{|z-i|}{|z+i|} \ .$$

Put $\varepsilon = 1-r$. Then the radius of C_r is of order ε^{-1}. The lower arc of the circle C_r is approximated by the parabola of the form

$$y = \varepsilon^2 + \varepsilon x^2 \ .$$

The number of fundamental domains in the parabolic sequence which have non-empty intersection with C_r is of order $O(\varepsilon^{-\frac{1}{2}})$. Therefore, the hyperbolic area of the portion of the parabolic sequence truncated by $\mathbb{D}(r)$ having non-empty intersection with $\partial\mathbb{D}(r)$ is at most of order $O(\varepsilon^{-\frac{1}{2}})$. On the other hand, the hyperbolic area of the rest of the parabolic sequence truncated by $\mathbb{D}(r)$ is

$$O\biggl(\int_{\varepsilon^{-\frac{1}{2}}}^{\frac{1}{\varepsilon N}} dx \int_{\varepsilon^2+\varepsilon x^2}^{\varepsilon^{-1}} \frac{dy}{y^2}\biggr) = O\biggl(\int_{\varepsilon^{-\frac{1}{2}}}^{\frac{1}{\varepsilon N}} \frac{dx}{\varepsilon^2+\varepsilon x^2}\biggr) = O(\varepsilon^{-2}) \ ,$$

where N is a large number but $\ll \varepsilon^{-\frac{1}{2}}$ which appears when we approximate C_r by the parabola $y = \varepsilon^2 + \varepsilon x^2$. This manipulation implies that the contribution to the hyperbolic area from the parabolic sequence is dominated by those fundamental domains whose Euclidean height is smaller than $1-r$.

To sum up, the counting function $N_{g,\infty}(r)$ (more generally, $N_{g,a}(r)$, where $-1/\overline{a}, a \notin \{g(P)\,|\,P \in \overline{M}\setminus M$ or $dg(P) = 0\}$) behaves in a special way, as soon as the original (g,ω) is replaced by a new $(\widetilde{g},\widetilde{\omega})$ by the application of the strong hyperbolic translation T having $\{a\}$ (resp. $\{-1/\overline{a}\}$) as the repelling (resp. absorbing) fixed point. The above manipulation is the mathematical ground for this statement. Indeed, as is shown in the following discussion, the growth of $N_{g,\infty}(r)$ (more generally, $N_{g,a}(r)$, where $\{a\}$ is as above) is strictly smaller than that of $\log\frac{1}{1-r}$ as soon as we take the effect of the application of T into account.

We count how many members of a parabolic sequence contribute to $N_{g,\infty}(r)$. It follows from [Part I, Lemma 3.4.3.9-10] that we have only to look at a parabolic sequence starting at a fundamental domain of Euclidean diameter $(1-r)^{\alpha(r)}$ where

$$\alpha(r) = 1 - \frac{2\log n_{\rm th}}{\log\frac{1}{1-r}} \ , \quad \text{i.e.,} \quad n_{\rm th} = (1-r)^{\frac{\alpha(r)-1}{2}} \ .$$

Applying the $n_{\rm th}$-th iteration of the parabolic translation to the initial fundamental domain the Euclidean diameter decays like $O((1-r)^{\frac{\alpha(r)+1}{2}})$ and the Euclidean diameter of the cluster part decays like $O(1-r)$. Therefore, it follows from the above manipulation that the number of fundamental domains which contribute to $N_{g,\infty}(r)$ is just $O((1-r)^{\frac{\alpha(r)-1}{2}})$. We have approximately $O((1-r)^{-\alpha(r)})$ such fundamental domains and the contribution from such ones dominates by [Part I, Lemma 3.4.3.10]. Therefore, the contribution to $N_{g,\infty}(r)$ is just of order

$$\int_{2^{-1}}^{r} \frac{dt}{t}(1-t)^{-\alpha}(1-t)^{\frac{\alpha-1}{2}} \approx \int_{2^{-1}}^{r} \frac{dt}{t}(1-r)^{-1}(1-t)^{\frac{1-\alpha}{2}} \approx n_{\rm th}^{-1}\log(1-r)^{-1} \ .$$

(2) We measure the area of the portion of $\mathbb{D}(r)$ occupied by fundamental domains corresponding to $|\mathcal{J}_r| := \cup_{J\in\mathcal{J}_r} J$. The contribution to the area from each $|\mathcal{J}_r|$ is bounded above by the number of intervals in $\mathcal{J}_r$. This is of order $n_{\rm th} = (1-r)^{\frac{\alpha(r)-1}{2}}$. Therefore the contribution to the area of the minimal covering of $\mathbb{D}(r)$ from $|\mathcal{J}_r|$ is bounded above by

$$(1-r)^{-\alpha(r)}n_{\rm th} = (1-r)^{-\frac{\alpha(r)+1}{2}} = (1-r)^{-1}(1-r)^{\frac{1-\alpha(r)}{2}} = (1-r)^{-1}n_{\rm th}^{-1} \ .$$

We have thus proved Theorem 2.8.3.1. □

2.9. From LLD to SMT.

Let $(M,(g,\omega))$ be a pseudo-algebraic minimal surface. We interpret $dg = g^{(1)}dz$ and $\omega = h(z)dz = h(\zeta)d\zeta$ as a map from $T'\mathbb{D}$ to $T'\mathbb{P}^1$, i.e., $dg = g^{(1)} : T'_z\mathbb{P}^1 \to T'_{g(z)}\mathbb{P}^1$ and $h : T'_z\mathbb{D} \to T'_{g(z)}\mathbb{P}^1$ $(\forall z \in \mathbb{D})$. Following the classical argument from LLD (Lemma on Logarithmic Derivative) to SMT (the Second Main Theorem) (cf. [N] and [Kod]), we have the following Nevanlinna calculus inequality.

Lemma 2.9.1 [KM2, Part II, Lemma 5.1.1] (Nevanlinna calculus inequality for the lifted Gauss map). *Let $(M,(g,\omega))$ be a pseudo-algebraic minimal surface and (g,ω). Let D be a finite collection of distinct points which is regarded as a divisor on $\mathbb{P}^1$. Then the Nevanlinna theory function $m_{g,D}(r) + N_{g,\rm Ram}(r)$ which counts the number of solutions of the equation $g(z) \in D$ (counted with multiplicity) satisfies the following estimate*

$$m_{g,D} + N_{g,\rm Ram}(r) \le_\delta \underbrace{(m_{h,S_0}(r) + N_{h,S_0}(r) - m_{h,S_\infty}(r))}_{T_{h,S_0-S_\infty}(r)=2T_g(r)} + (\kappa_g+\delta)\,T_g(r) - J(r)$$

asymptotically as $r \to 1$, where $J(r)$ is the Nevanlinna theoretic function defined by

$$J(r) := m_{g^{(1)},S_\infty}(r) - m^{\zeta}_{g^{(1)},S_\infty}(r) + m_{h,S_0}(r) - m^{\zeta}_{h,S_0}(r) \ .$$

Therefore, the RHS of Lemma 2.9.1 becomes (omitting δ)

$$2\,T_g(r) + \{\kappa_g T_g(r) - J(r)\} \ .$$

It follows from the proof in [KM2, Part II, Lemma 5.1.1] that the origin of 2 before $T_g(r)$ is topological, i.e., $2 = \chi(\mathbb{P}^1)$, the Euler number of the 2-sphere. This is the same as the classical Second Main Theorem. On the other hand, the origin of the quantity $\kappa_g T_g(r) - J(r)$ is Theorem 2.6.1 (the Effective geometric LLD) and has group theoretic and analytic in nature.

It therefore turns out that evaluating the Nevanlinna theoretic function $\kappa_g T_g(r) - J(r)$ is the main task toward establishing the Second Main Theorem for the lifted Gauss map $g : \mathbb{D} \to \mathbb{P}^1$.

2.10. Proof of the Second Main Theorem for the Lifted Gauss Map of an Algebraic Minimal Surface.

In **2.10**, we decode the period condition encoding pair $(e^H, \mathcal{D})$ by applying Theorem 2.6.1 (the Effective geometric LLD). We have two versions of the effective LLD applied to $(e^H, \mathcal{D})$. One is "without ζ", i.e., the derivative is taken w.r.to the linear coordinate z on $\mathbb{D} \subset \mathbb{C}$. The other is "with ζ", i.e., the derivative is taken w.r.to the local parameter ζ around a punctured point $P \in \overline{M}\backslash M = \{P_1, \ldots, P_{n(M)}\}$.

In order to understand the "hidden" geometry of algebraic minimal surface which should be extracted from the two types of effective LLD's applied to $(e^H, \mathcal{D})$, we introduce the following Nevanlinna theoretic functions

(2.10.1)
$$\begin{aligned} J(r) &:= (m_{g^{(1)},S_\infty}(r) - m^{\zeta}_{g^{(1)},S_\infty}(r)) + (m_{h,S_0}(r) - m^{\zeta}_{h,S_0}(r)) \ , \\ J_1(r) &:= m_{h,S_0}(r) - m^{\zeta}_{h,S_0}(r) \ , \\ J_2(r) &:= m_{h,S_\infty}(r) - m^{\zeta}_{h,S_\infty}(r) \end{aligned}$$

which are constructed from the singular coordinate transformation from the local parameter ζ at punctures $\{P_1, \ldots, P_{n(M)}\}$ to the linear coordinate z of $\mathbb{D} \subset \mathbb{C}$. These are Nevanlinna theoretic functions which essentially depend only on the conformal structure of the basic domain M.

For instance, Lemma 2.8.1 implies that

$$m_{g^{(1)},S_\infty}(r) - m^{\zeta}_{g^{(1)},S_\infty}(r) \geq \log \frac{1}{1-r}$$

holds. In terms of $J(r)$ and $J_1(r)$, this formula is rewritten as

(2.10.2)
$$J(r) \geq \ J_1(r) + \log \frac{1}{1-r} \ .$$

In the following sequence of Lemmas 2.10.1-4, we compare two types of effective LLD applied to the pair $(e^H, \mathcal{D})$ and extract useful information from the period condition. The inequalities which we will prove in Lemmas 2.101-4 involve small $\delta > 0$ and the associated occurrence of the exceptional intervals in the same sense as that explained at the beginning of **2.6**. For simplicity we use the symbol $\leq_\delta$ as in **2.6**.

Let us write

$$A_r := \{a_1, \ldots, a_{k(r)}\}$$

for the totality of the values of e^H at the poles of g inside $|z| = r$ $(0 < r < 1)$ where z is the linear coordinate of $\mathbb{D} \subset \mathbb{C}$. Here, for a fixed $r < 1$, the number $k(r)$ which appears in the definition of A_r is finite and $\lim_{r \to 1} k(r) = \infty$. The image curve $g(\partial \mathbb{D}(r))$ in M must be at distance from the set of all punctured points $\overline{M} \setminus M$ by some positive number. As $|e^H|$ is invariant under the action of $\pi_1(M)$, these values of e^H are expressed as

(2.10.3)
$$\underbrace{\{\alpha_1 e^{i\theta_{11}}, \ldots, \alpha_1 e^{i\theta_{1k_1(r)}}, \ldots, \alpha_p e^{i\theta_{p1}}, \ldots, \alpha_p e^{i\theta_{pk_p(r)}}\}}_{\text{finitely many variations for } |a_i| = \alpha_1, \ldots, \alpha_p \text{ even if } k(r) \to \infty}$$

where $k(r) = k_1(r) + \cdots + k_p(r)$ and $p \leq \sharp(g^{-1}(\infty))$ (equality, if counted with multiplicities) where $g : M \to \mathbb{P}^1$ is the Gauss map. The picture how these points (2.10.3) are distributed on the Riemann sphere $\mathbb{P}^1$ is as follows. There exist a **finitely many** circles

(2.10.4)
$$|w|_{\mathbb{P}^1} = \alpha_i \quad (i = 1, \ldots, p) \quad \text{in} \quad \mathbb{P}^1$$

and the points (2.10.3) distribute on these circles. Here the parameter w in the expression $|w|_{\mathbb{P}^1} = \alpha_i$ is a fixed affine coordinate of $\mathbb{P}^1$. What is essential is that the point $\{\infty\} \in \mathbb{P}^1$ is not an accumulation point of the point set (2.10.3). Indeed, as $p(g)$ varies in the unit sphere $\mathbb{S}(V)$ in V, the location of the circles $|w|_{\mathbb{P}^1} = \alpha_i$ in $\mathbb{P}^1$ is uniform and is uniformly apart from the point $\{\infty\}$. As $k(r) \to \infty$ as $r \to 1$, the points in (2.10.3) distribute more densely on finitely many circles (2.10.4), i.e., $\cup_i\{|w|_{\mathbb{P}^1} = \alpha_i\}$ $(1 \leq i \leq p)$ in $\mathbb{P}^1$ and never accumulate at ∞. This means that although the "divisor" $\mathcal{D}$ has potentially infinite degree "as $r \to 1$", we can imagine that the set consisting of the fixed circles $|w|_{\mathbb{P}^1} = \alpha_i$ $(1 \leq i \leq p)$ behaves as a usual "fixed target divisor" on $\mathbb{P}^1$ when we discuss the approximation of e^H to its values at the poles of the lifted Gauss map g.

Lemma 2.10.1 (Effective LLD "without ζ" applied to $(e^H, \mathcal{D})$ for algebraic minimal surfaces) (cf. [KM2, Part II, Lemma 5.4.1]). *Let $(e^H, \mathcal{D})$ be the period condition encoding pair of any algebraic minimal surface. Let the notations such as $\cup_i\{|w|_{\mathbb{P}^1} = \alpha_i\}$ be understood as in (2.10.4). The effective LLD without the superscript ζ for the function e^H with the divisor $\sum_{i=1}^{k(r)}\{a_i\}$ is the following inequality (δ being any small positive number).*

$$\begin{aligned}
&\mathfrak{M}_{p(g)\in\mathbb{S}}\{(m_{e^H,\cup_i\{|w|_{\mathbb{P}^1}=\alpha_i\}}(r) - m_{(e^H)^{(1)},\cup_i\cup_{|w|_{\mathbb{P}^1}=\alpha_i}\{z\}^{(1)}}(r)) + m_{(e^H)^{(1)},S_\infty}(r)\}\\
&\leq_\delta (1+\delta)\kappa_g T_g(r)\\
&= (1+\delta)\log\frac{1}{1-r} \quad \text{asymptotically as} \quad r\to 1\ .
\end{aligned}$$

Lemma 2.10.2 (Effective LLD "with ζ" applied to $(e^H, \mathcal{D})$ for algebraic minimal surfaces) (cf. [KM2, Part II, Lemma 5.4.2]). *Let $(e^H, \mathcal{D})$ be the period condition encoding pair of any algebraic minimal surface. Let the notations such as $\cup_i\{|w|_{\mathbb{P}^1} = \alpha_i\}$ be understood as in (2.10.4). The effective LLD with the superscript ζ for the function e^H with the divisor*

$$\cup_i\{|w|_{\mathbb{P}^1} = \alpha_i\} + \{0\} + \{\infty\}$$

is the following inequality (δ being any small positive number).

$$\begin{aligned}
&\mathfrak{M}_{p(g)\in\mathbb{S}}\{(m^\zeta_{e^H,\cup_i\{|w|_{\mathbb{P}^1}=\alpha_i\}}(r) - m^\zeta_{(e^H)^{(1)},\cup_i\cup_{|w|_{\mathbb{P}^1}=\alpha_i}\{z\}^{(1)}}(r))\\
&+ (m^\zeta_{e^H,\{0\}}(r) - m^\zeta_{(e^H)^{(1)},\{0\}^{(1)}}(r)) + (m^\zeta_{e^H,\{\infty\}}(r) - m^\zeta_{(e^H)^{(1)},\{\infty\}^{(1)}}(r))\\
&+ m^\zeta_{(e^H)^{(1)},S_\infty}(r)\}\\
&\leq_\delta (1+\delta)\,\kappa_g T_g(r)\\
&= (1+\delta)\,\log\frac{1}{1-r} \quad \text{asymptotically as} \quad r\to 1\ .
\end{aligned}$$

The proofs of Lemmas 2.10.1-2 are almost the same as those of [KM2, Part II, Lemmas 5.4.1-2]. We have only to replace the LLD-type formulas in [KM, Part II, (5.2.9)], which was proved under the assumption that the Cohn-Vossen ratio $R \leq 2$, with the usual one, i.e., Theorem 2.6.1 (the effective geometric LLD) of type

$$\begin{aligned}
&m_{g,D}(r) - m_{g^{(1)},D^{(1)}}(r) \leq \alpha_{\rm LLD}\kappa_g T_g(r)\ ,\\
&m_{g^{(1)},S_\infty}(r) \leq \alpha_{\rm LLD}\kappa_g T_g(r)\ .
\end{aligned}$$

This is the point which explains the reason why the case $R = 2$ is so special.

A small technical point is that the interval version of κ_g appears if we try to adapt the argument in [KM2] to the proof of Lemma 2.10.1. For instance, if the interval version appears, the definition $\kappa_b(I) := \frac{\kappa_g(I)}{k(r)}$ should be replaced by the definition $\kappa_b := \frac{\kappa_g}{k(r)}$ and the relation $(\kappa_b(I)\mathbf{1}(I))\cdot T_{g_b}(r) = \frac{|I|}{2\pi}\log\frac{1}{1-r}$ should be replaced by the relation $\kappa_b T_{g_b}(r) = \log\frac{1}{1-r}$.

Another difficulty is that $k(r)$ diverges as $r \to 1$. By using Lemma 2.3.4 [KM1, Part I, Lemmas 3.4.3.9-10], we settle the difficulty arising from the fact $k(r) \to \infty$ (as $r \to 1$) in Lemmas 2.10.1-2.

Our next task is to translate the estimates in Lemmas 2.10.1-2 into Nevanlinna theory statements by using the Nevanlinna theory interpretations of the parabolic localization principle explained in **2.8**. We have the following results.

Lemma 2.10.3 (Consequence of effective LLD "without ζ" applied to $(e^H, \mathcal{D})$) (cf. [KM2, Part II, Lemma 5.4.3]). *Let $(e^H, \mathcal{D})$ be the period condition encoding pair of any algebraic minimal surface. The effective LLD in Lemma 2.10.1 without the superscript ζ implies the following inequality (δ being any small positive number).*

$$J_2(r) + m_{g,\infty}(r) \leq_\delta (1+\delta)\,\kappa_g T_g(r) = (1+\delta)\,\log\frac{1}{1-r} \quad \text{asymptotically as} \quad r\to 1\ .$$

This implies

$$J_2(r) \leq_\delta (1+\delta)\,\kappa_g T_g(r) - T_g(r) + N_{g,\infty}(r) \quad \text{asymptotically as} \quad r \to 1\ .$$

As soon as we replace [KM2, Part II, Lemma 5.4.1] with Lemma 2.10.1, we can prove Lemma 2.10.3 exactly in the same way as [KM2, Part II, Lemma 5.4.3].

Lemma 2.10.4 [KM, Lart II, Lemma 5.4.4] (Consequence of effective LLD with ζ applied to $(e^H, \mathcal{D})$). *Let $(e^H, \mathcal{D})$ be the period condition encoding pair of any algebraic minimal surface. The effective LLD in Lemma 2.10.2 with the superscript ζ implies the following inequality (δ being any small positive number) :*

$$(8\kappa_g^{-1} - 2 + \delta)\cdot \log \frac{1}{1-r} \leq_\delta J_1(r) + 2\,J_2(r) + 2N_{g,\infty}(r) \quad \text{asymptotically as} \quad r \to 1\ .$$

Proof. The proof reduces to showing the following inequalities :

(2.10.5a)

$$\mathfrak{M}\,\{m^\zeta_{e^H,\cup_i|w|_{\mathbb{P}^1}=\alpha_i}(r) - m^\zeta_{(e^H)^{(1)},\cup_i\cup_{|w|_{\mathbb{P}^1}=\alpha_i}\{z\}^{(1)}}(r)\} \geq \frac{1}{2}m^\zeta_{H',S_0}(r) + 2m_{g,\infty}(r)\ ,$$

(2.10.5b)

$$(m^\zeta_{e^H,\{0\}}(r) - m^\zeta_{(e^H)^{(1)},\{0\}^{(1)}}(r)) + (m^\zeta_{e^H,\{\infty\}}(r) - m^\zeta_{(e^H)^{(1)},\{\infty\}^{(1)}}(r)) = m^\zeta_{h,S_\infty}(r)\ ,$$

(2.10.5c)

$$m^\zeta_{(e^H)^{(1)},S_\infty}(r) = O(1)\ .$$

In order to prove (2.10.5a) it suffices to prove

$$\sum_{i=1}^{k(r)}\{m^\zeta_{e^H,\{a_i\}}(r) - m^\zeta_{(e^H)^{(1)},\{a_i\}^{(1)}}(r)\} \geq \frac{1}{2}m^\zeta_{H',S_0}(r) + 2m_{g,\infty}(r)$$

for generic $p(g) \in \mathbb{S}(V)$. To prove Lemma 2.10.4, we first assume that the estimates (2.10.5a.b.c) hold. Then the consequence of Lemma 2.10.2 becomes

$$\frac{1}{2}m^\zeta_{H',S_0} + 2m_{g,\infty} + m^\zeta_{h,S_\infty}(r) \leq_\delta (1+\delta)\,\kappa_g T_g(r)\ . \tag{2.10.6}$$

Note that here we do not need to take the mean over $\mathbb{S}(V)$, because what contributes to the LHS of (2.10.6) is only the set of points of $\mathbb{D}$ where g has poles.

Whenever (effective) LLD is used there occurs a small constant δ and the inequality $\leq$ becomes $\leq_\delta$. With this understood, for simplicity, we simplify formulas by omitting small constant δ and replacing $\leq_\delta$ and $1+\delta$ by $\leq$ and 1.

The definition of $J_1(r)$ implies

$$m^\zeta_{H',S_0}(r) = m_{H',S_0}(r) - J_1(r)\ .$$

Lemma 2.8.2.1 (parabolic localization in terms of the completeness of the Riemannian metric of M induced from $\mathbb{R}^3$) imples

$$T_{H',S_\infty}(r) = 2\,T_g(r) + 2\,N_{g,\infty}(r)\ .$$

Using these two estimates, we have

$$\begin{aligned} m_{H',S_0}(r) &= T_{H',S_0}(r) \quad [\ N_{H',S_0}(r) = 0\ ] \\ &= T_{H',S_\infty + \mathcal{O}_{\mathbb{P}^1}(2)}(r) \quad [\ \text{linear equivalence } [S_0] = [S_\infty] + \mathcal{O}_{\mathbb{P}^1}(2)\ ] \\ &= 2\,T_{H',S_\infty}(r) + 2\,T_g(r) \\ &= 2\,T_g(r) + 2\,N_{g,\infty}(r) + 2\,T_g(r) \quad [\ \text{Lemma 2.8.2.1}\ ] \\ &= 4\,\kappa_g^{-1}\log\frac{1}{1-r} + 2N_{g,\infty}(r)\ . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} m_{h,S_\infty}(r) &= T_{h,S_\infty}(r) \\ &= T_{h,S_0-\mathcal{O}_{\mathbb{P}^1}(2)}(r) \quad [\text{ linear equivalence } [S_0]=[S_\infty]+\mathcal{O}_{\mathbb{P}^1}(2)\ ] \\ &= T_{h,S_0}(r) - 2\,T_g(r) \\ &= m_{h,S_0}(r) + N_{h,S_0}(r) - 2\,T_g(r) \quad [\text{ First Main Theorem }] \\ &= 2\,m_{g,\infty}(r) + 2\,N_{g,\infty}(r) - 2\,T_g(r) \quad [\text{ Regularity condition }] \\ &= O(1)\ . \end{aligned}$$

As

$$J_2(r) = m_{h,S_\infty}(r) - m^{\zeta}_{h,S_\infty}(r)\ ,$$

we have

$$m^{\zeta}_{h,S_\infty}(r) = m_{h,S_\infty}(r) - \ J_2(r) = -J_2(r) + O(1)\ .$$

Inserting these into (2.10.6) we have the consequence of Lemma 2.10.4. Indeed, we have

$$\begin{aligned} \kappa_g T_g(r) &\geq \frac{1}{2} m^{\zeta}_{H',S_\infty}(r) + 2\,m_{g,\infty} - J_2(r) \\ &\geq \frac{1}{2} m_{H',S_0}(r) - \frac{1}{2} J_1(r) + 2\,m_{g,\infty}(r) - J_2(r) \\ &= \frac{1}{2}(4\,T_g(r) + 2\,N_{g,\infty}(r)) - \frac{1}{2} J_1(r) + 2\,m_{g,\infty}(r) - J_2(r)\ . \end{aligned}$$

It follows that

$$\begin{aligned} J_1(r) + 2J_2(r) &\geq -2\,(\kappa_g T_g(r) + 4\,T_g(r) + 2\,N_{g,\infty}(r) + 4\,m_{g,\infty}(r) \\ &= -2\kappa_g T_g(r) + 6\,T_g(r) + 2\,m_{g,\infty}(r)\ . \end{aligned}$$

Adding $2N_{g,\infty}(r)$ to both sides, we have

$$\begin{aligned} J_1(r) + 2J_2(r) + 2\,N_{g,\infty}(r) &\geq -2\kappa_g T_g(r) + 8\,T_g(r) \\ &\geq (8\kappa_g^{-1} - 2)\cdot \log\frac{1}{1-r}\ . \end{aligned}$$

The rest of the proof consists of showing (2.10.5abc) and is the same as that of [KM2, Part II, Lemma 5.4.4]. The hardest is the proof of (2.10.5a). The effect of the group theoretic approximation, which arises from the application of the strong hyperbolic translation T with ∞ (resp. 0) the repelling (resp. absorbing) fixed point, should be taken into account. Because (2.10.5a) is highly non-trivial, we repeat its proof. The effect of the application T is present in the occurrence of $2m_{g,\infty}(r)$ in (2.10.5a). Suppose that $H' = p(g)\omega = 0$. Then, e^H stays longer at points where $p(g) = 0$. Hence the places where $p(g) = 0$ gives rise to the contribution to the proximity function $m_{(e^H)^{(1)},S_0}(r)$. If $p(g) = 0$, then the value of g is a solution of the equation $p(w) = 0$ (e.g., $p(w) = 1 - w^2$, $p(w) = i(1 + w^2)$ and so on), where $p \in \mathbb{S}(V)$. Therefore the value of g is generically far from 0 w.r.to the pillow case metric $\omega_{\mathrm{FS}}^{\mathrm{modified}}$. This means that if g solves the equation $p(w) = 0$ (e.g., if $p(w) = 1 - w^2$, then $w = g = \pm i$), then the solution of the equation $p(g) = 0$ is generically close to the place where g has a pole (in particular, far from the pre-image $g^{-1}(\{g(P)\,|\,P \in \overline{M}\setminus M\})$. Theorem 2.8.3.1 implies that the effect of the group theoretic approximation implies

$$m_{g,w}(r) = (1 - O(n_{\mathrm{th}}^{-1}))\cdot T_g(r)$$

for $\forall w = g(p)$ where p is contained in a sufficiently small neighborhood of the set of all points where g has a pole. Therefore, $p(g) = 0$ happens at places in the domain of g which are close to the poles of g in the sense that $p(g) = 0$ has solutions at places in a small punctured neighborhood of poles of g where $p \in \mathbb{S}(V)$ is generic. Thus, the approximation of $p(g)$ to 0 gives rise to the approximation of g to a small neighborhood of places where g has a pole and therefore, by the regularity condition, the "extra approximation of e^H to $\mathcal{D}$". Here, the "extra approximation of e^H to $\mathcal{D}$" should not be understood literally. Instead, this should be understood as the approximation of e^H to the values of e^H at places where $p(g) = 0$ which occurs via the approximation of $(e^H)^{(1)}$ to S_0. We have shown in [KM2, Part II, Lemma 5.3.3] that the approximation of g to ∞ contributes only to the approximation of e^H to $\mathcal{D}$. We can argue in the parallel way that the effect of this "extra approximation to $\mathcal{D}$" can be estimated in a similar way. Since the equation $p(w) = 0$ is a quadratic equation, we infer that the "extra approximation of e^H to $\mathcal{D}$" in question is generically of magnitude $2\, m_{g,\infty}(r)$. Indeed, we can work separately on two roots of the equation $p(w) = 0$. In this case, by using the argument in the proof of [KM2, Part II, (5.4.15) and (5.4.17)], we conclude that "the extra approximation to $\mathcal{D}$" is $m_{g,w_1}(r) + m_{g,w_2}(r)$ (w_1, w_2 being the places where $p(g) = 0$ occurs) which is generically of the magnitude $m_{g,\infty}(r)+m_{g,\infty}(r) = 2m_{g,\infty}(r)$ by the group theoretic approximation described in Theorem 2.8.3.1. Taking [KM2, Part II, (5.4.15) and (5.4.17)] (i.e., $\frac{3}{2}m^{\zeta}_{H',S_0}(r) - m^{\zeta}_{H',S_0}(r) = \frac{1}{2}m^{\zeta}_{H',S_0}(r)$) and the extra approximation stemming from the solution of $p(g) = 0$ (i.e., $2m_{g,\infty}(r)$) into account, we have the RHS of (2.10.5a). □

Lemma 2.10.5 (Bounding $\kappa_g T_g(r) - J(r)$) (cf. [KM2, Part II, Lemma 5.4.5]). *Let $(M, (g, \omega))$ be an algebraic minimal surface. Then we have*

$$\kappa_g T_g(r) - J(r) \le (4 - 10\kappa_g^{-1}) \log \frac{1}{1-r} \le (4\kappa_g - 10)\, T_g(r) \le (4e_+ - 10)\, T_g(r) \quad \text{asymptotically as} \quad r \to 1 \ .$$

Proof. For simplicity we omit small $\delta > 0$ which appears in applying LLD. Recall Lemmas 2.10.3-4, i.e.,

$$J_2(r) \le \kappa_g T_g(r) - T_g(r) + N_{g,\infty}(r)$$

and

$$(8\kappa_g^{-1} - 2) \cdot \log \frac{1}{1-r} \le J_1(r) + 2\, J_2(r) + 2N_{g,\infty}(r) \ .$$

It follows that

$$\begin{aligned} J_1(r) &\ge (8\kappa_g^{-1} - 2) \cdot \log \frac{1}{1-r} - 2\, J_2(r) - 2\, N_{g,\infty}(r) \\ &\ge (8\kappa_g^{-1} - 2) \cdot \log \frac{1}{1-r} - (2 - 2\kappa_g^{-1}) \cdot \log \frac{1}{1-r} - 4\, N_{g,\infty}(r) \\ &= (10\kappa_g^{-1} - 4) \cdot \log \frac{1}{1-r} - 4\, N_{g,\infty}(r) \ . \end{aligned}$$

We have from Lemma 2.8.1.1 (LLD type interpretation of parabolic localization) the estimate

$$J(r) \ge \ J_1(r) + \log \frac{1}{1-r} \ .$$

Therefore, we have

$$J(r) \ge \ J_1(r) + \log \frac{1}{1-r} \ge (10\kappa_g^{-1} - 3)\ \log \frac{1}{1-r} - 4\, N_{g,\infty}(r) \ .$$

Therefore

$$\begin{aligned} \kappa_g T_g(r) - J(r) &\le \log \frac{1}{1-r} - (10\kappa_g^{-1} - 3) \cdot \log \frac{1}{1-r} + 4\, N_{g,\infty}(r) \\ &= (4 - 10\kappa_g^{-1}) \cdot \log \frac{1}{1-r} + 4\, N_{g,\infty}(r) \\ &\le (4\kappa_g - 10) \cdot T_g(r) + 4\, N_{g,\infty}(r) + O(n_{\text{th}}^{-1}) T_g(r) \ . \end{aligned}$$

Finally, we use the Theorem 2.8.3.1 (group theoretic approximation arising from parabolic localization), namely, we use the estimate

$$N_{g,\infty}(r) \le O(n_{\text{th}}^{-1}) T_g(r) \ .$$

We have

$$\kappa_g T_g(r) - J(r) \le (4\, e - 10)\, T_g(r) + O(n_{\text{th}}^{-1}) T_g(r) \ ,$$

which finishes the proof of Lemma 2.10.5. □

Theorem 2.10.6 (Second Main Theorem for Gauss map of algebraic minimal surfaces) (cf. [KM2, Part II, Theorem 5.4.6]). *The lifted Gauss map of an algebraic minimal surface $(M,(g,\omega))$ satisfies the Second Main Theorem*

$$m_{g,D}(r) + N_{g,\mathrm{Ram}}(r) \leq \{4(e-2) + O(n_{\mathrm{th}}^{-1})\}\, T_g(r)\ .$$

The inequality should be understood that $\leq$ (resp. $4(e-2)$) means $\leq_\delta$ (resp. $(4(e-2)+\delta)$).

Proof of Theorem 2.10.6. By rotating a given algebraic minimal surface in $\mathbb{R}^3$, we may assume without loss of generality that

(i) $\{\infty, 0\}$ is NOT contained in the union of $g(\overline{M}\setminus M)$ and the g-image of the set P s.t. $dg(P)=0$,

(ii) $\{\infty, 0\}$ is NOT contained in $\mathrm{Supp}(D)$.

In this setting, we may use Theorem 2.8.3.1. Combining Lemma 2.9.1 and Lemma 2.10.6 we have Theorem 2.10.6. □

In particular, Theorem 2.10.6 implies that the totally ramified value number ν_g (resp. the number D_g of the exceptional values of $g : M \to \mathbb{P}^1$) satisfies

$$D_g \leq \nu_g \leq 4(e-2) \leq 2.88\ .$$

This settles Osserman's problem, i.e., the Gauss map of an algebraic minimal surface can omit at most two values.

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