第18回 岡　シンポジュウム

モジュライ理論と可積分系

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14:00-16:00, 2019年11月30日

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Supported by JSPS Grand-in-Aid for Scientific Research (S)17H06127

数学における経歴

- 1980年　京都大学理学部卒業
- 1982年　京都大学理学研究科　数学専攻修士課程修了
- 1985年　同　博士後期課程修了 博士号取得
- 1985年　学術振興会ＰＤ(10月から）
- 1986年　滋賀大学
- 1989年　北海道大学
- 1991年　京都大学
- 1996年　神戸大学理学部　現在に至る。23年目
- 2017年　神戸大学数理・データサイエンスセンター

海外渡航歴

- 1987/10-1988/09 Max Planck 数学研究所 (Bonn) 研究員
- 1990/19-1991/05 Johns Hopkins 大学 日米数学研究所研究員
- 1994/10–1995/01 Cambridge 大学数学教室客員教授
- 1995/02–1996/06, 2003/09–2003/10 Utrecht 大学数学研究所客員教授

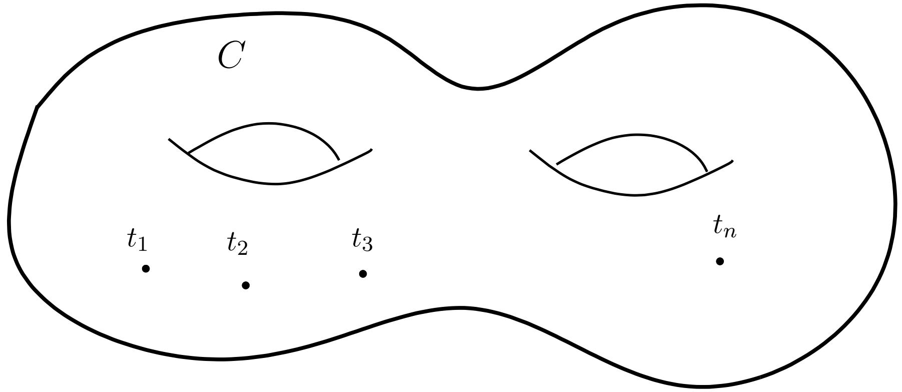
1. Moduli spaces of stable λ-parabolic connectios

1.1. Settings.

- C: a nonsingular projective curve of genus $g \geq 0$
- $\mathbf{t} = \{t_1, \ldots, t_n\}$, a set of n-distinct points on C.

$$D(\mathbf{t}) = \sum_{i=1}^{n} t_i = t_1 + \cdots + t_n.$$

- $M_{g,n} = \{(C, \mathbf{t}) \quad \text{as above}\}/\simeq$: The moduli of (ordered) n-pointed curves of genus g.



1.2. λ-connections. Fix $\lambda \in \mathbf{C}$.

Definition 1.1. (E, ∇) is called a λ-connection if

- E : An algebraic vector bundle on C of rank r and of degree d.
- $\nabla : E \longrightarrow E \otimes \Omega^1_C(D(\mathbf{t}))$: A logarithmic λ-connection. $a \in \mathcal{O}_C, \sigma \in E$

$$\nabla(a\sigma) = \lambda\sigma \otimes da + a\nabla(\sigma) \quad \boxed{\lambda\text{-twisted Leibniz rule}}$$

We denote by

$$\boxed{L = \Omega^1_C(D(\mathbf{t}))}$$

the line bundle or the invertible sheaf of meromorphic 1 form on C having poles on $D(\mathbf{t}) = t_1 + t_2 + \cdots + t_n$ at most order 1. Later we may allow the higher order pole $D(\mathbf{t}) = m_1t_1 + m_2t_2 + \cdots m_nt_n$ with $m_i \geq 1$. $\boxed{\deg L = 2g - 2 + n}$. We assume that $n \geq 1$ by a technical reason.

- $\lambda \neq 0$: **linear connection**:

  (E, ∇): λ-connection $\Rightarrow$ $(E, \frac{1}{\lambda}\nabla)$: a usual connection

  Locally near at $z = t_i$, taking a local frame of E near $z = t_i$, $E \simeq \mathcal{O}_{C,t_i}^{\oplus r} \ni (a_k(z))_{k=1}^r$, $A(z)\frac{dz}{z-t_i} \in \mathrm{M}_r(\mathcal{O}_{C,t_i}) \otimes \Omega_C^1(D(\mathbf{t}))$

$$\nabla((a_k(z))) = \lambda(da_k(z)) + A(z)(a_k(z))\frac{dz}{z - t_i}$$

- $\lambda = 0$: **Higgs bundle**: Denote $\nabla = \Phi$.

  (E, Φ): 0-connection $\Rightarrow$ (E, Φ) :a Higgs bundle, Φ:Higgs field

  Twisted Leibniz rule leads: for a local section $a \in \mathcal{O}_C, \sigma \in E$

$$\Phi(a\sigma) = a\Phi(\sigma) \quad \boxed{\text{an } \mathcal{O}_C\text{-linear hom.}}$$

  $\Phi \in \mathrm{End}(E) \otimes L$. Locally near $z = t_i$, $B(z)\frac{dz}{z-t_i} \in \mathrm{M}_r(\mathcal{O}_{C,t_i}) \otimes L$.

$$\Phi((a_k(z))) = B(z)(a_k(z))\frac{dz}{z - t_i}$$

1.3. Residues and Local exponets.

- (E, ∇), (E, Φ) as above.
- $\text{res}_{t_i}(\nabla) = A(t_i), \text{res}_{t_i}(\Phi) = B(t_i) \in \mathrm{End}(E_{|t_i})$: residue homomorphisms. $A(t_i) = (a_{kl})_{1\le k,l\le r}$, $B(t_i) = (b_{kl})_{1\le k,l\le r}$: complex $r \times r$ matrices.
- We put an order of eigenvalues of $\text{res}_{t_i}(\nabla)$ and $\text{res}_{t_i}(\Phi)$ respectively, and denote them as

$$\{\nu_0^{(i)}, \nu_1^{(i)}, \cdots, \nu_{r-1}^{(i)}\}$$

local exponents of ∇ at t_i.

- We denote the local exponents of ∇ and Φ by

$$\boldsymbol{\nu} = (\nu_j^{(i)})_{0\le j\le r-1}^{1\le i\le n}$$

1.4. Fuchs relation.

Lemma 1.1. For a λ-connection (E, ∇) (resp. a Higgs bundle (E, Φ)), with singularity at $D(\mathbf{t})$ as above, we have the following relation.

$$\sum_{i=1}^{n}\left(\sum_{j=0}^{r-1} \nu_j^{(i)}\right) = -\lambda \deg E = -\lambda d$$

$$\left(\text{resp.} \quad \sum_{i=1}^{n}\left(\sum_{j=0}^{r-1} \nu_j^{(i)}\right) = 0\right)$$

1.5. The space of local exponents of λ-connections.

$$\mathcal{N}_{r,\lambda}^{n}(d) := \left\{ \boldsymbol{\nu} = (\nu_j^{(i)})_{0 \le j \le r-1}^{1 \le i \le n} \in \mathbf{C}^{nr} \;\middle|\; \lambda d + \sum_{1 \le i \le n} \sum_{0 \le j \le r-1} \nu_j^{(i)} = 0 \right\}.$$

$$\mathcal{N}_{r,H}^{n} = \mathcal{N}_{r}^{n}(0) \quad \boxed{\text{Higgs bundle case}}$$

1.6. **Genericity for local exponents.**

Definition 1.2. Let $\boldsymbol{\nu} = \{\nu_j^{(i)}\}_{1\le i\le n}^{0\le j\le r-1} \in \mathcal{N}_{r,\lambda}^n(d)$.

(1) $\boldsymbol{\nu}$ is called *resonant*, if for some i and $j_1 \neq j_2$, $\nu_{j_1}^{(i)} - \nu_{j_2}^{(i)} \in \lambda\mathbf{Z}$.

(2) $\boldsymbol{\nu}$ is called *reducible* if there exists a subset $\boldsymbol{\nu}' = \{\nu_{j'}^{(i)}\}$ of $\boldsymbol{\nu}$ such that for each $i, 1 \le i \le n$, the number of $\nu_{j'}^{(i)} \in \boldsymbol{\nu}'$ is a fixed number k, $1 \le k \le r-1$ and $\sum_{\boldsymbol{\nu}'} \nu_{j'}^{(i)} \in \lambda\mathbf{Z}$ where the last sum is taken over $\boldsymbol{\nu}'$. If $\boldsymbol{\nu}$ is not reducible, $\boldsymbol{\nu}$ is called *irreducible*

(3) If $\boldsymbol{\nu}$ is neither resonant, nor reducible, we call $\boldsymbol{\nu}$ is *generic*.

Remark 1.1. If a λ-connection (E, ∇) has a subconnection $(F, \nabla_{|F})$ is with $0 < \operatorname{rank} F < \operatorname{rank} E$, the local exponents of (E, ∇) is reducible.

1.7. Parabolic connections.

Definition 1.3. Fix $(C, \mathbf{t}) \in M_{g,n}$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$

- $(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})$: a *$\boldsymbol{\nu}$-parabolic connection of rank r and degree d* on C

$$\Longleftrightarrow$$

- (E, ∇): a logarithmic connection of rank r and degree d

$$\nabla : E \longrightarrow E \otimes \Omega_C^1(D(\mathbf{t}))$$

- $l_*^{(i)} : E_{|t_i} = l_0^{(i)} \supset l_1^{(i)} \supset \cdots \supset l_{r-1}^{(i)} \supset l_r^{(i)} = 0$: a filtration of $E_{|t_i}$ for each $i, 1 \leq i \leq n$ such that

(1) $\dim(l_j^{(i)}/l_{j+1}^{(i)}) = 1$ and

(2) $(\mathrm{res}_{t_i}(\nabla) - \nu_j^{(i)})(l_j^{(i)}) \subset l_{j+1}^{(i)}$ for $j = 0, 1, \cdots, r-1$.

1.8. Parabolic stability. Next, we define $\boldsymbol{\alpha}$-stability condition on the $\boldsymbol{\nu}$-parabolic connections $(E, \nabla, \{l_*^{(i)}\}_{1\leq i\leq n})$.

- Fix a sequence of rational numbers $\boldsymbol{\alpha} = (\alpha_j^{(i)})_{1\leq j\leq r}^{1\leq i\leq n}$ such that

(1) $$0 < \alpha_1^{(i)} < \alpha_2^{(i)} < \cdots < \alpha_r^{(i)} < 1$$

  for $i = 1, \ldots, n$ and $\alpha_j^{(i)} \neq \alpha_{j'}^{(i')}$ for $(i, j) \neq (i', j')$.

- $(E, \nabla, \{l_*^{(i)}\}_{1\leq i\leq n})$: a $\boldsymbol{\nu}$-parabolic connection.
- $0 \subsetneqq F \subset E$, $\nabla(F) \subset F \otimes \Omega_C^1(D(\mathbf{t}))$. Define integers $\mathrm{length}(F)_j^{(i)}$ by

(2) $$\mathrm{length}(F)_j^{(i)} = \dim(F|_{t_i} \cap l_{j-1}^{(i)})/(F|_{t_i} \cap l_j^{(i)}).$$

  Note that $\mathrm{length}(E)_j^{(i)} = \dim(l_{j-1}^{(i)}/l_j^{(i)}) = 1$ for $1 \leq j \leq r$.

Definition 1.4. • A $\boldsymbol{\nu}$-parabolic connection $(E, \nabla, \{l_*^{(i)}\}_{1\leq i\leq n})$: is $\boldsymbol{\alpha}$-stable

$$\Longleftrightarrow$$

$0 \underset{\neq}{\subsetneq} F \underset{\neq}{\subsetneq} E$, $\nabla(F) \subset F \otimes \Omega^1_C(D(\mathbf{t}))$,

$$\frac{\deg F + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \operatorname{length}(F)_j^{(i)}}{\operatorname{rank} F} < \frac{\deg E + \sum_{i=1}^n \sum_{j=1}^r \alpha_j^{(i)} \operatorname{length}(E)_j^{(i)}}{\operatorname{rank} E}$$

We can define the notion of:

- a $\boldsymbol{\nu}$-parabolic Higgs bundle $(E, \Phi, \{l_*^{(i)}\}_{1\leq i\leq n})$ and
- the $\boldsymbol{\alpha}$-stability conditions for a $\boldsymbol{\nu}$-parabolic Higgs bundle as in the same way above.

1.9. Moduli spaces of stable parabolic connections and stable parabolic Higgs bundles.

- Fix $(C, \mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}_r^n(d)$. We can define the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections

$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d) = \{(E, \nabla, \{l_*^{(i)}\}_{1 \leq i \leq n})\}/ \simeq . \tag{3}$$

- Moreover for $\boldsymbol{\nu} \in \mathcal{N}_{r,H}^n$, we can define the moduli space of $\boldsymbol{\alpha}$-stable parabolic Higgs bundles:

$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)_H = \{(E, \Phi, \{l_*^{(i)}\}_{1 \leq i \leq n})\}/ \simeq . \tag{4}$$

1.10. Existence of algebraic moduli space of α-stable $\boldsymbol{\nu}$-parabolic connections.

Theorem 1.1. (Inaba-Iwasaki-Saito RIMS2006 [6], ASPM2006 [7], Inaba, JAG2013 [5]). There exists the relative fine moduli scheme

$$\pi : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/\tilde{M}_{g,n}\times\mathcal{N}^n_r(d)}(r,d,n) \longrightarrow \tilde{M}_{g,n} \times \mathcal{N}^n_r(d)$$

such that π is smooth and quasi-projective.

Corollary 1.1. For fixed $(C, \mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}^n_r(d)$, the moduli space

$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)$$

is a smooth quasi-projective algebraic scheme (most case irreducible) of dimension

$$2r^2(g-1) + nr(r-1) + 2 = 2N.$$

Moreover $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)$ admits the natural algebraic symplectic structure.

1.11. As in the similar way, we can obtain the existence of algebraic moduli space of α-stable $\boldsymbol{\nu}$-parabolic Higgs bundles ($K(D)$-pairs of Boden and Yokogawa).

Theorem 1.2. There exists the relative fine moduli scheme

$$\pi : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/\tilde{M}_{g,n}\times\mathcal{N}^n_r(d)}(r,d,n)_H \longrightarrow \tilde{M}_{g,n} \times \mathcal{N}^{n,H}_r$$

such that π is smooth and quasi-projective.

Corollary 1.2. For fixed $(C,\mathbf{t})$ and $\boldsymbol{\nu} \in \mathcal{N}^n_{r,H}$, the moduli space

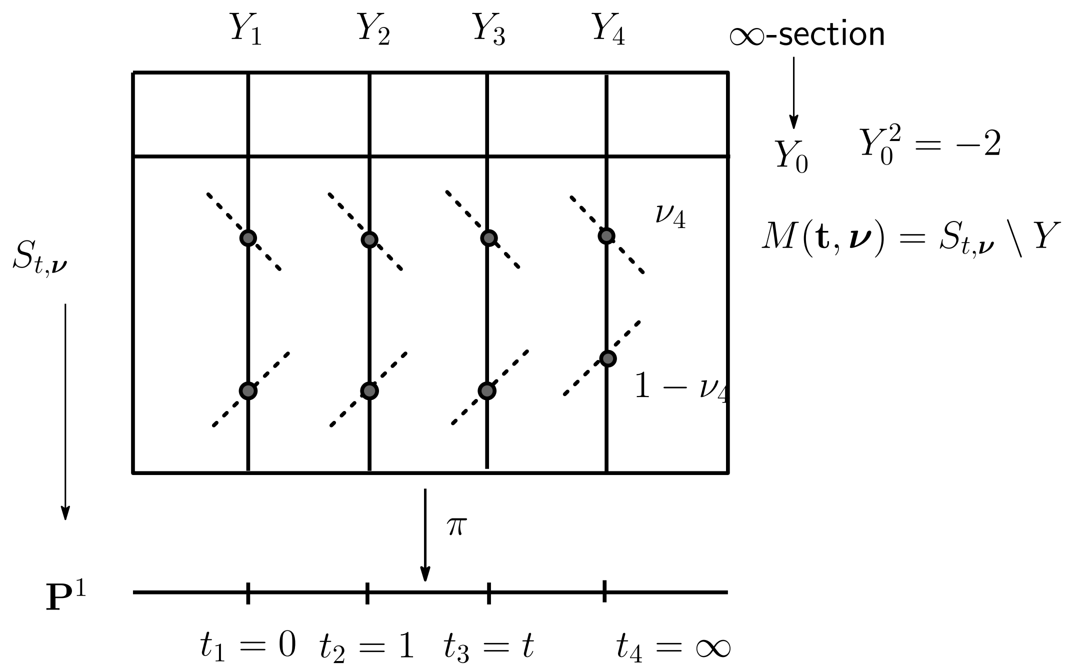
$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,n,d)_H$$

is a smooth quasi-projective algebraic scheme (most case variety) of dimension

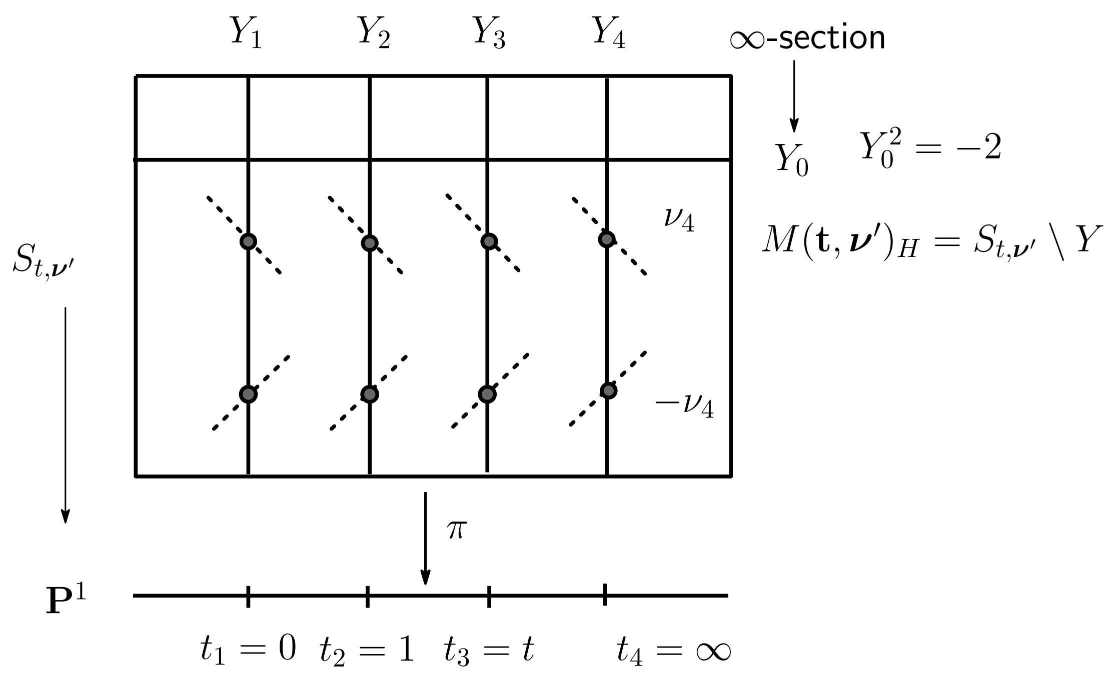
$$2r^2(g-1)+nr(r-1)+2=2N.$$

Moreover $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu},r,n,d)_H$ admits the natural algebraic symplectic structure.

1.12. **Example: Moduli space of connections, Painlevé VI case.** Consider the case: $C = \mathbf{P}^1, r = 2, n = 4, d = -1$ and a generic $\boldsymbol{\nu} \in \mathcal{N}_2^4(-1)$. We can normalize $\mathbf{t} = \{t_1, t_2, t_3, t_4\} = \{0, 1, t, \infty\}$ and $\boldsymbol{\nu} = \{\pm\nu_1, \pm\nu_2, \pm\nu_3.\nu_4, 1-\nu_4\}$. Then the moduli space $M(\mathbf{t}, \boldsymbol{\nu}) = \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1, \mathbf{t})}(\boldsymbol{\nu}, 2, 4, -1)$ is an algebraic surface. $\dim M(\mathbf{t}, \boldsymbol{\nu}) = 2N = 4(0-1) + 4 \times 2 + 2 = 2$. $M(\mathbf{t}, \boldsymbol{\nu})$ has a nice compactification $S_{\mathbf{t},\boldsymbol{\nu}} = \overline{M(\mathbf{t}, \boldsymbol{\nu})}$. $S_{\mathbf{t},\boldsymbol{\nu}}$ is a 8-points blowing up of $\Sigma_2 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$. The points of blowing up depends on the local exponents $\boldsymbol{\nu}$. See below. The anti-canonical divisor of $S_{\mathbf{t},\boldsymbol{\nu}}$ is given $-K_{S_{\mathbf{t},\boldsymbol{\nu}}} = 2Y_0 + Y_1 + Y_2 + Y_3 + Y_4$.
$\boxed{M(\mathbf{t}, \boldsymbol{\nu}) = S_{\mathbf{t},\boldsymbol{\nu}} \setminus Y}$.



1.13. **Example: Moduli space of parabolic Higgs bundles.** Consider the case: $C = \mathbf{P}^1, r = 2, n = 4, d = -1$ and a generic $\boldsymbol{\nu}' \in \mathcal{N}_2^4(0)$. We can normalize $\mathbf{t} = \{t_1, t_2, t_3, t_4\} = \{0, 1, t, \infty\}$ and $\boldsymbol{\nu}' = \{\pm\nu_1, \pm\nu_2, \pm\nu_3. \pm\nu_4\}$. Then $M(\mathbf{t}, \boldsymbol{\nu}')_H = \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1, \mathbf{t})}(\boldsymbol{\nu}', 2, 4, -1)_H$ is also an algebraic surface. $\dim M_H(\mathbf{t}, \boldsymbol{\nu}') = 2N = 4(0-1) + 4 \times 2 + 2 = 2$. $M_H(\mathbf{t}, \boldsymbol{\nu}')$ has a nice compactification $S_{\mathbf{t},\boldsymbol{\nu}'} = \overline{M(\mathbf{t}, \boldsymbol{\nu}')_H}$. $S_{\mathbf{t},\boldsymbol{\nu}'}$ is a 8-points blowing up of $\Sigma_2 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-2))$. $-K_{S_{\mathbf{t},\boldsymbol{\nu}'}} = 2Y_0 + Y_1 + Y_2 + Y_3 + Y_4$. $\boxed{M(\mathbf{t}, \boldsymbol{\nu}')_H = S_{\mathbf{t},\boldsymbol{\nu}'} \setminus Y}$. We can see that algebraic structures of $M(\mathbf{t}, \boldsymbol{\nu})$ and $M(\mathbf{t}, \boldsymbol{\nu}')_H$ are different.



2. The Riemann-Hilbert correspondence

2.1. Moduli space of representations of $\pi_1(C \setminus D(\mathbf{t}), *)$. Define:

$$\mathcal{RP}^r_{(C,\mathbf{t})} = \mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), GL_r(\mathbf{C}))//Ad(GL_r(\mathbf{C}))$$

or

$$\mathcal{RP}^{r,s}_{(C,\mathbf{t})} = \mathrm{Hom}(\pi_1(C \setminus D(\mathbf{t}), *), SL_r(\mathbf{C}))//Ad(SL_r(\mathbf{C}))$$

By definition, $\mathcal{RP}^r_{(C,\mathbf{t})}$ and $\mathcal{RP}^{r,s}_{(C,\mathbf{t})}$ are affine varieties associated to the invariant ring of matrices.

Replacing $T = \mathcal{M}'_{g,n}$ by a certain finite étale covering $u : T' \longrightarrow T$ and varying $((C, \mathbf{t}), \nu) \in T' \times \mathcal{N}_r^{(n)}(d)$ we can define a morphism

$$\mathbf{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\mathbf{t})/T'}(r, n, d) \longrightarrow \mathcal{RP}^r_{n,T'} \tag{5}$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T'}(r, n, d) & \stackrel{\mathbf{RH}}{\longrightarrow} & \mathcal{RP}^r_{n,T'} \\ \Phi_{r,n,d}\big\downarrow & & \big\downarrow \phi^r_n \\ T' \times \mathcal{N}_r^{(n)}(d) & \stackrel{Id \times rh}{\longrightarrow} & T' \times \mathcal{A}_r^{(n)} \end{array} \tag{6}$$

commute.

2.2. Riemann-Hilbert correspondences.

Theorem 2.1. (Inaba-Iwasaki-Saito, RIMS2006 [6], ASPM2006[7], Inaba JAG2013[5]). Assume that $\boldsymbol{\alpha}$ is generic. The Riemann-Hilbert correspondence

$$\mathbf{RH} : \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathcal{C},\tilde{\mathbf{t}})/T'}(r,n,d) \longrightarrow \mathcal{RP}^{r}_{n,T'} \times_{\mathcal{A}_r^{(n)}} \mathcal{N}_r^{(n)} \tag{7}$$

is a *proper surjective bimeromorphic analytic* morphism. In particular, for each $((C,\mathbf{t}),\boldsymbol{\nu}) \in T' \times \mathcal{N}_r^{(n)}(d)$, the restricted morphism

$$\mathbf{RH}_{((C,\mathbf{t}),\boldsymbol{\nu})} : \mathcal{M}^{\boldsymbol{\alpha}}_{((C,\mathbf{t}),\boldsymbol{\nu})}(r,n,d) \longrightarrow \mathcal{RP}^{r}_{(C,\mathbf{t}),\mathbf{a}} \tag{8}$$

gives an analytic resolution of singularities of $\mathcal{RP}^{r}_{(C,\mathbf{t}),\mathbf{a}}$ where $\mathbf{a} = rh(\boldsymbol{\nu})$ is a image of small Riemann-Hilbert correspondence rh.

3. General schemes of the geometry of Riemann-Hilbert correspndences

Consider the following diagram:

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\mathbf{RH}} & \tilde{\mathcal{R}} \\
\tilde{\pi} \Big\downarrow & & \Big\downarrow \tilde{\phi} \\
\tilde{T} \times N & \xrightarrow{(1 \times \mu)} & \tilde{T} \times \ \mathcal{A}.
\end{array}
$$

Theorem 3.1. If the Riemann-Hilbert map

$$\mathbf{RH}_{t,\boldsymbol{\nu}} : \tilde{M}_{t,\boldsymbol{\nu}} \longrightarrow \tilde{\mathcal{R}}_{t,\mu(\boldsymbol{\nu})}$$

is a **proper, surjective bimeromorphic holomorphic map** for any $(t, \boldsymbol{\nu}) \in \tilde{T} \times N$. Then the corresponding isomonodromic differential equations satisifies **the geometric Painlevé property**.

Isomonodromic Flows: $\boldsymbol{\nu}$ Generic Case

The Riemann-Hilbert correspondence $\mathbf{RH}_{\nu}$ induce an analytic isomorphisms for all $t \in \tilde{T}_n$. Pulling back the constant section on the right hand side, we have the isomonodromic flows on the left hand side. These isomondromic flows satisfy the Geometric Painlevé property.

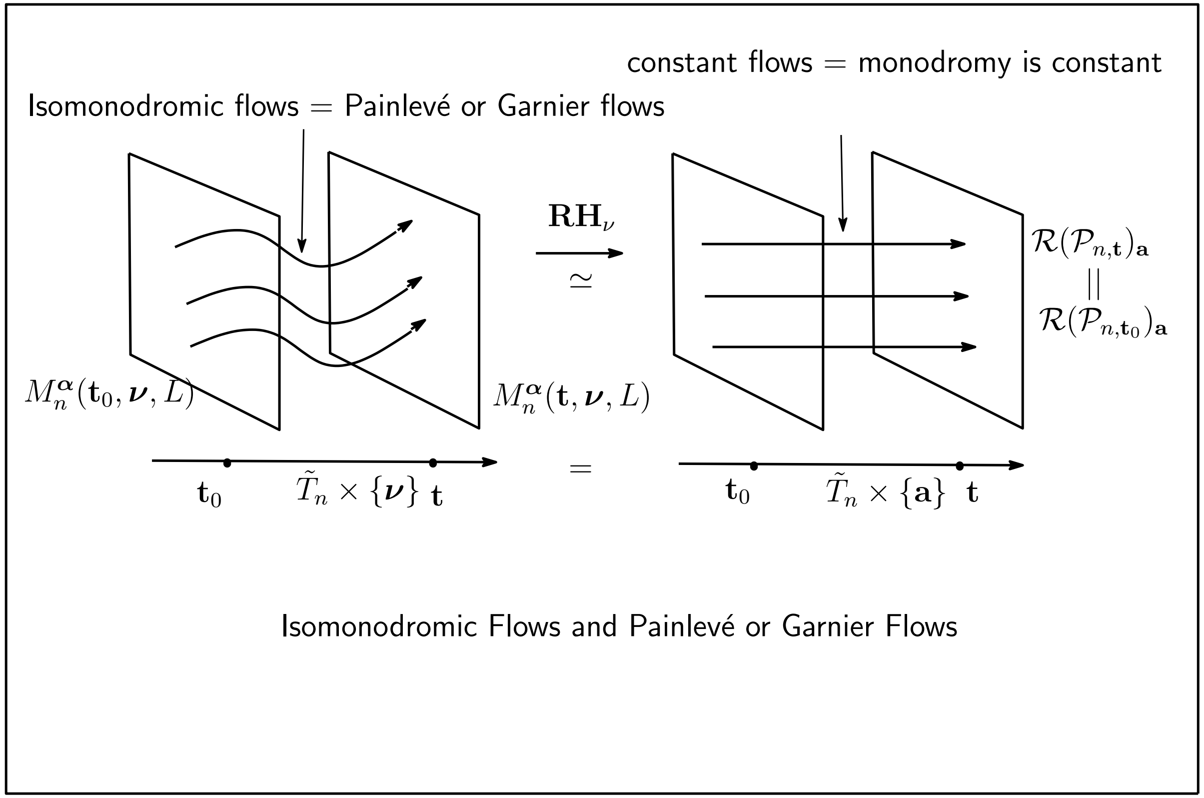
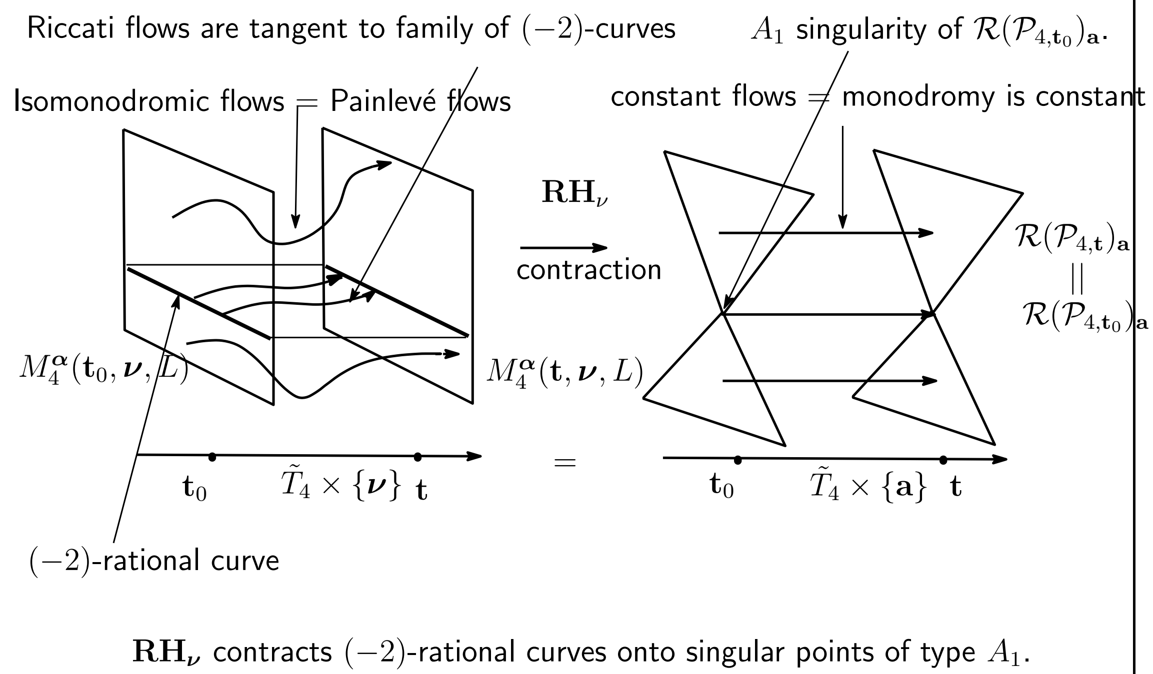


FIGURE 1. Riemann-Hilbert correspondence and isomonodromic flows for generic $\boldsymbol{\nu}$

Isomonodromic Flows: Special Case

If $\boldsymbol{\nu}$ is special (resonant, reducible), the right hand side have singularity. On the other hand, the left hand side is always nonsingular, hence $\mathbf{RH}_{\nu}$ gives a simultaneous resolution of singularities. Riccati flows.



Case of Painlevé VI

FIGURE 2. Riemann-Hilbert correspondence and isomonodromic flows for special $\boldsymbol{\nu}$

3.1. Geometric Painlevé property of the NDFE arrising from Isomonodromic deformation of LODE.

Corollary 3.1. ([6], [7], [5]) Differential equations arrising from isomonodromic deformations of linear connections with regular singularities over a curve satisfies the geometric Painlevé property.

Remark 3.1. We can extend the above result in the following cases;

- Connections of any rank with generic unramified irregular singularity on smooth projective curves. (Inaba-Saito, Kyoto JM2012 [9])
- Logarithmic connections of any rank with fixed spectral type with multiplicities. (Inaba-Saito, Math. Soc. Japan 2018).
- Generic ramified irregular singular case (Inaba. in preparation).

3.2. **Moduli spaces of monodromy representations and generalized Stokes data related to Painlevé equations. Monodromy variety for Painlevé VI case**

Define

$$
\begin{aligned}
\mathcal{RP}_4^{2,s} &= \mathrm{Hom}(\pi_1(\mathbf{P}^1 \setminus \{t_1, t_2, t_3, t_4\}, SL(2, \mathbf{C}))//Ad(SL_2(\mathbf{C})) \\
&= \{(M_1, M_2, M_3, M_4) \in SL_2(\mathbf{C}), M_1M_2M_3M_4 = I_2\}//Ad(SL_2(\mathbf{C})) \\
&= \{(M_1, M_2, M_3) \in SL_2(\mathbf{C})\}//Ad(SL_2(\mathbf{C}))
\end{aligned}
$$

We can describe the moduli space as follows.

Take $M_i \in SL_2(\mathbf{C})$ for $i = 1, 2, 3$ and set

$$a_i = \mathrm{Tr}[M_i], i = 1, 2, 3 \quad a_4 := \mathrm{Tr}[M_4] = \mathrm{Tr}[M_4^{-1}] = \mathrm{Tr}[M_1M_2M_3]$$

For a circle permutation (i, j, k) of $(1, 2, 3)$, set

$$x_i = \mathrm{Tr}[M_jM_k].$$

Then the invariant ring is given by

$$\mathbf{C}[M_1, M_2, M_3]^{SL_2(\mathbf{C})} = \mathbf{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f(\mathbf{x}, \mathbf{a}))$$

where we set the cubic polynomial given by Fricke-Klein, Jimbo and Iwasaki.

$$f(\mathbf{x}, \mathbf{a}) = x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\mathbf{a})x_1 - \theta_2(\mathbf{a})x_2 - \theta_3(\mathbf{a})x_3 + \theta_4(\mathbf{a})$$

$$\begin{aligned}\theta_i(\mathbf{a}) &= a_i a_4 + a_j a_k, \quad (i,j,k) = \text{a cyclic permutation of } (1,2,3), \\ \theta_4(\mathbf{a}) &= a_1 a_2 a_3 a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4.\end{aligned}$$

Theorem 3.2. *The monodromy variety of Painlevé VI is isomorphic to the affine variety*

$$\begin{aligned}\mathcal{X} = \mathcal{RP}_4^{2,s} &= SL_2(\mathbf{C})^3 // Ad(SL_2(\mathbf{C})) \\ &= \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3, a_1.a_2, a_3, a_4]/(f(\mathbf{x}, \mathbf{a})) \\ &= \{(\mathbf{x}, \mathbf{a}) \in \mathbf{C}^7, f(\mathbf{x}, \mathbf{a}) = 0\} \subset \mathbf{C}^7\end{aligned}$$

Moreover for a fixed $\mathbf{a} = (a_1, a_2, a_3, a_4) \in \mathbf{C}^4$

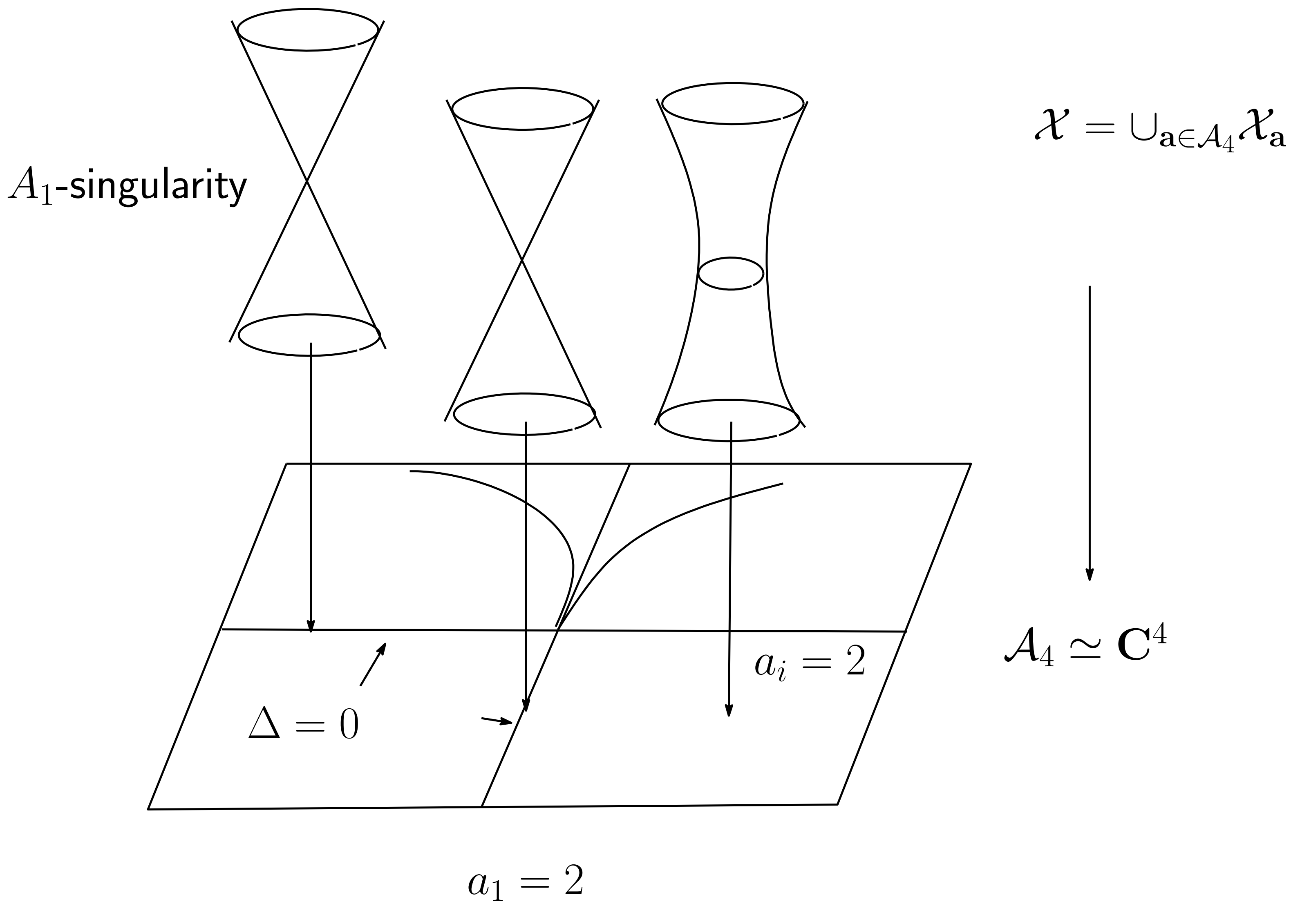
$$\mathcal{X}_{\mathbf{a}} = \mathcal{RP}_{4,\mathbf{a}}^{2,s} = \mathrm{Spec}(\mathbf{C}[x_1, x_2, x_3]/(f(\mathbf{x}, \mathbf{a}))) = \mathcal{X}_{\mathbf{a}} \subset \mathbf{C}^3 \subset \mathbf{P}^3.$$

The Riemann-Hilbert correspondence induces an analytic isomorphism for generic $\boldsymbol{\nu} = (\pm\nu_i, i = 1, 2, 3, \nu_4, 1 - \nu_4)$. $a_i = 2\cos(-2\pi\nu_i)$.

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\nu}} : M(\mathbf{t}, \boldsymbol{\nu}) \xrightarrow{\simeq} \mathcal{X}_{\mathbf{a}}$$

For special $\boldsymbol{\nu}$, we have a proper bimeromorphic analytic morphism (analytic resolution of singularities).

$$\mathbf{RH}_{\mathbf{t},\boldsymbol{\nu}} : M(\mathbf{t}, \boldsymbol{\nu}) \longrightarrow \mathcal{X}_{\mathbf{a}}$$



A_1-singularity
$\mathcal{X} = \cup_{\mathbf{a} \in \mathcal{A}_4} \mathcal{X}_{\mathbf{a}}$
$\mathcal{A}_4 \simeq \mathbf{C}^4$
$a_i = 2$
$\Delta = 0$
$a_1 = 2$

4. Types of Singularities of Linear connetions

Let us list up the types of irregular singular points of lin. connetions of rank 2 on $\mathbf{P}^1$ which induces iso-Stokes-Monodromy differential equations (=Lax equations)isomorphic to the Painlevé equations of the types in the table. This results follows from original result due to Garnier, Okamoto, Miwa-Jimbo-Ueno and Ohyama, Kawamuko, Sakai and Okamoto. (Moreover Flaschka and Newell obtained $PII(FN)$.)

Dynkin	Painlevé equation	$s(0)$	$s(1)$	$s(\infty)$	$s(t)$	no. of parameters
$\tilde{D}_4$	PVI	0	0	0	0	4
$\tilde{D}_5$	PV	0	0	1	-	3
$\tilde{D}_6$	deg PV= PIII(D6)	0	0	1/2	-	2
$\tilde{D}_6$	PIII(D6)	1	-	1	-	2
$\tilde{D}_7$	PIII(D7)	1/2	-	1	-	1
$\tilde{D}_8$	PIII(D8)	1/2	-	1/2	-	0
$\tilde{E}_6$	PIV	0	-	2	-	2
$\tilde{E}_7$	PII(FN)=PII	0	-	3/2	-	1
$\tilde{E}_7$	PII	-	-	3	-	1
$\tilde{E}_8$	PI	-	-	5/2	-	0

Table 1. The type of singularities for linear problems and Pailevé equations

Equations of Moduli space of Stokes-Monodromy data

The following result is due to a joint work with Marius van der Put ([21]).

(1) PVI $x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\mathbf{a})x_1 - \theta_2(\mathbf{a})x_2 - \theta_3(\mathbf{a})x_3 + \theta_4(\mathbf{a}) = 0$,
$\theta_i(\mathbf{a}) = a_ia_4 + a_ja_k, \quad (i,j,k) =$ a cyclic permutation of $(1,2,3)$,
$\theta_4(\mathbf{a}) = a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4.$ with $a_1, a_2, a_3, a_4 \in \mathbb{C}$.

(2) PV $x_1x_2x_3 + x_1^2 + x_2^2 - (s_1 + s_2s_3)x_1 - (s_2 + s_1s_3)x_2 - s_3x_3 + s_3^2 + s_1s_2s_3 + 1 = 0$
with $s_1, s_2 \in \mathbb{C},\ s_3 \in \mathbb{C}^*$.

(3) deg PV $x_1x_2x_3 - x_1^2 - x_2^2 + s_0x_1 + s_1x_2 - 1 = 0.$
with $s_0, s_1 \in \mathbb{C}$.

(4) PIII(D6) $x_1x_2x_3 + x_1^2 + x_2^2 + (1 + \alpha\beta)x_1 + (\alpha + \beta)x_2 + \alpha\beta = 0$
with $\alpha, \beta \in \mathbb{C}^*$.

(5) PIII(D7) $x_1x_2x_3 + x_1^2 + x_2^2 + \alpha x_1 + x_2 = 0$
with $\alpha \in \mathbb{C}^*$.

(6) **PIII(D8)** $x_1x_2x_3 + x_1^2 - x_2^2 - 1 = 0$. **wrong.**

(7) PIV $x_1x_2x_3 + x_1^2 - (s_2^2 + s_1s_2)x_1 - s_2^2x_2 - s_2^2x_3 + s_2^2 + s_1s_2^3$
with $s_1 \in \mathbb{C},\ s_2 \in \mathbb{C}^*$.

(8) PII(FN) $x_1x_2x_3 + x_1 - x_2 + x_3 + s_1 = 0$, with $s_1 \in \mathbb{C}$.

(9) PII $x_1x_2x_3 + x_1 + x_2 + \alpha x_3 + \alpha + 1 = 0$ with $\alpha \in \mathbb{C}^*$.

(10) PI=PI($\tilde{E}_8$) $x_1x_2x_3 + x_1 + x_2 + 1 = 0.$

4.1. Correction of PIII(D8) and $P = W$ conjecture.

PIII(D8) $S_1 : x_1x_2x_3 + x_1^2 - x_2^2 - 1 = 0$. **wrong.**

Simple mistake:

PIII(D8) $S_2 : x_1x_2x_3 + x_1^2 + x_2^2 - 1 = 0$.

More seriously, one should mode out by S_2 by the involution σ induced by

$$\sigma : (x_1, x_2, x_3) \longrightarrow (-x_1, -x_2.x_3)$$

Then one has the following equation for $S_3 = S_2/ < \sigma >$:

$$S_3 : y_1y_2x_3 + y_2^2 + y_1^2 - y_1 = 0$$

S_3 has a compactication $\overline{S_3} \subset \mathbf{P}^3$ as a singular cubic surface with A_4 singularity at the boundary $\overline{S_3} \setminus S_3$ and one can see easily the weight filtration of $H^2(S_3, \mathbf{Q})$ coming from $H^2(\widehat{\overline{S_3}}, \mathbf{Q})$ is 0 where $\widehat{\overline{S_3}}$ is the minimal resolution of $\overline{S_3}$. This corresponds to the fact that the corresponding pervese Leray sequence of Hitchin fibration $\pi : M_{Dol} \longrightarrow \mathbf{C}$ has no contribution from $H^1(\mathbf{C}, R^1\pi_*\mathbf{Q})$. This is a special case of $P = W$ conjecture which was proved by S. Szabo (arXiv:1802.03798). Actually, he checked the conjecture for all the cases of 10 types.

5. Apparent singularities (a joint work with S. Szabo)

5.1. Apparent singularities of connections and Higgs bundles.

- $C, \mathbf{t}$ as before.
- We set $L = \Omega^1_C(t_1 + \cdots + t_n)$. We assume that $n \geq 1$ and $\deg L = 2g - 2 + n > 0$.

Consider the moduli spaces

(9) $$M_{DR}(\boldsymbol{\nu}) = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)_{=}\{(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq n})\}/\simeq .$$

(10)
$$M_H(\boldsymbol{\nu}) = \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, n, d)_H = \{(E, \Phi, \{l^{(i)}_*\}_{1\leq i\leq n})\}/\simeq .$$

For simplicity, we assume that $\boldsymbol{\nu} \in \mathcal{N}^n_r(d)$ or $\boldsymbol{\nu} \in \mathcal{N}^n_{r,H}$ are non-resonant and so generic such that all members of moduli spaces are irreducible.

Proposition 5.1. Assume that $\exists \sigma \in H^0(C,E)\setminus\{0\}$ and $\deg L = 2g-2+n \geq 1$ and $\deg D = n \geq 1$. Moreover assume that (E,∇) (resp. (E,Φ)) is irreducible. Set

$$F = \oplus_{j=0}^{r-1} L^{-j} = \mathcal{O}_C \oplus L^{-1} \oplus \cdots \oplus L^{-(r-1)}. \tag{11}$$

$\exists$ a natural embedding $F \hookrightarrow E$ such that $H^0(C,F) \simeq \mathbf{C}\sigma \subset H^0(C,E)$. Define the torsion sheaf T_A by the exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0, \tag{12}$$

Then

$$\operatorname{length} T_A = d - r(g-1) + r^2(g-1) + n\frac{r(r-1)}{2}.$$

Definition 5.1. For an irreducible parabolic connection (E, ∇, l) (resp. irreducible parabolic Higgs bundles (E, Φ, l)) and a non-zero section σ, we call the support of T_A *apparent singular points* of the parabolic connection (E, ∇, l) (resp. (E, Φ, l)) with *the cyclic vector* σ.

Now assume that $\deg E = d = r(g-1)+1$. We have $\dim H^0(C, E) = \dim H^1(C, E) + 1$ by Riemann-Roch. If moreover $H^1(C, E) = 0$, we have a non-zero section $\sigma \in H^0(C, E) \simeq \mathbf{C}\sigma$ unique up to non-zero scalar multiplications.

Theorem 5.1. Under the same notation and assumption as before, let us assume that

$$d = \deg E = r(g-1) + 1, \tag{13}$$

$$H^1(C, E) = 0. \tag{14}$$

Then we have a natural unique embedding $F \hookrightarrow E$ which yields

$$0 \longrightarrow F \longrightarrow E \longrightarrow T_A \longrightarrow 0. \tag{15}$$

Then the sheaf T_A is a torsion sheaf of length

$$N = r^2(g-1) + n\frac{r(r-1)}{2} + 1. \tag{16}$$

5.2. The case of parabolic Higgs bundles.

- Let (E, Φ, l) be the $\boldsymbol{\nu}$-parabolic Higgs bundles of degree $d = \deg E = r(g-1)+1$ and assume that $\dim H^0(C, E) = 1$. Again we set $L = \Omega^1_C(D)$.
- We have a canonical exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0$$

with $F = \oplus_{j=1}^r L^{-(j-1)}$ and with apparent singularities

$$\mathrm{supp} T = \{q_1, \cdots, q_N\}$$

where

$$N = r^2(g-1) + n\frac{r(r-1)}{2} + 1 = \frac{1}{2}\dim M_H(\boldsymbol{\nu})$$

5.2.1. *Spectral curves.* Let

$$p : \mathbf{P} = \mathbb{P}(\mathcal{O}_C \oplus L^{-1}) \longrightarrow C$$

be the $\mathbb{P}^1$-bundle over C which is a relative compactification of the total space of $L \longrightarrow C$. The canonical section $x \in H^0(P, \mathcal{O}_P(1) \otimes p^*(L))$ can be used to define the spectral curve

$$C_s : \det(xI_r - \Phi) = x^r - s_1 x^{r-1} - s_2 x^{r-2} - \cdots s_r = 0 \subset L \subset P$$

with the natural map $\pi : C_s \longrightarrow C$ and $s_i \in H^0(C, L^i)$.

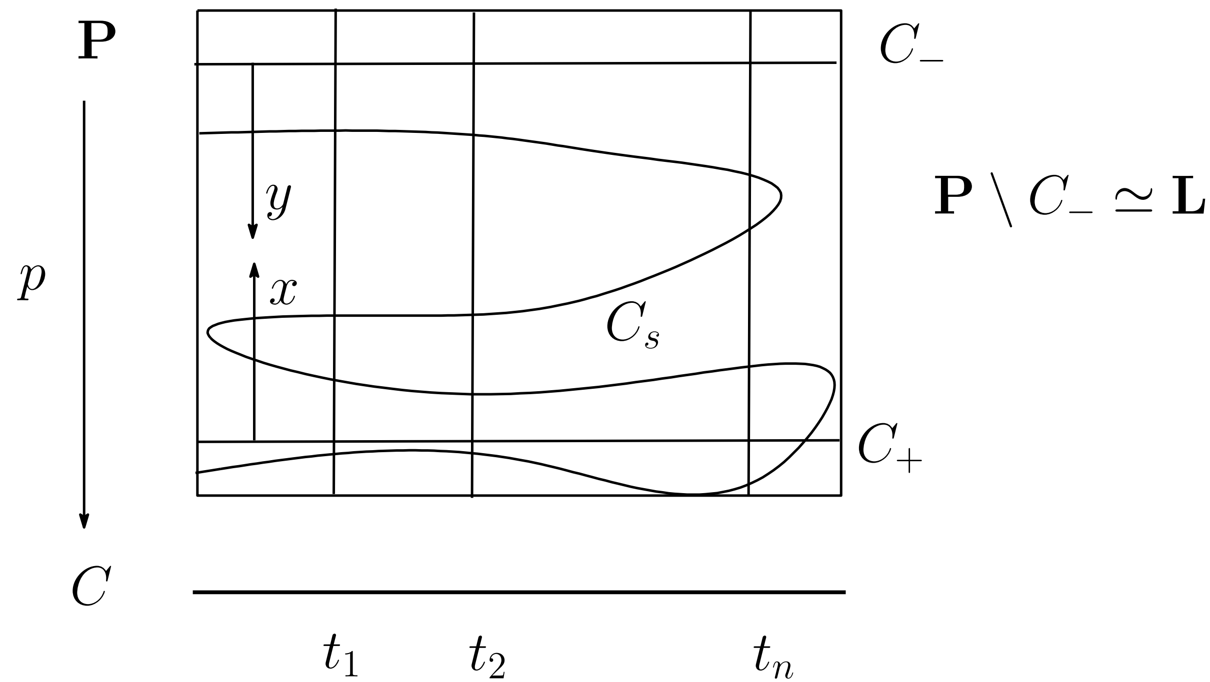


FIGURE 3. The ruled surface and the curve

Proposition 5.2. [BNR, [3]]. Assume that C_s is a smooth and irreducible Then there exists one to one correspondence

$$(E, \Phi, l) \Leftrightarrow (\pi : C_s \longrightarrow C, \xi)$$

where ξ is a line bundle on C_s. The correspondence $\Longleftarrow$is given by $\pi_* \xi = E$ and the structure of $\pi_* \mathcal{O}_{C_s}$-algebra.

Since $H^0(C_s, \xi) = H^0(C, E) = \mathbf{C}$, we see that a unique nonzero effective diviosr δ of degree

$$\deg \xi = \deg E - \deg F = r(g-1) + 1 + (2g - 2 + n)\frac{r(r-1)}{2} = N.$$

We have the natural exact sequence

$$0 \longrightarrow \mathcal{O}_{C_s} \longrightarrow \xi \longrightarrow \tilde{T} \longrightarrow 0$$

$$0 \longrightarrow \pi_* \mathcal{O}_{C_s} \longrightarrow \pi_* \xi \longrightarrow \pi_* \tilde{T} \longrightarrow 0$$

and $\pi_* \mathcal{O}_{C_s} \simeq F$, $\pi_* \xi = E$ and $\pi_* \tilde{T} = T$.

$$0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0$$

5.3. **Higgs case.** For (E, Φ, l), take the data of spectral curve and the line bundle $(\pi : C_s \longrightarrow C, \xi)$.

Since $H^0(C,E)$ a nonzero section σ, there exist a non-zero section $\tilde{\sigma} \in H^0(C_s,\xi)$ such that $\pi_*(\tilde{\sigma}) = \sigma$. Let $\delta = p_1 + \cdots + p_N$ be the zero divisor of $\tilde{\sigma}$. We have the exact sequence of sheaves on C_s

$$0 \longrightarrow \mathcal{O}_{C_s} \xrightarrow{\tilde{\sigma}} \mathcal{O}_{C_s}(\delta) \longrightarrow T_\delta \longrightarrow 0$$

The pushforward of this sequence

$$0 \longrightarrow \pi_*\mathcal{O}_{C_s} \longrightarrow \pi_*\xi \longrightarrow \pi_*T_\delta \longrightarrow 0$$

is isomorphic to

$$0 \longrightarrow F \longrightarrow E \longrightarrow T \longrightarrow 0$$

So we have

$$\pi(\delta) = \sum_{\imath=1}^{N} \pi(p_i) = \sum_{i=1}^{N} q_i.$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow \Phi & & \downarrow \Phi & & \downarrow \oplus\Phi_{q_i} & & \\ 0 & \longrightarrow & F \otimes L & \longrightarrow & E \otimes L & \longrightarrow & T \otimes L & \longrightarrow & 0 \end{array}$$

The dual coordinates $\{p_1, \cdots p_N\}$.

$$p_i = \Phi(q_i) \in L_{q_i}$$

5.4. **Geometric aspects of Higgs cases.** Let us set

$$M_H(\boldsymbol{\nu})^0 = \{(E, \Phi, l), \deg E = r(g-1)+1, H^0(C,E) \simeq \mathbf{C}\}.$$

Then we have the following

$$\begin{array}{ccccc} M_H(\boldsymbol{\nu})^0 & \simeq & M^0 := \{(C_s, \xi)\} & \stackrel{\phi}{\longrightarrow} & \mathrm{Hilb}^N(L) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hilb}^N(C) & & |C_s| & & \mathrm{Hilb}^N(C) \end{array}$$

apparent map   Hitchin fibration

$\phi((C_s, \xi)) = I_\delta$: Ideal sheaf of $\delta \subset C_s \subset L$

In many known cases, we can check that

ϕ is a dominant birational morphism,

and we expect that this statement is always true.

6. Moduli Spaces of Parabolic Bundles

For simplicity, we will consider only the full flag case.

- $\mathcal{P}^r_d$: the moduli stack of quasiparabolic bundles $(E, l) = (E, \{l^{(i)}_*\}_{1 \leq i \leq n})$ of rank r and degree d over $(C, \mathbf{t})$.

Definition 6.1. (1) (E, l) is simple if $H^0(C, \mathcal{F}^0) = \mathbf{C}$.

(2) (E, l) is *decomposable* if there exit quasiparabolic bundles (F_1, l_1) and (F_2, l_2) such that $(E, l) \simeq (F_1, l_1) \oplus (F_2, l_2)$ (after renumbering of filtrations).

(3) (E, l) is *indecomposable* if it is not decomposable.

Let us also denote by $\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)$ the moduli stack of $\boldsymbol{\alpha}$-stable connection (E, ∇, l) with the given invariants.

7. The image of $\boldsymbol{\nu}$-parabolic connections

For simplicity, we propose the following:

Assumption 7.1. The local exponents $\boldsymbol{\nu}$ is generic so that all (E, ∇, l) is irreducible.

Then we have the morphism from the stack to the coarse moduli space of $\boldsymbol{\alpha}$-stable connections

$$\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d) \longrightarrow M^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d).$$

Moreover, we have a natural forgetful morphism of stacks

$$\pi : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d) \longrightarrow \mathcal{P}^r_d, \quad \pi((E, \nabla, l)) = (E, l).$$

Question 7.1. Determine the image $\mathcal{P}^{r,flat}_d$ of π

$$\pi(\mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)) = \mathcal{P}^{r,flat}_d \subset \mathcal{P}^r_d$$

Theorem 7.1. If (E, l) is simple, $(E, l) \in \mathcal{P}_d^{r,flat}$

Let $\mathcal{P}_d^{r,s}$ the moduli stack of simple quasiparbolic bundles. We obtain an open embedding

$$\mathcal{P}_d^{r,s} \subset \mathcal{P}_d^{r,flat}.$$

For $C = \mathbf{P}^1$ and $r = 2, d = 0$, Arinkin and Lysenko showed that

Theorem 7.2. For $C = \mathbf{P}^1$, $(E, l) \in \mathcal{P}_0^2$, the following are equivalent.

(1) (E, l) is simple.
(2) (E, l) is undecomposable.
(3) (E, l) is flat, that is, $(E, l) \in \overline{\mathcal{P}_d^r}$.

So in the case of $C = \mathbf{P}^1$ with $(t_1, \cdots, t_n), n \geq 4$

$$\mathcal{P}_0^{2,s} = \mathcal{P}_0^{2,ud} = \mathcal{P}_0^{2,flat}$$

Moreover let us assum that the coarse moduli space of $\mathcal{P}_d^{r,flat}$ exists, and we obtain the natural morphism

$$\mathcal{P}_d^{r,flat} \longrightarrow P_d^{r,flat}$$

which has a $\mathbf{G}_m$-torsor structure.

In good case, $P_d^{r,flat}$ becomes a sheme, but it may be nonseparated schme.

We have the following commutative diagram.

$$\begin{array}{ccc} \pi : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)^0 & \longrightarrow & \mathcal{P}_d^{r,flat} \\ \downarrow & & \downarrow \\ \pi_1 : \mathcal{M}^{\boldsymbol{\alpha}}_{(C,\mathbf{t})}(\boldsymbol{\nu}, r, d)^0 & \longrightarrow & P_d^{r,flat} \end{array}$$

7.1. **The coarse moduli for $C = \mathbf{P}^1$, $n = 4$ Painlevé VI case.** Take $C = \mathbf{P}^1, r = 2, n = 4, d = -1$ and a generic $\boldsymbol{\nu} \in \mathcal{N}_2^4(-1)$. We can normalize $\mathbf{t} = \{t_1, t_2, t_3, t_4\} = \{0, 1, t, \infty\}$ and $\boldsymbol{\nu} = \{\pm\nu_1, \pm\nu_2, \pm\nu_3.\nu_4, 1 - \nu_4\}$. Then $\mathcal{M}(\mathbf{t}, \boldsymbol{\nu}) = \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1,\mathbf{t})}(\boldsymbol{\nu}, 2, 4, -1)$ is an algebraic surface.

We have isomorphisms

$$P^{2,flat}_{-1} = P^{2,s}_{-1} = P^{2,ud}_{-1} = P$$

and a natural morphism

$$M = \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1,\mathbf{t})}(\boldsymbol{\nu}, 2, 4, -1) \longrightarrow P.$$

Theorem 7.3. The moduli space of quasiparbolic bundles P is a nonseparated scheme obtained by two copies of $\mathbf{P}^1$ identifying at $\mathbf{P}^1 \setminus \{t_1, \cdots, t_4\}$. There are two points $t_i^{\pm} \in P$ for each i.

$7.2.$ $C = \mathbf{P}^1$ **and** $\mathbf{t} = (t_1, \cdots, t_5)$**.** Consider the case of $C = \mathbf{P}^1$ with 5 singular points and $r = 2, d = -1$. Frank Loray an I described $P = P_{-1}^{2,flat}$ as follows.

Theorem 7.4.

$$P = \hat{V} \cup V \cup V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5$$

Here $V \simeq \mathbf{P}^2$ and there is a natural embedding of $\mathbf{P}^1 \longrightarrow \Delta \subset \mathbf{P}^2$, whose image is a conic Δ. Blowing up the image of five points $t_1, \cdots, t_5 \in \Delta \subset \mathbf{P}^2$, we obtain the

$$\hat{V} \longrightarrow V \simeq \mathbf{P}^2.$$

Then we have 16-(-1) curves on $\hat{V}$ and we have 5 blowing downs

$$\hat{V} \longrightarrow V_i \simeq \mathbf{P}^2$$

besides original blowing up. Patching these projective surfaces by these biratinal map, one can obtain the moduli space P.

8. A result of Arinkn and Lysenko

In case of $C = \mathbf{P}^1, \mathbf{t} = (t_1, \cdots, t_4)$, $r = 2, d = 0$.

- (E, ∇, φ, l) such that $(E, \nabla, l) \in \mathcal{M}^{\boldsymbol{\alpha}}_{(\mathbf{P}^1, \mathbf{t})}(\boldsymbol{\nu}, 2, 4, 0)$ with an isomorphism $\varphi : \wedge^2 E \longrightarrow \mathcal{O}_{\mathbf{P}^1}$.
- $\mathcal{M}$: the moduli stack of (E, ∇, φ, l)
- P is the moduli space of undcomposable rank 2-bundles.
- $j : U = \mathbf{P}^1 \setminus \{t_1, \cdots, t_4\} \hookrightarrow P$. Consider $[\boldsymbol{\nu}] := \sum_{i=1}^4 \nu_i(t_i^+ - t_i^-) \in div(P) \otimes_{\mathbf{Z}} \mathbf{C}$. Denote by $D_{\boldsymbol{\nu}}$ the TDO ring corresponding to the divisor $[\boldsymbol{\nu}]$. For each $(E, \nabla, \varphi, l) \in \mathcal{M}$, Denote by $E_{[\boldsymbol{\nu}]}$ the $D_{\boldsymbol{\nu}}$ module defined by $E_{[\boldsymbol{\nu}]} = j_{*!}(E_{|U})$. Varying $(E, \nabla) \in \mathcal{M}$, we obatin $\xi_{[\boldsymbol{\nu}]}$ are $\mathcal{M}$-family of $D_{[\boldsymbol{\nu}]}$-modules on P.
- $\sigma : P \longrightarrow P$ is an isomorphism of P with $\sigma(t_i^{\pm}) = t_i^{\mp}$.
- $\mathcal{M}$ is a μ_2-gerbe. the derived category $D_qc(\mathcal{M})$ of quasicoherent sheaves on $\mathcal{M}$ naturally decomposes as $D_{qc}(\mathcal{M}) = D_{qc}(\mathcal{M})^+ \times D_{qc}(\mathcal{M})^-$, where $\mathcal{F} \in D_{qc}(\mathcal{M})^{\pm}$ if and only if $-1 \in \mu_2$ acts as ± 1 on $H^i(\mathcal{F})$ for any i.

$$\begin{array}{ccc} \mathcal{M} \times P & \xrightarrow{p_2} & P \\ p_1 \downarrow & & \\ \mathcal{M} & & \end{array}$$

The following is a theorem due to D. Arinkin around 2001.

Theorem 8.1. *The functor*

$$\Phi_{\mathcal{M}\longrightarrow P} : \mathcal{F} \longrightarrow \mathbf{R}p_{2,*}(\xi_{[\nu]} \otimes_{\mathcal{O}_{\mathcal{M}\times P}} p_1^*(\mathcal{F}))[1]$$

is an equivalent between $D_{qc}(\mathcal{M})^-$ and the derived category of $D_{[\boldsymbol{\nu}]}$-modules. The inverse funtor is given by

$$\Phi_{P\longrightarrow\mathcal{M}} : \mathcal{F} \longrightarrow \mathbf{R}_{p_1*}\mathbb{D}\mathbb{R}_P((id_{\mathcal{M}} \times \sigma)^*\xi_{[\boldsymbol{\nu}]} \otimes_{\mathcal{O}_{\mathcal{M}\times O}} p_2^*\mathcal{F})[1].$$

9. Mandala of related moduli spaces

Players

- $(C, t_1, t_2, \cdots, t_n)$: A base curve.
- $L = \Omega_C(t_1 + \cdots + t_n)$: the extended cotangent line bundle on C.
- g, n, r, d: Numbers
- $N = r^2(g-1) + \frac{r(r-1)}{2} n + 1$: The half of dimension of the moduli spaces.
- M_{DR}: the moduli space of parabolic connections.
- M_{Dol}: the moduli space of parabolic Higgs bundles.
- $\mathcal{P}$: the moduli space of parabolic bundles.
- $\mathcal{X}$: the moduli space of generalized monodromy data (Character variety)
- $S^N(C) = \underbrace{C \times \cdots \times C}_{N} / \mathfrak{S}_N$: N-th Symmmetric Product of C
- $\mathrm{Hilb}^N(\mathbb{L})$:Hilbert space of N-points of the total space of L.

Relations of Players

(1) non abelian Hodge theory and Riemann-Hilbert correspondence

$$\begin{array}{ccccccc}
M_{Dol} & \Leftrightarrow & M_{DR} & \xrightarrow{\text{RH}} & \mathcal{X} & & \text{dim 2N} \\
 & \text{nonabelian Hodge} & & & & & \\
\downarrow & \text{forget full map} & \downarrow & \text{Lagangian fibrations} & & & \\
\mathcal{P} & = & \mathcal{P} & & & & \text{dim N}
\end{array}$$

(2) Hitchin fibration and apparent map

$$\begin{array}{ccccccc}
M_{Dol} & \longrightarrow & \mathrm{Hilb}^N(\mathbb{L}) & \leftarrow \cdots & M_{DR} & & \text{dim 2N} \\
\text{Hitchin fibration} & \text{BNR-map} & & \text{apparent map} & & & \\
\text{spectral curve} & & & & & & \\
\downarrow & \searrow & \downarrow & \swarrow & & & \\
B & & S^N(C) & & & & \text{dim N}
\end{array}$$

Related Problems

- Geometry of Riemann-Hilbert correspondences and Isomonodromic Deformations of Linear connections = Differential Equations of Painlevé type. Tau-functions.
- Explicit description of $M_{Dol}, M_{DR}, \mathcal{P}, \mathcal{X}$.
- Geometric propety of moduli spaces $M_{Dol}, M_{DR}, \mathcal{X}$ such as their Mixed Hodge polynomials, Simpson conjectures and $P = W$ conjecture.
- Transversaility of Lagangian fibration $M_{DR} \longrightarrow \mathcal{P}$ and $M_{DR} \longrightarrow S^N(C)$.
- Special Kähler Geometry and Topological Recursion related to $M_{Dol} \longrightarrow B$. (as in the work of Baraglia).
- Geometric Langlands by Fourie-Mukai transoform $M_{DR} \times \mathcal{P}$ (as in the work of Arinkin-Lysenko).

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