

Aspects of symmetries of projective varieties in the
view of birational geometry, cone conjecture and
algebraic dynamics - Primitive birational
automorphisms of Calabi-Yau manifolds and finiteness
problem of real forms

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Aim and Setting

Aim: Birational Algebraic Geometry in the view of Algebraic Dynamics - Direct (e.g. primitive selfmaps) and Indirect (real form problem)

V : a smooth projective variety of dimension $d = d(V)$ over $k := \mathbb{C}$;
 $f \in \text{Rat}(V)$, i.e., $f : V \dashrightarrow V$ a dominant rational selfmap over k , mostly, $\text{ord } f = \infty$, $f \in \text{Bir}(V)$ or $f \in \text{Aut}(V)$.

Interest: behaviour of n -times iterations $f^n := f \circ \cdots \circ f$ under $n \rightarrow \infty$

$$V_f := \{x \in V \mid \forall n \in \mathbb{Z}_{\geq 0} f^n(x) \notin I(f)\}$$

$f^n(x) \in V$ ($\forall n \in \mathbb{Z}_{\geq 0}$) well-defined points and $f^{n+m}(x) = f^m(f^n(x))$ for $\forall x \in V_f$ and $\forall m, n \in \mathbb{Z}_{\geq 0}$.

E.g. For $x \in V_f$, $\overline{\{f^n(x)\}_{n \geq 0}} = ?$, when $\overline{\{f^n(x)\}_{n \geq 0}} = V$? or

$$Z_f := \{x \in V_f \mid \overline{\{f^n(x)\}_{n \geq 0}} = V\} \neq \emptyset?$$

For $A \subset V$, $\overline{A}$ is the Zariski closure of A in V .

Primitivity and Zariski dense orbit conjecture (ZDOC)

$Z_f = \emptyset$ if $\exists W$ with $0 < \dim W < \dim V$ and $\exists$ dominant $\varphi : V \dashrightarrow W$ s.t.
 $\varphi \circ f = \varphi$ ($\dim W \neq \dim V$ if $\text{ord } f = \infty$):

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V \\ \downarrow \varphi & & \downarrow \varphi \\ W & \xrightarrow{\quad \text{Id}_W \quad} & W \end{array}$$

We call $f : V \dashrightarrow V$ **imprimitive** if $\exists W$ with $0 < \dim W < d = \dim V$,
 $\exists$ dominant $f_W : W \dashrightarrow W$ and $\exists$ dominant $\varphi : V \dashrightarrow W$ s.t.

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V \\ \downarrow \varphi & & \downarrow \varphi \\ W & \xrightarrow{\quad f_W \quad} & W \end{array}$$

We call $f : V \dashrightarrow V$ **primitive** if it is not imprimitive (D.Q.Zhang).

Zariski Dense Orbit Conjecture (ZDOC)

Conjecture 1 (ZDOC (Medvedev-Scanlon))

$Z_f \neq \emptyset$ iff $k(V)^{f^*} = k$, or equivalently, iff $\nexists$ first diagram in the previous page for $f \in \text{Rat}(V)$.

The next theorem is fundamental to this conjecture:

Theorem 2 (Amerik-Campana, Bell-Ghioca-Reichstein)

Let $f \in \text{Rat}(V)$. Then there exist a dense open subvariety $V_0 \subset V$, a *countable* collection of proper closed f -invariant subvarieties Y_i ($i \in I$) of V_0 and a dominant morphism $\varphi : V_0 \rightarrow W$ s.t.

- (1) $\varphi \circ f = \varphi$ (or more strongly, $k(W) = k(V)^{f^*}$);
- (2) $f^n(x)$ ($\forall n \in \mathbb{Z}_{\geq 0}$) are well-defined for all $x \in V_0 \setminus \bigcup_{i \in I} Y_i$;
- (3) for $x, y \in V_0 \setminus \bigcup_{i \in I} Y_i$, $\varphi(x) = \varphi(y)$ iff $\overline{\{f^n(x)\}_{n \geq 0}} = \overline{\{f^n(y)\}_{n \geq 0}}$; and
- (4) $\overline{\{f^n(x)\}_{n \geq 0}} = \varphi^{-1}(\varphi(x))$ for all $x \in V_0 \setminus \bigcup_{i \in I} Y_i$ in V_0 .

Some constraints of existence of Zariski-dense orbit

Corollary 3 (Amerik-Campana, Bell-Ghioca-Reichstein)

ZDOC is true and Z_f is Zariski dense in V over $k = \mathbb{C}$ (uncountable). In particular, *primitivity of f implies $Z_f \neq \emptyset$ and also the property $Z_f \neq \emptyset$ (or not) is a birational invariant property over $\mathbb{C}$.*

Rem. If k is countable (e.g $k = \overline{\mathbb{Q}}$), it might happen $V_0 = \cup_{i \in I} Y_i$ and DZOC is **open** (cf. Matsuzawa-Long Wang, Matsuzawa-Xie).

Let $\kappa(V) := \max_{m \in \mathbb{Z}_{>0}} \dim \phi_{|mK_V|}(V)$ be the Kodaira dimension of V .

Theorem 4 (Deligne-Nakamura-Ueno, Nakayama-D.Q.Zhang)

Let V be a smooth projective variety and $f \in \text{Rat}(V)$. Let

$$\phi_{|mK_V|} : V \dashrightarrow W \subset |mK_V|^*$$

be the pluricanonical map of V (when $\kappa(V) \geq 0$). Then f descends to $f_W \in \text{Aut } W$ *of finite order*. In particular, if $Z_f \neq \emptyset$ then $\kappa(V) = 0, -\infty$.

Existence of Zariski-dense orbit

Recall that $0 \leq q(V) := \dim H^0(V, \Omega_V^1) \leq d := \dim V$ if $\kappa(V) = 0$ (Kawamata); and $0 \leq q(V) := \dim H^0(V, \Omega_V^1) \leq \dim V - 1$ if $Z_f \neq \emptyset$ and $\kappa(V) = -\infty$ (based on Ueno, Mori). Set:

$$\mathcal{S}_d := \{(0, 0), \dots, (0, d-1), (0, d), (-\infty, 0), \dots, (-\infty, d-1)\}.$$

The following is a kind of converse of Theorem (Coarse Classification):

Theorem 5 (J. Chen, H.-Y. Lin, Oguiso)

Let V be a smooth projective variety and $f \in \text{Bir}(V)$. Assume that $d := \dim(V) \geq 2$. Then:

- (1) If $Z_f \neq \emptyset$, then $(\kappa(V), q(V)) \in \mathcal{S}_d \setminus \{(0, d-1)\}$.
- (2) Conversely, for each $(\kappa, q) \in \mathcal{S}_d \setminus \{(0, d-1)\}$, there exist V and $f \in \text{Aut}(V)$ s.t. $(\kappa(V), q(V)) = (\kappa, q)$ and $Z_f \neq \emptyset$.

Target varieties V with primitive $f \in \text{Bir}(V)$

Theorem 6 (D.Q.Zhang)

A smooth projective variety V with a primitive $f \in \text{Bir}(V)$, necessarily of $\text{ord } f = \infty$, satisfies one of:

- (1) $\kappa(V) = -\infty$ and $q(V) = 0$;*
- (2) $\kappa(V) = 0$ and $q(V) = 0$; or*
- (3) birational to an abelian variety ($\Leftrightarrow \kappa(V) = 0$ and $q(V) = \dim V$ by Kawamata).*

Remark 7 (D.Q.Zhang)

Assume further that the minimal model conjecture (MMP) and the abundance conjecture (AC) in dimension d hold. Then V of $\dim V = d$ in Theorem (1) and (2) will be:

- (1)' (RC) a rationally connected manifold;*
- (2)' (WCY) birational to a minimal Calabi-Yau variety, i.e., a $\mathbb{Q}$ -factorial terminal variety V' with $K_{V'} \equiv 0$ and $q(V') = 0$.*

Dynamical Degrees

Let V be a smooth projective variety of $\dim V = d$ with very ample divisor H and $f \in \text{Rat}(V)$. We define the p th dynamical degree of f by

$$d_p(f) = \lim_{n \rightarrow \infty} ((f^n)^*(H^p) \cdot H^{d-p})_V^{1/n} \geq 1,$$

where $(f^n)^*$ is defined as $(f^n)^* = (p_n)_* \circ q_n^*$ via resolution of $I(f^n)$:
 $V \xleftarrow{p_n} \tilde{V}_n \xrightarrow{q_n} V$ s.t. $f^n = q_n \circ p_n^{-1}$.

Theorem 8 (Dinh-Sibony, Truong, Dang)

- (1) $d_p(f)$ exists and is independent of H and a resolution of $I(f^n)$.
- (2) $d_p(f)$ is a birational invariant, i.e., $d_p(f) = d_p(f')$ for a birational map $\pi : V' \dashrightarrow V$ and $f' = \pi^{-1} \circ f \circ \pi$:

$$\begin{array}{ccc} V' & \xrightarrow{f'} & V' \\ \downarrow \pi & & \downarrow \pi \\ V & \xrightarrow{f} & V \end{array}$$

Basic Properties

The notion of dynamical degree comes from:

Theorem 9 (Gromov-Yomdin, Dinh-Sibony, Truong)

For a surjective morphism $f : V \rightarrow V$, $d_p(f)$ is the spectral radius $r_p(f)$ of $f^*|N^1(V)$ and the topological entropy $h_{\text{top}}(f) \geq 0$ satisfies:

$$(0 \leq) h_{\text{top}}(f) = \text{Max}_p \log d_p(f) = \text{Max}_p \log r_p(f).$$

Moreover $h_{\text{top}}(f) > 0 \Leftrightarrow d_1(f) > 1$.

Remark 10

- (1) The set $\{d_p(f)\}_{V, f \in \text{Rat}(V)}$ is countable (Urech). *Open for Kähler case.*
- (2) $d_p(f) = r_p(f)$ are real alg. integers for a surj. *morphism* $f : V \rightarrow V$.
- (3) For $\forall d \geq 3$, $\exists$ rational V of $\dim V = d$ and $\exists f \in \text{Bir}(V)$ s.t. $d_1(f)$ is transcendental (Bell-Diller-Jonsson-Krieger) and for $\forall d \geq 6$, $\exists$ rational W of $\dim W = d$ and $\exists f \in \text{Bir}(W)$ s.t. $d_p(f)$ ($1 \leq \forall p \leq d-1$) are all transcendental (*Yutaro Sugimoto*).

Primitive birational automorphisms of CY mfd

Theorem 11 (Oguiso, in preparation with a correction of a proof of Oguiso's papers in MRL2018 and IMRN2019)

Let V be a minimal Calabi-Yau variety with $d = \dim V \geq 3$ and $\rho(V) \geq 2$. Then $f \in \operatorname{Bir}(V)$ is primitive provided that

- (1) For any movable effective divisor D , there are a minimal Calabi-Yau variety V' and a birational map $g : V' \dashrightarrow V$ s.t. $D' = g^*D$ is semi-ample on V' ; and*
- (2) The characteristic polynomial $\Phi_f(t)$ of $f^*|N^1(V)_{\mathbb{Q}}$ is either non-cyclotomic and irreducible over $\mathbb{Q}$, or factored as $\Phi_f(t) = \varphi(t)(t + a)$ where $\varphi(t)$ is non-cyclotomic and irreducible over $\mathbb{Q}$ of degree ≥ 2 and $a \in \mathbb{Q}_{>0}$.*

Remark 12

In Theorem above]

- (1) is unconditional if log MMP and log AC holds in dimension d .*
- (2) is invariant under $\mathbb{Q}$ -factorial birational modification isomorphic in*

Examples of primitive birational automorphisms of CY mfd

Let V be a general hypersurface of multi-degree $(2, 2, \dots, 2)$ in $(\mathbb{P}^1)^{d+1}$ with $d \geq 3$. Then V is a d -dimensional strict Calabi-Yau manifold (sCY mfd) with $\rho(V) = d + 1$, called **CY mfd of Wehler type**.

As an application of the previous Theorem, we find

Theorem 13 (Oguiso, in preparation (Cf. Oguiso, IMRN 2019))

Any CY mfd of Wehler type has a primitive birational automorphism.

Remark 14 (Cantat-Oguiso)

Let V be a CY mfd of Wehler type.

Then $\text{Bir}(V) = (\mathbb{Z}/2\mathbb{Z})^{*(d+1)}$ (free product) and $\text{Aut}(V) = \{\text{id}_V\}$.

Moreover, V is a unique minimal model of V , $\text{Bir}(V)$ preserves a non-degenerate integral quadratic form of signature $(1, d)$ on $N^1(V)$ and satisfies Kawamata-Morrison's movable cone conjecture.

Primitive birational automorphisms of HK manifolds

Theorem 15 (Lo Bianco)

Any birational automorphism f of a HK manifold is primitive if $d_1(f) > 1$.

Besides HK geometry (e.g. existence of BBF form on $N^1(V)$), the following observation using BCHM is crucial:

Theorem 16 (Lo Bianco with some clarification)

Let V be a projective variety, $f \in \text{Bir}(V)$ and $\varphi : V \dashrightarrow W$ an f -equivariant dominant rational map whose general fibers after desingularization, say $\tilde{V}_w$ ($w \in W$), are of general type of positive dimension. Then $Z_f = \emptyset$:

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & V \\ \downarrow \varphi & & \downarrow \varphi \\ W & \xrightarrow{\quad f_W \quad} & W \end{array}$$

with $\kappa(\tilde{V}_w) = \dim \tilde{V}_w > 0$ for general $w \in W$.

Application for primitive biregular automorphisms

Proposition 17 (Oguiso)

- (1) $f \in \text{Aut}(A)$ of any simple abelian variety A is primitive iff $\text{ord } f = \infty$, e.g. a translation by a non-torsion point (application of idea of proof).
- (2) For any even $2d$, there is a smooth rational projective variety of dimension $2d$ with a primitive biregular automorphism.
- (3) For any even $2d$, there is a sCY mfd of dimension $2d$ with a primitive biregular automorphism.

Remark 18 (A brief idea of construction in (3))

Let S be an Enriques surface and $T \rightarrow S$ be the universal covering K3 surface. The universal cover V of $\text{Hilb}^d(S)$ is a sCY mfd of dimension $2d$ (Schröer-Oguiso), whereas $\text{Hilb}^d(T)$ is a HK mfd and we have natural dominant rational maps $\text{Hilb}^d(T) \dashrightarrow V \rightarrow \text{Hilb}^d(S)$, being equivariant for automorphisms naturally induced from $\text{Aut}(S)$.

Strict CY 3-fold (sCY3) case - two examples X_3 and X_7

$\zeta_n := \exp(2\pi\sqrt{-1}/n)$. We recall two sCY3 X_3 and X_7 .

Example 19 (Beauville, Roan, Yau, Oguiso, N.-H. Li, Truong, Gachet)

Let $A_3 := E_3^3$ where $E_3 := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\zeta_3)$. Let $g_3 = \zeta_3 \cdot I_3 \in \text{Aut}_{\text{group}}(A_3)$. Then $\dim A_3 = 3$ and g_3 is Gorenstein, of $\text{ord } g_3 = 3$. Consider:

$$X_3 \xrightarrow{\nu_3} \overline{X}_3 := A_3 / \langle g_3 \rangle \xleftarrow{q_3} A_3.$$

Here ν_3 is a unique crepant projective resolution of $\overline{X}_3$ and at the same time the unique birational c_2 -contraction of sCY3 X_3 (up to isom).

Here a surjective morphism $f : V \rightarrow W$ from CY3 to a normal projective variety is a **c_2 -contraction** if $(f^*N^1(W).c_2(V)) = 0$.

Strict CY 3-fold (sCY3) case - two examples X_3 and X_7

Example 20 (Roan, Yau, Oguiso, Gachet)

$A_7 := \text{Jac}(C)$ the Jacobian variety of $C := (xy^3 + yz^3 + zx^3 = 0) \subset \mathbb{P}^2$.

Let $g_7 \in \text{Aut}_{\text{group}}(A_7)$ induced from $g \in \text{Aut}(C)$ given by $g([x : y : z]) = [\zeta_7 x : \zeta_7^2 y : \zeta_7^4 z]$.

Then $\dim A_7 = 3$ and g_7 is Gorenstein, of $\text{ord } g_7 = 7$. Consider:

$$X_7 \xrightarrow{\nu_7} \overline{X}_7 := A_7 / \langle g_7 \rangle \xleftarrow{q_7} A_7.$$

Here ν_7 is the unique crepant projective resolution of $\overline{X}_7$ and at the same time the unique birational c_2 -contraction of sCY3 X_7 (up to isom).

Theorem 21 (Oguiso-Truong, Oguiso)

Both X_3 and X_7 have (a lot of) primitive automorphisms among

$\text{Aut}(X_3) \simeq (\mathbb{Z}/3)^{\oplus 3} \rtimes \text{GL}(3, \mathbb{Z}[\zeta_3]) / \langle \zeta_3 \cdot I_3 \rangle$, and

$\text{Aut}(X_7) \simeq (\mathbb{Z}/7) \rtimes ((\mathbb{Z}/2 \oplus \mathbb{Z}^{\oplus 2}) \rtimes \mathbb{Z}/3); \mathbb{Z}[\zeta_7]^\times / \langle \zeta_7 \rangle \simeq \mathbb{Z}/2 \oplus \mathbb{Z}^{\oplus 2}$.

Some general facts on a strict CY 3-fold (sCY3)

Let V be a **sCY3** and $f \in \text{Aut}(V)$.

Theorem 22 (Shing-Tung Yau, Miyaoka)

$c_2(V) > 0$ on the ample cone $\text{Amp}(V) \subset N^1(V)_{\mathbb{R}}$, i.e., $(c_2(V).x) > 0$ for all $x \in \text{Amp}(V)$. Hence $c_2(V) \geq 0$ on the nef cone $\text{Nef}(V) = \overline{\text{Amp}(V)}$.

Note that for $f \in \text{Aut}(V)$, f^* preserves $\text{Amp}(V)$, $\text{Nef}(V)$ and also the hyperplane $c_2(V)^{\perp} = ((c_2(V).x) = 0) \subset N^1(V)_{\mathbb{R}}$.

Proposition 23 (Wilson)

$(c_2(V)^{\perp} \cap \text{Nef}(V))(\mathbb{R}) \setminus \{0\} \neq \emptyset$ if $|\text{Aut}(V)| = \infty$.

Theorem 24 (Oguiso)

Let $\nu : X \rightarrow \overline{X}$ be a birational c_2 -contraction of a sCY3 X . Then:
 $(\nu : X \rightarrow \overline{X}) \simeq (\nu_3 : X_3 \rightarrow \overline{X}_3)$ or $(\nu_7 : X_7 \rightarrow \overline{X}_7)$.

An observation of CY3 V with primitive $f \in \text{Aut}(V)$

Question 25 (Rationality Question and Nonvanishing Conjecture)

- (1) $(c_2(V)^\perp \cap \text{Nef}(V))(\mathbb{Q}) \setminus \{0\} \neq \emptyset$ for a sCY3 V with $|\text{Aut}(V)| = \infty$?
- (2) For $\forall D \in \text{Nef}(V)(\mathbb{Q}) \setminus \{0\}$ on a sCY3 V , $\exists m \in \mathbb{Z}_{>0}$ s.t. $|mD| \neq \emptyset$?

Proposition 26 (Oguiso)

If Questions (1) and (2) are affirmative, then a sCY3 V with $f \in \text{Aut}(V)$ s.t. $Z_f \neq \emptyset$, or a sCY3 V with a primitive $f \in \text{Aut}(V)$, is isomorphic to either X_3 or X_7 .

Proposition 27 (Oguiso-D.Q.Zhang in the study of wild autom.)

If Question (2) are affirmative, then there is no sCY3 V with $f \in \text{Aut}(V)$ s.t. $\overline{\{f^n(x)\}_{n \geq 0}} = V$ for *all* $x \in V$.

Question 28 (Open)

Can one prove/disprove these two propositions in unconditional ways?

Some (indirect) application - Real form problem

$V = (F_0 = F_1 = \cdots = F_m = 0) \subset \mathbb{P}_{\mathbb{C}}^n$: a complex projective variety of dimension d (or more generally a quasi-projective variety).

Eg. $P_{\mathbb{C}}^1 \simeq L \simeq C_0 \simeq C_1$ over $\mathbb{C}$, where

$L = (X_2 = 0)$, $C_0 = (X_0^2 - X_1X_2 = 0)$, $C_1 = (X_0^2 + X_1^2 + X_2^2 = 0)$ in $\mathbb{P}_{\mathbb{C}}^2$.

Question 29 (Real Form Problem (Intuitive Formulation))

*How many different ways can one describe a variety V (up to isom. over $\mathbb{C}$) by a variety of V_0 defined by a system of equations of *real* coefficients (up to isom. given by polynomials with *real* coefficients, i.e., isom. over $\mathbb{R}$)?*

Ex. L, C_0, C_1 are all real forms of $\mathbb{P}_{\mathbb{C}}^1$. (Equations are all real coefficients).

Are L_0, C_0, C_1 different as real forms up to isom over $\mathbb{R}$ or not?

Ex. L_0 and C_0 are isom as real forms by $[X_0 : X_1 : 0] \mapsto [X_0X_1 : X_0^2 : X_1^2]$ (isomorphism given by polynomials of real coefficients). But not L_0 and C_1 ; as otherwise $[p : q : 0] \in L_0$ ($p, q \in \mathbb{R}$; many such points) must map to $[a : b : c] \in C_1$ with $a, b, c \in \mathbb{R}$ but no such points on C_1 .

Precise Definition of Real Form and Real Structure

Definition 30

(1) A **real form** of $V \xrightarrow{\pi} \operatorname{Spec} \mathbb{C}$ is a real algebraic variety $W \rightarrow \operatorname{Spec} \mathbb{R}$ s.t.

$$V \simeq W \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C} \text{ over } \operatorname{Spec} \mathbb{C}.$$

$\operatorname{RF}(V) :=$ (the set of isomorphism classes of real forms of V over $\mathbb{R}$).

(2) A **real structure** of $V \xrightarrow{\pi} \operatorname{Spec} \mathbb{C}$ is an involution ι of V as a scheme s.t. $\pi \circ \iota = c \circ \pi$ where c is the involution of $\operatorname{Spec} \mathbb{C}$ given by the **complex conjugation** $z \mapsto \bar{z}$, and we say that real structures ι and ι' of $V \xrightarrow{\pi} \operatorname{Spec} \mathbb{C}$ are equivalent iff $\exists \varphi \in \operatorname{Aut}(V/\mathbb{C})$ s.t. $\iota' = \varphi \circ \iota \circ \varphi^{-1}$:

$$\begin{array}{ccc} V & \xrightarrow{\iota} & V \\ \downarrow \pi & & \downarrow \pi \\ \operatorname{Spec} \mathbb{C} & \xrightarrow{c} & \operatorname{Spec} \mathbb{C} \end{array}$$

Some basic result of Borel-Serre

Theorem 31 (Borel-Serre)

Assume that $W \in \text{RF}(V)$ and ι_W is the real structure s.t. $W = V/\iota_W$.

Then, there are natural bijective correspondences among

- (1) $\text{RF}(V)$, the set of real forms of V up to isomorphisms over $\mathbb{R}$;*
- (2) $\text{RS}(V)$, the set of real structures on V up to equivalence; and*
- (3) the Galois cohomology set $H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(V/\mathbb{C}))$ with the action of $\text{Gal}(\mathbb{C}/\mathbb{R}) = \langle c \rangle$ on $\text{Aut}(V/\mathbb{C})$ is given by $f \mapsto \iota_W \circ f \circ \iota_W = c(f)$.*

This theorem is more or less tautological but important as a general framework. For instance, from (1) $\Leftrightarrow$ (3), we obtain:

Corollary 32 (Within Complex Algebraic Geometry)

- (1) If $|\text{Aut}(V/\mathbb{C})| < \infty$, then $|\text{RF}(V)| < \infty$.*
- (2) If the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$ on $\text{Aut}(V)$ in (3) is trivial, then $\text{RF}(V)$ bijectively corresponds to the set of conjugacy classes of involutions and id_V in $\text{Aut}(V/\mathbb{C})$.*

Some elementary examples

Ex 1. $\text{RF}(\text{Spec } \mathbb{C}[X]/((X - a_1)(X - a_2) \cdots (X - a_n)))$ ($a_i \neq a_j$ if $i \neq j$) is $\{\coprod_k \text{Spec } \mathbb{R} \amalg \coprod_l \text{Spec } \mathbb{R}[X]/(X^2 + 1) \mid k + 2l = n\}$

Ex 2. RF of affine line $\mathbb{A}_{\mathbb{C}}^1 = \text{Spec } \mathbb{C}[X]$ is $\{\mathbb{A}_{\mathbb{R}}^1\}$ (no exotic one).

Ex 3. RF of the projective line $\mathbb{P}_{\mathbb{C}}^1 = \text{Proj } \mathbb{C}[X, Y]$ is $\{\mathbb{P}_{\mathbb{R}}^1 = \text{Proj } \mathbb{R}[X, Y], C_1 = \text{Proj } \mathbb{R}[X_0, X_1, X_2]/(X_0^2 + X_1^2 + X_2^2)\}$
(L and C_0 are isomorphic to $\mathbb{P}_{\mathbb{R}}^1$ as real forms)

Ex 4. If V is a smooth projective curve of genus ≥ 2 , then $|\text{Aut}(V/\mathbb{C})| < \infty$ and $|\text{RF}(C)| < \infty$.

Ex 5. There are (plenty of) smooth projective curves C with no real form.
E.g., for a complex elliptic curve C , $|\text{RF}(C)| = \emptyset \Leftrightarrow J(C) \in \mathbb{C} \setminus \mathbb{R}$.

Rem 1. $\text{RF}(\mathbb{A}_{\mathbb{C}}^2) = \{\mathbb{A}_{\mathbb{R}}^2\}$ (Kambayashi, very difficult to prove) and it is completely open if $\mathbb{A}_{\mathbb{C}}^n$ ($n \geq 3$) have a real form other than $\mathbb{A}_{\mathbb{R}}^n$ or not.

Rem 2. On the other hand,

$\text{RF}(\mathbb{P}_{\mathbb{C}}^n) = \{\mathbb{P}_{\mathbb{R}}^n\}$ (n is even) and $\text{RF}(\mathbb{P}_{\mathbb{C}}^n) = \{\mathbb{P}_{\mathbb{R}}^n, \text{Hilb}_{\mathbb{R}}^n(C_1)\}$ (n is odd).

Basic problems and the first result due to Lesieutre

Question 33 (the main theme of the rest of my talk is Question (3).)

- (1) Describe $\text{RF}(V)$ for each given V .
- (2) Describe $\text{RS}(V)$ for each given V .
- (3) *Is $|\text{RF}(V)| < \infty$? Otherwise for which V , $|\text{RF}(V)| = \infty$?*

(1): **Yuya Sasaki** determines $\text{RF}(V_d^n)$ for the Fermat hypersurface V_d^n of $\dim V = n \geq 1$ and of $\deg V = d \geq 2$ **except $(n, d) = (2, 4)$** (after kind corrections/suggestions from Professor János Kollár).

(3) is natural and fundamental but open for long time. The **first definite answer is given by John Lesieutre in this century**:

Theorem 34 (Lesieutre; Invent. Math. 2018)

$\exists$ a smooth projective complex variety V of $\dim V = 6$ s.t. $|\text{RF}(V)| = \infty$.

Rem. Lesieutre's V is uniruled (hence $\kappa(V) = -\infty$), but not rationally connected (hence not rational).

Main Results (for surfaces)

Goal of the rest of this talk: To give some fairly satisfactory generalization of Lesieutre's theorem (at least as a first step).

The first main result is the following theorem finally obtained by Dinh-Gachet-Lin-Oguiso-Long Wang-Yu (DGLOWY) after several preceding works including my preceding joint works Dinh-Oguiso (Duke Math. J. 2019), Dinh-Oguiso-Yu (Crelle J. 2023):

Theorem 35 (DGLOWY; Ann. Ecole Norm. Sup. Pisa 2025)

- (1) *Let V be a smooth complex projective surface. Then $|\mathrm{RF}(V)| < \infty$ unless V is either a rational surface, a **non-minimal** K3 surface, or a **non-minimal** Enriques surface.*
- (2) *Conversely, there are a rational surface, a non-minimal K3 surface, and a non-minimal Enriques surface with infinitely many real forms.*

Rem. If $|\mathrm{RF}(V)| = \infty$, then $\dim V \geq 2$ (discuss later)..

Rem. (2) also gives a **first definite answer to a question of Kharlamov**:
Is there a rational surface V with $|\mathrm{RF}(V)| = \infty$?

Main Results (for higher dimensional case)

By taking a suitable product and further blowing-ups, we also obtained:

Theorem 36 (Dinh-Oguiso-Yu (Math. Ann. 2022) and DGLOWY)

- (1) $\forall$ integer $d \geq 3$, $\exists$ a smooth complex projective *rational* variety V of $\dim V = d$, with $|\mathrm{RF}(V)| = \infty$.
- (2) Let V be a smooth complex projective variety of $\dim V = d \geq 3$. If $|\mathrm{RF}(V)| = \infty$, then $\kappa(V) \in \{-\infty, 0, 1, \dots, d-2\}$.
- (3) Conversely, for $\forall$ integer $d \geq 3$ and $\forall \kappa \in \{-\infty, 0, 1, \dots, d-2\}$, $\exists$ a smooth complex projective variety V with $\dim V = d$ and $\kappa(V) = \kappa$ s.t. $|\mathrm{RF}(V)| = \infty$.

Rem 1. These two theorems give a coarse but fairly satisfactory answer to Question (3) in any dimension ≥ 2 .

Rem 2. It is also remarkable that $|\mathrm{RF}(V)|$ is *far from being a birational invariant*. It might be interesting to find the set $I_n := \{|\mathrm{RF}(V)|\}$ when V runs through all complex (smooth) rational variety of dimension n .

Finiteness of real forms and dynamical degree

From now V is a smooth projective variety/ $\mathbb{C}$, $\text{Aut}(V) := \text{Aut}(V/\mathbb{C})$.
 $d_1(f) := (\text{spectral radius of } f^*|_{H^2(V^{\text{an}}, \mathbb{Z})}) = (\text{spectral radius of } f^*|_{N^1(V)})$
(first dynamical degree, fundamental invariant in algebraic dynamics).

Theorem 37 (cf. DGLOWY)

$|\text{RF}(V)| < \infty$ if one of the following holds:

- (1) $\text{RF}(V) \neq \emptyset$ and $|H^1(\text{Gal}(\mathbb{C}/\mathbb{R}), \text{Aut}(V)/\text{Aut}^0(V))| < \infty$;
- (2) $\text{Aut}(V)/\text{Aut}^0(V)$ is virtually solvable or arithmetic; or
- (3) $\text{Aut}(V)$ is of zero entropy, i.e., $d_1(f) = 1 \ \forall f \in \text{Aut}(V)$.

The last condition (3), which relates the finiteness of real forms with algebraic dynamics, is unexpected. This theorem was inspired by preceding works by Borel-Serre, by Degtyarev - Itenberg - Kharlamov (DIK), by Benzerga, and by Cattaneo - Fu (CF). From any one of (1), (2), (3):

Corollary 38 (Classical(Revisited))

$|\text{RF}(V)| < \infty$ if $\dim V = 1$.

Finiteness of real forms and cone conjecture

Theorem 39 (DGLOWY after DIK, CF)

$|\mathrm{RF}(V)| < \infty$ if $\exists$ rational polyhedral cone $\Sigma \subset \mathrm{Nef}^+(V) \subset N^1(V)_{\mathbb{R}}$ s.t. $\mathrm{Aut}^*(V) \cdot \Sigma \supset \mathrm{Amp}(V)$.

Eg., This is the case when $\mathrm{Nef}^+(V)$ is a rational polyhedral cone, or more generally when V satisfies Kawamata-Morrison's nef cone conjecture.

The statement which relates the finiteness of real forms with Kawamata-Morrison's nef cone conjecture (of mirror symmetry origin) is also unexpected.

Question 40 (Open)

$|\mathrm{RF}(V)| < \infty$ for any CY mfd V ?

Rem. $|\mathrm{RF}(V)| < \infty$ for smooth **minimal** projective surfaces V with $\kappa(V) = 0$, abelian varieties and HK mfd's, as they satisfy Kawamata-Morrison's nef cone conjecture (by Kawamata after Sterk and Namikawa, Prendergast-Smith, Markman-Yoshioka, Amerik-Verbitsky).

Brief Outline of Proof of Main Result (1)

Smooth complex projective surfaces V are classified by the **Kodaira dimension** $\kappa(V) := \dim V_m$ of the image $\Phi_{mK_V} : V \dashrightarrow V_m \subset \mathbb{P}^N$ for large divisible m (maps given by $H^0(V, \mathcal{O}(mK_V)) = H^0(V, (\wedge^2 \Omega_V^1)^{\otimes m})$):

(1) $\kappa(V) = 2$ (V general type): $|\mathrm{RF}(V)| < \infty$ by $|\mathrm{Aut}(V)| < \infty$ or weaker $\mathrm{Aut}(V)$ is zero entropy;

(2) $\kappa(V) = 1$ ($\Phi_{mK_V} : V \rightarrow C$ with general fiber of genus 1); $|\mathrm{RF}(V)| < \infty$ by $\mathrm{Aut}(V)$ is zero entropy;

(3) $\kappa(V) = 0$ and **minimal** (K3 surface, Enriques surface, abelian surface, bielliptic surfaces); $|\mathrm{RF}(V)| < \infty$ by the cone conjecture;

(4) $\kappa(V) = -\infty$ and $q(V) = \dim H^0(V, \Omega_V^1) > 0$ (unique Albanese fibration $V \rightarrow C$ with general fiber of genus 0); $|\mathrm{RF}(V)| < \infty$ by $\mathrm{Aut}(V)$ is zero entropy.

Remaining cases: rational surfaces ($\Leftrightarrow \kappa(V) = -\infty$ and $q(V) = 0$), non-minimal K3, non-minimal Enriques, (after showing that

$|\mathrm{RF}(V)| < \infty$ for non-minimal abelian surfaces and non-minimal bielliptic surfaces)

Brief Outline of Proof of Main Result (2)

We choose $S = \text{Km}(E_1 \times E_2)$ where E_i ($i = 1, 2$) are elliptic curves defined by $y^2 = x(x-1)(x-\lambda_i)$ s.t. $\lambda_i \in \mathbb{R}$ and λ_i ($i = 1, 2$) are algebraically independent over $\mathbb{Q}$.

By construction, S has a natural real form $S_{\mathbb{R}}$. Let $E_{11} \subset S$ be one of 16 exceptional curves of $S \rightarrow (E_1 \times E_2)/\langle -1_{E_1 \times E_2} \rangle$, $P \in E_{11}(\mathbb{R})$ a very general real point w.r.t. the real structure ι given by $S_{\mathbb{R}}$, and $\tilde{S} \rightarrow S$ the blow-up of S at P . Then $|\text{RF}(\tilde{S})| = \infty$ for this $\tilde{S}$.

For proof, we observe that ι acts on $\text{Aut}(\tilde{S}) \subset \text{Aut}(S)$ as identity.

It thus suffices to find a set of involutions of $\tilde{S}$ whose conjugacy classes under $\text{Aut}(\tilde{S})$ is infinite. We found such a set by lifting infinitely many involutions of S such that all conjugate under $\text{Aut}(S)$ but the conjugacy classes under much smaller $\text{Aut}(\tilde{S}) (\subset \text{Aut}(S))$ are infinite, by using various elliptic fibrations on S and the genericity of P .

We construct $\tilde{W}$ and $\tilde{T}$ as a one point blow-up of the étale quotient $W = S/\tau_W$ and the smooth non - étale quotient $T = S/\tau_T$ by suitable involutions τ_W and τ_T on S respectively.

Thanks

Thank you very much for your attention!