

Invariant Partial Differential Operators Arising in Classical Fields*

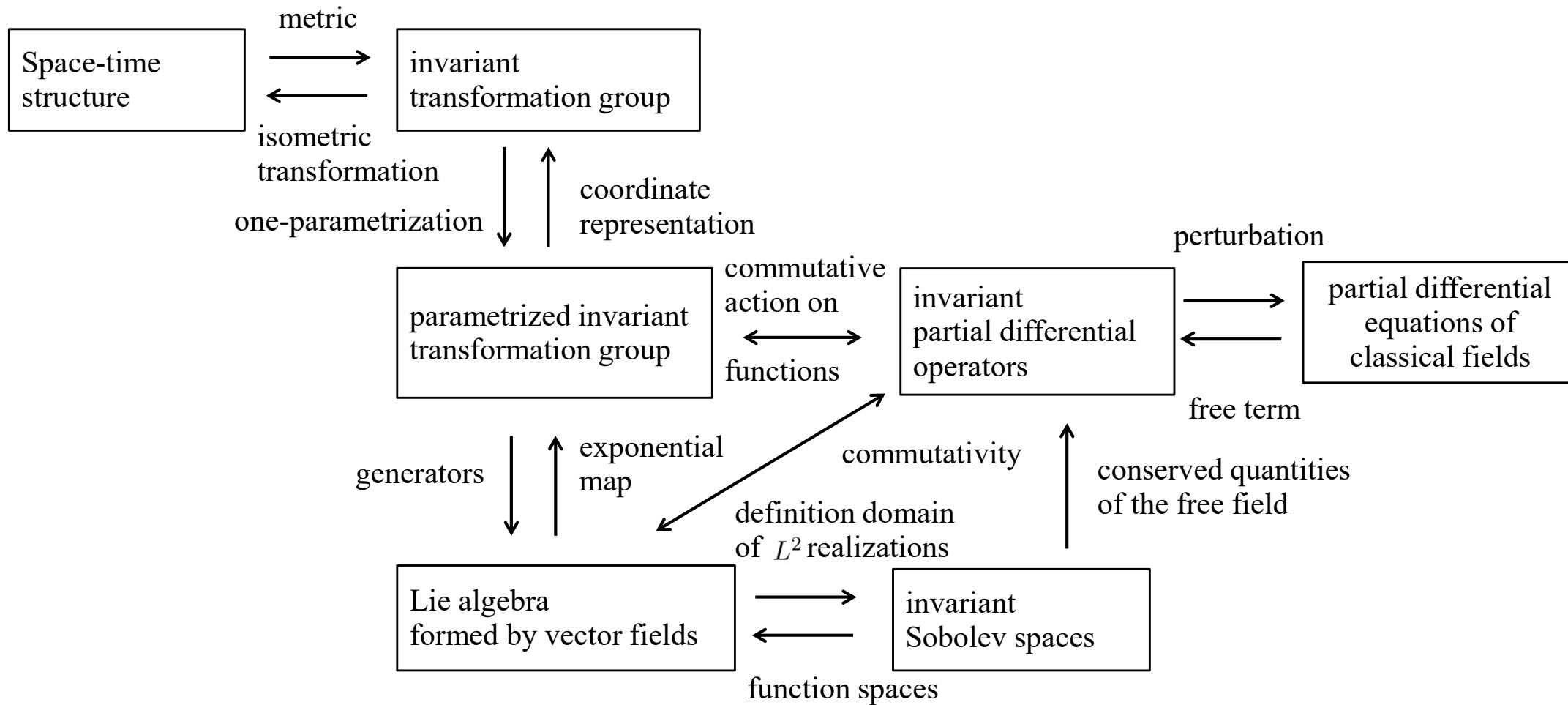
Tohru Ozawa
Department of Applied Physics, Waseda University

December 7, 2025
The 23rd Oka Symposium

1. Summary
 2. Euclid space (Review)
 3. Minkowski space-time
 4. Galilei space-time
 5. A derivation of the Schrödinger equation
-

*Based on a recent joint-work with Hiromichi Nakazato, Department of Physics, Waseda University

1. Summary



1. Summary

Space-time	Motion Group	Invariant Partial Differential Operator
Euclid space $\mathbb{R}^n \ni \mathbf{x} = (x_1, \dots, x_n)$ $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n$	Euclid	Helmholtz (Laplace) $\Delta + \kappa^2$
Minkowski space-time $\mathbb{R} \times \mathbb{R}^n \ni x = (t, \mathbf{x})$ $x \cdot y = ts - \mathbf{x} \cdot \mathbf{y}$	Poincaré	Klein-Gordon (D'Alembert) $\partial_t^2 - \Delta + m^2$
Galilei space-time $\mathbb{R} \times \mathbb{R}^n \ni x = (t, \mathbf{x})$ $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n$	Galilei	Schrödinger $i\partial_t + \frac{1}{2}\Delta$

2. Euclid space (Review)

島倉紀夫『楕円型偏微分作用素』紀伊國屋書店, 1978.

Norio Shimakura, “Partial Differential Operators of Elliptic Type,”
Translations of Math. Monographs 99, AMS, 1992.

野村隆昭『球面調和函数と群の表現』日本評論社, 2018.

- Translation by $\mathbf{y} \in \mathbb{R}^n$: $T_{\mathbf{y}} : \mathbf{x} \mapsto T_{\mathbf{y}}(\mathbf{x}) := \mathbf{x} - \mathbf{y}$.
- Rotation by $R \in O(n)$: $R : \mathbf{x} \mapsto R\mathbf{x}$.

- Action on smooth functions:

$$T_{\mathbf{y}}^* f := f \circ T_{\mathbf{y}}, \quad R^* f := f \circ R, \quad f \in C^\infty(\mathbb{R}^n; \mathbb{C}).$$

- Partial differential operators of order m : $L = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$,

$$(a_\alpha; |\alpha| \leq m) \subset C(\mathbb{R}^n; \mathbb{C}), \quad \sum_{|\alpha|=m} |a_\alpha| \neq 0.$$

- L is translation invariant. $\stackrel{\text{def.}}{\iff} T_{\mathbf{y}}^* L f = L T_{\mathbf{y}}^* f \quad \forall \mathbf{y} \in \mathbb{R}^n \quad \forall f \in C^\infty(\mathbb{R}^n; \mathbb{C}).$
- L is rotation invariant. $\stackrel{\text{def.}}{\iff} R^* L f = L R^* f \quad \forall R \in O(n) \quad \forall f \in C^\infty(\mathbb{R}^n; \mathbb{C}).$
- L is Euclid invariant. $\stackrel{\text{def.}}{\iff} L$ is translation and rotation invariant.

Theorem 1. The following statements are equivalent.

- (1) L is Euclid invariant.
- (2) Coefficients of L are constants and L is rotation invariant.
- (3) L is a polynomial in Δ of order $[m/2]$.

Corollary. Any Euclid invariant 2nd order partial differential operator L is uniquely represented as

$$L = \alpha \Delta + \beta \text{ with } \alpha \in \mathbb{C} \setminus \{0\} \text{ and } \beta \in \mathbb{C}.$$

Proof of Theorem 1. For $\xi \in \mathbb{R}^n$ let $e_\xi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ be defined by $e_\xi(x) = \exp(\xi \cdot x)$, $x \in \mathbb{R}^n$.

Then $(Le_\xi)(x) = p(x, \xi)e_\xi(x)$, where $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$.

- translation invariance of $L \Leftrightarrow p(x, \xi) = p(0, \xi) \quad \forall x, \xi \in \mathbb{R}^n$
 $\Leftrightarrow a_\alpha(x) = a_\alpha(0) \quad \forall \alpha \in \mathbb{Z}_{\geq 0}^n, \forall x \in \mathbb{R}^n$
- $q \in \mathbb{C}[\xi]$: homogeneous of degree j

$$R^*q = q \quad \forall R \in O(n) \Leftrightarrow \begin{cases} q(\xi) = 0 \quad \forall \xi & \text{for } j \text{ odd,} \\ \exists! a \in \mathbb{C} \setminus \{0\} : q(\xi) = a|\xi|^j \quad \forall \xi & \text{for } j \text{ even.} \end{cases}$$

3. Minkowski space-time

- Lorentz metric: $x \cdot y = ts - \mathbf{x} \cdot \mathbf{y}$, $x = (t, \mathbf{x})$, $y = (s, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^n$.
- Translation by $y \in \mathbb{R} \times \mathbb{R}^n$: $T_y : x \mapsto T_y(x) := x - y = (t - s, \mathbf{x} - \mathbf{y})$.
- Lorentz rotation by $\Lambda \in O(1, n)$: $\Lambda : x \mapsto \Lambda x$,
 $O(1, n) := \{\Lambda \in GL(1 + n; \mathbb{R}); {}^t\Lambda g \Lambda = \Lambda g {}^t\Lambda = g := \text{diag}(1, -1, \dots, -1)\}$.
- Partial differential operator of order m : $L = \sum_{j+|\alpha| \leq m} a_{j\alpha} \partial_t^j \partial^\alpha$,
 $(a_{j\alpha}; j + |\alpha| \leq m) \subset C(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $\sum_{j+|\alpha|=m} |a_{j\alpha}| \neq 0$.
- L is space-time translation invariant.
 $\stackrel{\text{def.}}{\iff} T_y^* L u = L T_y^* u \quad \forall y \in \mathbb{R} \times \mathbb{R}^n, \forall u \in C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$.
- L is Lorentz invariant.
 $\stackrel{\text{def.}}{\iff} \Lambda^* L u = L \Lambda^* u \quad \forall \Lambda \in O(1, n), \forall u \in C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$.
- L is Poincaré invariant. $\stackrel{\text{def.}}{\iff} L$ is space-time translation and Lorentz invariant.

Theorem 2. The following statements are equivalent.

- (1) L is Poincaré invariant.
- (2) Coefficients of L are constants and L is Lorentz invariant.
- (3) L is a polynomial in $\square$ of order $[m/2]$.

Corollary. Any Poincaré invariant 2nd order partial differential operator L is uniquely represented as $L = \alpha \square + \beta$ with $\alpha \in \mathbb{C} \setminus \{0\}$ and $\beta \in \mathbb{C}$.

Proof of Theorem 2. $e_\xi(x) := \exp(\tau t + \xi \cdot \mathbf{x})$, $x = (t, \mathbf{x})$, $\xi = (\tau, \xi)$.

$$p(x, \xi) := \sum_{j+|\alpha| \leq m} a_{j\alpha}(x) \tau^j \xi^\alpha \Rightarrow (Le_\xi)(x) = p(x, \xi) e_\xi(x).$$

- translation invariance of $L \Leftrightarrow p(x, \xi) = p(0, \xi) \quad \forall x, \xi \in \mathbb{R} \times \mathbb{R}^n$
 $\Leftrightarrow a_{j\alpha}(x) = a_{j\alpha}(0) \quad \forall (j, \alpha) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n, \forall x \in \mathbb{R} \times \mathbb{R}^n$.
- $p_j(\xi) := \sum_{|\alpha| \leq m-j} a_{j\alpha}(0) \xi^\alpha \quad (0 \leq j \leq m)$.
 $R^* p_j = p_j \quad \forall R \in O(n) \Leftrightarrow p_j \in \mathbb{C}[|\xi|^2]$.
- $q \in \mathbb{C}[\xi]$: homogeneous of degree ℓ

$$\Lambda^* q = q \quad \forall \Lambda \in O(1, n) \Leftrightarrow \begin{cases} q_\ell(\xi) = 0 \quad \forall \xi \text{ for } \ell \text{ odd,} \\ \exists! a \in \mathbb{C} \setminus \{0\} : q(\xi) = a(\tau^2 - |\xi|^2)^{\ell/2} \quad \forall \xi \text{ for } \ell \text{ even.} \end{cases}$$

4. Galilei space-time

- Translation by $y \in \mathbb{R} \times \mathbb{R}^n$: $T_y : x \mapsto T_y(x) := x - y = (t - s, \mathbf{x} - \mathbf{y})$.
- Space rotation by $R \in O(n)$: $1 \otimes R : x \mapsto (1 \otimes R)(x) := (t, R\mathbf{x})$.
- Galilei transformation by $\mathbf{v} \in \mathbb{R}^n$: $G_{\mathbf{v}} : x \mapsto G_{\mathbf{v}}(x) := (t, \mathbf{x} - t\mathbf{v})$.
- Partial differential operator of order m : $L = \sum_{j+|\alpha| \leq m} a_{j\alpha} \partial_t^j \partial^\alpha$,
 $(a_{j\alpha}; j + |\alpha| \leq m) \subset C(\mathbb{R} \times \mathbb{R}^n; \mathbb{C})$, $\sum_{j+|\alpha|=m} |a_{j\alpha}| \neq 0$.
- L is space rotation invariant. $\stackrel{\text{def.}}{\iff} (1 \otimes R)^* Lu = L(1 \otimes R)^* u \quad \forall R \in O(n) \quad \forall u \in C^\infty$.
- L is pure Galilei invariant.
 $\stackrel{\text{def.}}{\iff} \forall \mathbf{v} \in \mathbb{R}^n \exists \theta_{\mathbf{v}} \in C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}) : e^{i\theta_{\mathbf{v}}} G_{\mathbf{v}}^* Lu = L e^{i\theta_{\mathbf{v}}} G_{\mathbf{v}}^* u \quad \forall u \in C^\infty$.
- L is Galilei invariant. $\stackrel{\text{def.}}{\iff} L$ is space-time translation, space rotation, and pure Galilei invariant.

Theorem 3. For $m = 2$, the following three statements are equivalent.

- (1) L is Galilei invariant.
- (2) Coefficients of L are constants and L is space rotation and pure Galilei invariant.
- (3) Coefficients of L are constants and 0 except

$$(j, \alpha) = (0, \mathbf{0}), (1, \mathbf{0}), (0, 2\mathbf{e}_1), \dots, (0, 2\mathbf{e}_n),$$

$$a_{0,2\mathbf{e}_1} = \dots = a_{0,2\mathbf{e}_n} (=:\alpha) \neq 0, \quad ia_{10}/\alpha \in \mathbb{R}, \text{ and}$$

$$L = \alpha(2i\lambda\partial_t + \Delta) + \beta,$$

$$\text{where } \lambda := -\frac{ia_{10}}{2\alpha} \in \mathbb{R}, \quad \beta = a_{00} \in \mathbb{C},$$

$$\theta_{\mathbf{v}}(t, \mathbf{x}) = \begin{cases} \theta_{\mathbf{v}}(0, \mathbf{0}) + \lambda \mathbf{v} \cdot \mathbf{x} - \frac{\lambda}{2} t |\mathbf{v}|^2 & \text{if } \lambda \neq 0, \\ \theta_{\mathbf{v}}(t, \mathbf{0}) & \text{if } \lambda = 0. \end{cases}$$

Proof of Theorem 3.

$$e_\xi(x) := \exp(i(\tau t + \xi \cdot x)), \quad p(x, \xi) := \sum_{j+|\alpha| \leq m} a_{j\alpha}(x) (i\tau)^j (i\xi)^\alpha$$

$$\Rightarrow (Le_\xi)(x) = p(x, \xi) e_\xi(x).$$

- space-time translation invariance of L

$$\Leftrightarrow p(x, \xi) = p(0, \xi) \quad \forall x, \xi \in \mathbb{R} \times \mathbb{R}^n$$

$$\Leftrightarrow a_{j\alpha}(x) = a_{j\alpha}(0) \quad \forall (j, \alpha) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^n, \forall x \in \mathbb{R} \times \mathbb{R}^n.$$

- $p_j(\xi) = \sum_{|\alpha| \leq 2-j} a_{j\alpha}(0) (i\xi)^\alpha \quad (0 \leq j \leq 2).$

$$R^* p_j = p_j \quad \forall R \in O(n) \Leftrightarrow p_j \in \mathbb{C}[|\xi|^2]$$

$$\Rightarrow \begin{cases} p(0, \xi) = p_0(\xi) + p_1(\xi) i\tau + p_2(\xi) (i\tau)^2 = a_{00} - \alpha |\xi|^2 + i a_{10} \tau - a_{20} \tau^2, \\ L = \beta + a \Delta + a_{10} \partial_t - a_{20} \partial_t^2 \end{cases}$$

- Let $u = e_\xi$ and compare both sides of $e^{i\theta} G_v^* L e_\xi = L e^{i\theta} G_v^* e_\xi$.

5. A derivation of the Schrödinger equation

$$L = i\partial_t + \frac{1}{2}\Delta, \quad f \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{C}; \mathbb{C}), \quad V \in C(\mathbb{R} \times \mathbb{R}^n : \mathbb{C})$$

$$\mathcal{X}_f := \{\psi \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n; \mathbb{C}); \forall (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n, (L\psi)(t, \mathbf{x}) = f(t, \mathbf{x}, \psi(t, \mathbf{x}))\}.$$

$$\dot{\mathcal{X}}_V := \{\psi \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n; \mathbb{C}); \forall (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n, (L\psi)(t, \mathbf{x}) = V(t, \mathbf{x})\psi(t, \mathbf{x})\}.$$

Theorem 4. Let $\mathcal{X}_f \neq \emptyset$. Then, the following statements are equivalent.

- (1) $\mathcal{X}_f$ is a complex vector space.
- (2) $\exists 1 V \in C(\mathbb{R} \times \mathbb{R}^n; \mathbb{C}) : \mathcal{X}_f = \dot{\mathcal{X}}_V$.

In this case, for any $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$, $V(t, \mathbf{x}) = f(t, \mathbf{x}, 1)$.

Theorem 5. Let $\dot{\mathcal{X}}_V \neq \{0\}$. Then, the following statements are equivalent.

- (1) $\dot{\mathcal{X}}_V$ is a space-time translation invariant complex vector space.
- (2) For any $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$, $V(t, \mathbf{x}) = V(t, \mathbf{0})$.
- (3) $\exists 1 \lambda \in \mathbb{C}; \dot{\mathcal{X}}_V = \dot{\mathcal{X}}_\lambda$.
- (4) $\exists 1 \lambda \in \mathbb{C}; e_\lambda \dot{\mathcal{X}}_V = \dot{\mathcal{X}}_0$, where $e_\lambda : \mathbb{R} \ni t \mapsto e^{i\lambda t} \in \mathbb{C}$.

In this case, λ is given by $\lambda = V(0, \mathbf{0})$.