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On the Topological Aspect in General-Dimensional Dynamical Systems

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In higher than 3-dimensional spaces, projecting a periodic solution into 3-dimensional subspaces, we define quantities which are natural extension of topological quantities in 3-dimensional systems. We confirm the rule in the change of these quantities under the generic bifurcation and find the absence of the rewinding rule in period doubling cascades in general-dimensional dynamical systems.

Deterministic equations are often used to describe natural and even social phenomena. When the parameters in such systems of equation are tuned, the appearance of new solutions or the disappearance of a pair of solutions (so-called bifurcation phenomena) occurs. There are four kinds of solutions: fixed points, periodic solutions, quasi-periodic solutions and chaos. Particularly, oscillatory phenomena existing in many systems (for example, rhythms in far from equilibrium systems, population variation in prey and predator systems) are interesting subjects to study. In this paper, we focus on periodic solutions to ordinary differential equation systems, especially from the viewpoint of topological nature. There are several studies on this, for example, the study of knots in the Lorenz model,[1] the study of topological characterization of periodic orbits and their rules under several bifurcations,[2)~7),12)] and the study of mutual links in coexisting periodic orbits.[8] These studies have been done in the 3-dimensional dissipative dynamical systems (2-dimensional non-autonomous systems or 3-dimensional autonomous or non-autonomous systems). Further, the study in a 2-dimensional mapping was done.[9] In Hamiltonian systems, there are also topological aspects. The Maslov index which appears in semi-classical quantization is a phase factor. This is proved to be a winding number of stable and unstable manifolds of a periodic orbit, and is defined in Hamiltonian systems of any dimension.[10] In dissipative 3-dimensional forced systems, which are regarded as 4-dimensional autonomous systems, the rule for change of topological quantities under bifurcation was found.[3),5)] However, there is no research from the viewpoint of topology in general dimensions. Also, the research in conservative non-Hamiltonian systems has not yet been done. In this paper, we attempt the former investigation. The latter will be reported elsewhere. First, we summarize results in 3-dimensional systems. Let r_i, n_i and L be a relative torsion number, a torsion number and a self-linking number, respectively. They are assumed to belong to a real eigenvalue of a periodic orbit. See Ref. 5) for definitions. The relation $n_i = r_i + L$ holds. The following rule I can be proved:

Rule I (a) $n_i' = n_i$, $r_i' = r_i$ and $L' = L$ for pitchfork bifurcations,
(b) $n_i' = 2(n_1 + n_i)$, $r_i' = 2r_i$ for period doubling bifurcations,

(c) $n'=m^2n$ for Hopf bifurcations.

In rule I(c) an eigenvalue is $\lambda=\exp[2\pi il/m]$, l and m are relatively prime integers. Quantities which are primed correspond to bifurcating solutions. Among these rules, (a) and (b) have been numerically checked in 3-dimensional autonomous and periodically forced systems, and (c) has in a 2-dimensional mapping. In the case of 3-dimensional periodically forced systems, these systems are regarded as dynamical systems in $\boldsymbol{R}^3\times S$, where S is the 1-dimensional torus. Rule I is confirmed in these systems looking at orbits in $\boldsymbol{R}^3$. That is, rule I is valid in the projected $\boldsymbol{R}^3$ space of 4-dimensional $\boldsymbol{R}^3\times S$ space. The proof of rule I is based on the assumption of the continuity of bifurcation and dimension of an unstable manifold, being one for cases (a) and (b) or two for case (c), respectively. Thus it is apparent that rule I holds in general-dimensional dissipative or conservative systems considering a 3-dimensional projection, except for non-generic one. Hereafter, for simplicity we consider 4-dimensional autonomous systems. The extension to higher than 4-dimensional cases is straightforward. The differential equations are

$$\frac{d}{dt}\boldsymbol{x}=\boldsymbol{F}(\boldsymbol{x}), \quad \boldsymbol{x}=(x_1, x_2, x_3, x_4). \tag{1}$$

Let us consider a projected orbit into any 3-dimensional subspace.*)

For example, we consider 3-dimensional space (x_2, x_3, x_4). In this space, n_i and r_i are well defined unless $(d/dt)(x_2, x_3, x_4)$ is parallel to the eigenvector $\boldsymbol{e}_i=(e_{i,2}, e_{i,3}, e_{i,4})$. Provided an unstable direction is not perpendicular to this space in any points on the orbit, then rule I also holds in this space. In 4-dimensional space, there are four combinations of three components each. Let $n_i(k)$, $r_i(k)$ and $L(k)$ be the torsion number, the relative torsion number and the self-linking number of the orbit in the projected space, respectively. k runs from 1 to 4 and distinguishes projected spaces. For example, $n_i(1)$ is regarded as the torsion number in (x_2, x_3, x_4) space, etc. Thus, for a given eigenvector, there corresponds one 'torsion vector' $\boldsymbol{n}_i=(n_i(1), n_i(2), n_i(3), n_i(4))$. Likewise, $\boldsymbol{r}_i$ and $\boldsymbol{L}$ are defined. Then $\boldsymbol{n}_i=\boldsymbol{r}_i+\boldsymbol{L}$ and the following rule II holds:

Rule II (a) $\boldsymbol{n}_i'=\boldsymbol{n}_i$, $\boldsymbol{r}_i'=\boldsymbol{r}_i$ and $\boldsymbol{L}'=\boldsymbol{L}$ for pitchfork bifurcations,
(b) $\boldsymbol{n}_i'=2(\boldsymbol{n}_1+\boldsymbol{n}_i)$, $\boldsymbol{r}_i'=2\boldsymbol{r}_i$ for period doubling bifurcations,
(c) $\boldsymbol{n}'=m^2\boldsymbol{n}$ for Hopf bifurcations.

We check these rules numerically. First, we study case (a). We investigate the 4-dimensional modified Lorenz model,

$$\frac{d}{dt}x=-\sigma(x-y), \quad \frac{d}{dt}y=rx-y-x(z+w), \quad \frac{d}{dt}z=xy-bz,$$
$$\frac{d}{dt}w=kx^2-cw, \tag{2}$$

where $\sigma=16$, $b=4$, $c=1$ and $k=0.02$. r is a bifurcation parameter. This model is invariant under the symmetry operation S,

*) This rule does not apply to projections in which the projected orbit lies on a 2-dimensional plane.

Table I. Pitchfork bifurcation in the 4-dimensional modified Lorenz model.

		1st	2nd	3rd
$r=535.0$	eigenvalue	0.967	0.761	$4.65*10^{-3}$
	$\boldsymbol{r}$	(0, 0, 0, 1)	(0, 0, −1, 0)	(0, 0, 0, 1)
	$\boldsymbol{L}$	(1, −1, 1, 0)	(1, −1, 1, 0)	(1, −1, 1, 0)
	$\boldsymbol{n}$	(1, −1, 1, 1)	(1, −1, 0, 0)	(1, −1, 1, 1)
$r=527.0$	eigenvalue	0.994	0.759	$4.34*10^{-3}$
	$\boldsymbol{r}$	(0, 0, 0, 1)	(0, 0, −1, 0)	(0, 0, 0, 1)
	$\boldsymbol{L}$	(1, −1, 1, 0)	(1, −1, 1, 0)	(1, −1, 1, 0)
	$\boldsymbol{n}$	(1, −1, 1, 1)	(1, −1, 0, 0)	(1, −1, 1, 1)

$$S:\ x\to -x\,,\quad y\to -y\,.$$

For large values of r, there is a stable symmetric periodic orbit. The pitchfork bifurcation takes place at $r=r_c(\simeq 530.0)$. For $r<r_c$, two asymmetric limit cycles are stable. Numerical results are shown in Table I. For the period doubling bifurcation, case (b), we study the following coupled rotator model,

$$\frac{d}{dt}W_1=(r_1-i\omega_1)W_1-(c_1-id_1)W_1|W_1|^2+D(W_2-W_1)+R\,,$$

$$\frac{d}{dt}W_2=(r_2-i\omega_2)W_2-(c_2-id_2)W_2|W_2|^2+D(W_1-W_2)\,, \tag{3}$$

where the W_i's are complex variables and $r_1=r_2=0.4$, $\omega_1=\omega_2=c_1=c_2=d_1=d_2=1$, $D=0.04$. As the bifurcation parameter R decreases, a period-doubling bifurcation cascades. Numerical results are shown in Table II. The results in Tables I, II and III confirm rules II(a) and (b). From the above results, we notice that n_i changes in the parameter region where eigenvalues λ_i are real. This is because any link can be removed in 4-dimensional space.

As to rule II(c), we seek a 3-dimensional mapping in which a Hopf bifurcation takes place. We make an approach similar to in Ref. 5). The mapping function $\boldsymbol{f}(\boldsymbol{x})$ is assumed to be the following form,

$$\boldsymbol{f}(\boldsymbol{x})=\rho R(\theta)\boldsymbol{x}+\boldsymbol{g}(\boldsymbol{x})\,,$$

$$\boldsymbol{x}=(x,y,z)^T\,,$$

$$\boldsymbol{g}=(g_1,g_2,h)^T\,,$$

$$R(\theta)=\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$g_1=zx^2+a_{12}xy\,,$$

$$g_2=b_{11}x^2+b_{12}xy\,,$$

$$h=\mu(1+x)(z-a_{11})^2+a_{11}+cy\,, \tag{4}$$

Table II. Period doubling bifurcation in the coupled rotators model. T is period and $*$ denotes undetermined figure in the precision used.

		1st	2nd	3rd
$R=0.28887$ $T=10.0533$	eigenvalue	-0.974	$-4.87*10^{-2}$	$7.18*10^{-4}$
	$\boldsymbol{r}$	$\left(-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)$	$\left(-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$(-1, 0, 1, 0)$
	$\boldsymbol{L}$	$(1, 0, -1, 0)$	$(1, 0, -1, 0)$	$(1, 0, -1, 0)$
	$\boldsymbol{n}$	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$	$(0, 0, 0, 0)$
$R=0.28900$ $T=20.1074$	eigenvalue	0.975	$2.32*10^{-3}$	$5.38*10^{-7}$
	$\boldsymbol{r}$	$(-3, -1, 3, -1)$	$(-3, 1, 1, -1)$	$(-2, 0, 2, 0)$
	$\boldsymbol{L}$	$(1, -1, -1, -1)$	$(1, -1, -1, -1)$	$(1, -1, -1, -1)$
	$\boldsymbol{n}$	$(-2, -2, 2, -2)$	$(-2, 0, 0, -2)$	$(-1, -1, 1, -1)$
$R=0.29885$ $T=20.0685$	eigenvalue	-0.992	$-3.01*10^{-3}$	$2.99*10^{-7}$
	$\boldsymbol{r}$	$\left(-\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right)$	$\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$(-1, 0, 1, 0)$
	$\boldsymbol{L}$	$(0, -1, 0, -1)$	$(0, -1, 0, -1)$	$(0, -1, 0, -1)$
	$\boldsymbol{n}$	$\left(-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)$	$\left(-\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right)$	$(-1, -1, 1, -1)$
$R=0.298885$ $T=40.1369$	eigenvalue	0.998	$8.50*10^{-6}$	$-4.96*10^{-7}$
	$\boldsymbol{r}$	$(-3, 1, 3, 1)$	$(-3, -1, 1, -1)$	$(-2, *, *, *)$
	$\boldsymbol{L}$	$(-3, -3, 3, -3)$	$(-3, -3, 3, -3)$	$(-3, -3, 3, -3)$
	$\boldsymbol{n}$	$(-6, -2, 6, -2)$	$(-6, -4, 4, -4)$	$(-5, *, *, *)$
$R=0.30086$ $T=40.1474$	eigenvalue	-0.996	$-1.01*10^{-5}$	$5.39*10^{-7}$
	$\boldsymbol{r}$	$\left(-\frac{5}{2}, -\frac{3}{2}, \frac{5}{2}, \frac{1}{2}\right)$	$\left(-\frac{5}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right)$	$(-2, *, *, *)$
	$\boldsymbol{L}$	$(-4, -3, -1, -3)$	$(-4, -3, -1, -3)$	$(-4, -3, -1, -3)$
	$\boldsymbol{n}$	$\left(-\frac{13}{2}, -\frac{9}{2}, \frac{3}{2}, -\frac{5}{2}\right)$	$\left(-\frac{13}{2}, -\frac{7}{2}, -\frac{1}{2}, -\frac{9}{2}\right)$	$(-6, *, *, *)$
$R=0.300868$ $T=80.2948$	eigenvalue	0.982	$4.11*10^{-7}$	$3.36*10^{-7}$
	$\boldsymbol{r}$	$(-5, -3, 5, 1)$	$(-5, *, *, *)$	$(-4, *, *, *)$
	$\boldsymbol{L}$	$(-21, -15, 1, -11)$	$(-21, -15, 1, -11)$	$(-21, -15, 1, -11)$
	$\boldsymbol{n}$	$(-26, -18, 6, -10)$	$(-26, *, *, *)$	$(-25, *, *, *)$

where $a_{11}=-0.2$, $a_{12}=1.0$, $b_{11}=-1.0$, $b_{12}=0.5$, $c=0.1$ and $\mu=1.0$. T denotes transposition. We take $\theta=2\pi(l/m)$ with $l=1$ and $m=4$. In this mapping, a fixed point $\boldsymbol{x}_F$, $x_F=y_F=0$ and $z_F=a_{11}$, is stable for $\rho<1$, and for $\rho>1$ it becomes unstable. Taking ρ as a bifurcation parameter, we performed numerical calculation of the mapping. We found that at $\rho=1.0$, a stable periodic solution $\boldsymbol{x}^*$ with period 4 bifurcates from the fixed point $\boldsymbol{x}_F$ by the Hopf bifurcation. $\boldsymbol{x}^*$ exists for $1<\rho<\rho_c\approx1.22$. We obtained real eigenvalues for $d\boldsymbol{f}^4(\boldsymbol{x}^*)$ and a pair of complex eigenvalues for $d\boldsymbol{f}(\boldsymbol{x}^*)$. This implies that our assumptions to derive the rule are satisfied.

In 3-dimensional autonomous systems, when period doubling bifurcations cascade, the rewinding rule holds.[3),5)] The rewinding rule is

$$\Delta n_i^{(k)} \equiv n_i^{(k)}(f) - n_i^{(k)}(i) = (-1)^k \frac{\gamma}{2},$$

where $n_i^{(k)}(i)$ is the i-th torsion number in the positive eigenvalue region and $n_i^{(k)}(f)$ is the i-th torsion number in the negative eigenvalue region. Usually γ takes ± 1 and k denotes the period 2^k. In our high-dimensional systems, it seems that this rule does not hold in general. For example, in Table II for the first eigenvector, although $\Delta n_1(1)$ changes from $1/2$ to $-1/2$ and the rule holds, others do not satisfy the rule, e.g., $\Delta n_1(2)$ changes from $3/2$ to $-5/2$. Since our model has no special features, we

Table III. Torsion numbers in the rotated coupled rotators model. The notation $xy(\pi/6)$ implies the rotation of the system in the xy plane by $\pi/6$.

R	rotation	1st eigenvector	2nd eigenvector	3rd eigenvector
0.2887	$xy\left(\frac{\pi}{6}\right)$	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$	(0, 0, 0, 0)
	$yz\left(\frac{\pi}{3}\right)$	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	(0, −1, 0, 0)
	$yz\left(\frac{\pi}{5}\right)$	$\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$	(0, 0, 0, 0)
0.289	$xy\left(\frac{\pi}{6}\right)$	(−2, 0, 2, −2)	(−2, 2, 0, −2)	(−1, −1, 1, −1)
	$yz\left(\frac{\pi}{3}\right)$	(−2, −2, 2, −2)	(−2, −2, 2, −2)	(−1, −3, 1, −1)
	$yz\left(\frac{\pi}{5}\right)$	(−2, 2, 2, −2)	(−2, 2, 2, −2)	(−1, 1, 1, −1)
0.29885	$xy\left(\frac{\pi}{6}\right)$	$\left(-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right)$	$\left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right)$	(−1, −1, 0, −1)
	$yz\left(\frac{\pi}{3}\right)$	$\left(-\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right)$	$\left(-\frac{3}{2}, \frac{5}{2}, \frac{3}{2}, -\frac{3}{2}\right)$	(−1, 1, 1, −1)
	$yz\left(\frac{\pi}{5}\right)$	$\left(-\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, -\frac{1}{2}\right)$	$\left(-\frac{3}{2}, \frac{5}{2}, \frac{1}{2}, -\frac{3}{2}\right)$	(−1, 1, 0, −1)
0.298885	$xy\left(\frac{\pi}{6}\right)$	(−8, −2, 6, −2)	(−8, −2, 4, −4)	(−7, −2, 4, −4)
	$yz\left(\frac{\pi}{3}\right)$	(−6, 6, 6, −2)	(−6, 8, 6, −4)	(−5, 8, 5, −4)
	$yz\left(\frac{\pi}{5}\right)$	(−6, 6, 6, −2)	(−6, 8, 5, −4)	(−5, 8, 4, −4)
0.30086	$xy\left(\frac{\pi}{6}\right)$	$\left(-\frac{13}{2}, -\frac{7}{2}, \frac{3}{2}, -\frac{5}{2}\right)$	$\left(-\frac{13}{2}, -\frac{3}{2}, -\frac{1}{2}, -\frac{9}{2}\right)$	(−6, −2, 0, −4)
	$yz\left(\frac{\pi}{3}\right)$	$\left(-\frac{13}{2}, \frac{15}{2}, \frac{13}{2}, -\frac{5}{2}\right)$	$\left(-\frac{13}{2}, \frac{19}{2}, \frac{13}{2}, -\frac{9}{2}\right)$	(−7, 9, 5, −5)
	$yz\left(\frac{\pi}{5}\right)$	$\left(-\frac{13}{2}, \frac{13}{2}, \frac{13}{2}, -\frac{5}{2}\right)$	$\left(-\frac{13}{2}, \frac{17}{2}, \frac{9}{2}, -\frac{9}{2}\right)$	$\left(-\frac{13}{2}, \frac{15}{2}, \frac{11}{2}, -\frac{9}{2}\right)$
0.300868	$xy\left(\frac{\pi}{6}\right)$	(−26, −14, 6, −10)	(−17.5, −14, 3, −13)	(−21, −15, 4, −12)
	$yz\left(\frac{\pi}{3}\right)$	(−26, 29, 26, −10)	(−19.5, 32, 23.5, −12.5)	(−26, 29, 22.5, −12)
	$yz\left(\frac{\pi}{5}\right)$	(−26, 26, 26, −10)	(−19.5, 25, 22, −13)	(−22, 26, 23, −12)

consider these results reflect situations in general higher-dimensional spaces. Indeed, similar results were obtained in the forced Lorenz system.[3),5)] In this case, for the first eigenvector, Δn_1 changes $-3/2$, $3/2$, $-1/2$ in the xyz space. Since this system is considered to be 4-dimensional autonomous system, the linking of colsed curves can be removed, and then the n_i's are not 'good quantum number'. Instead of considering vectors $\boldsymbol{n}$, we may need continuous parametrizations of projections to obtain enough information from 4-dimensional objects.

These facts cause the question about validity of the template picture by Holmes et al.[9)] in higher than 3-dimensional dynamical systems, because the template picture also essentially depends on the three dimensionality. Especially in the construction of a template from any experimental data, the problem is crucial. The template organization by Takens imbedding may not have any meaning from a topological aspect, since experimental systems are usually very high-dimensional. Of course, there is a possibility that the higher than 3-dimensional systems shrink to 3-dimensional systems asymptotically, and the template picture becomes valid in the asymptotic regime. The extension of the template theory to higher-dimensional spaces requires the construction of the topological invariants in higher dimensions.

Here, we give a brief comment about an attempt to obtain any invariants. We try to seek any invariants in terms of a torsion vector $\boldsymbol{n}_i$ under any 4-dimensional rotation of the orbit in the region where λ_i is real.[*)] Calculations are done about the coupled rotator model(3). Periodic orbits are rotated and torsion vectors, etc. are estimated. Results are listed in Table III. Up to now, we have not succeeded in the construction of any invariants from these data. To make conclusive statements about the problem, it seems that further systematic numerical calculations and theoretical investigation are necessary for developing a general theory of invariants.

The mechanism of the rewinding in period doubling cascades and the existence of invariants in higher than 3-dimensional spaces are difficult problems and beyond the scope of this paper. We leave them open and future problems.

References

1) J. Birman and R. Williams, Topology **22** (1983), 47; Contemporary Mathematics **20** (1983), 1.
2) Y. Aizawa and T. Uezu, Prog. Theor. Phys. Lett. **67** (1982), 982.
3) T. Uezu and Y. Aizawa, Prog. Theor. Phys. **68** (1982), 1907.
4) T. Uezu, Phys. Lett. **93A** (1983), 161.
5) T. Uezu, Prog. Theor. Phys. **83** (1990), 850 and references cited therein.
6) P. Beiersdorfer, J-M. Wersinger and Y. Treve, Phys. Lett. **96A** (1983), 269.
7) P. Beiersdorfer, Phys. Lett. **100A** (1984), 379.
8) H. G. Solari and R. Gilmore, Phys. Rev. **A37** (1988), 3096.
9) A. Sannami, Jpn. J. Appl. Math. **6** (1989), 291.
10) J. M. Robbins, Nonlinearity **4** (1991), 343.
11) P. Holmes, Physica **21D** (1986), 7.
12) T. Motoike, T. Arimitsu and H. Konno, Phys. Lett. **182A** (1993), 373.

*) Torsion numbers can change even in the region where λ_i is real. See Ref. 5). So, the region should be restricted to that where such change does not occur.