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On the Bifurcation of Trapezoid Maps

—— *An Example of a Class of One-Dimensional Maps with a Superconvergent Period Doubling Cascade* ——

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We investigate the period doubling cascade in symmetric and asymmetric trapezoid maps. We prove that as the slope a of the trapezoid is increased, there occurs a cascade of period doubling bifurcations. Further we prove that the symbolic sequence is the Metropolis-Stein-Stein sequence R^{*m} and that the ratio δ_m of the interval of a, where the 2^m-cycle is stable, to the interval where the 2^{m+1}-cycle is stable scales as $\delta_m \simeq \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}$, where a_c is the accumulation point of the period doubling cascade, and γ is the ratio of the two slopes of the trapezoid. That is, convergence of the onset point of the 2^m-cycle to a_c is extremely rapid, and the Feigenbaum constant $\delta \equiv \lim_{m\to\infty}\delta_m$ is infinite. Further, we give approximate expressions for the accumulation point as functions of b, the length of the smaller side of the trapezoid. The agreement between theoretical results and numerically obtained data is excellent.

§1. Introduction

It is well known that in a class of one-dimensional map, as a parameter is changed, there occurs a cascade of period doubling bifurcations, and there exist several universal properties.[1] One of the universal quantities is the so-called Feigenbaum constant δ, which depends only on the exponent z characterizing the behavior of the map around the critical point.

About a decade ago, there was controversy regarding the $z \to \infty$ limit of $\delta(z)$. As a typical example, the map $x_{n+1} = f(x_n) = 1 - a|x_n|^z$ has been extensively studied, and two different values were conjectured for this limit, one finite and the other infinite.[2),3)] This problem was solved by Eckmann and Epstein, who proved that the limit is finite.[4)]

In this paper, instead of considering map with finite z and taking $z \to \infty$, we investigate a map with $z = \infty$. We treat the trapezoid map $x_{n+1} = T_{a,b}(x_n)$,

$$T_{a,b}(x) = \begin{cases} ax & \text{for } 0 \le x \le (1-b)/2, \\ a(1-b)/2 & \text{for } (1-b)/2 \le x \le (1+b)/2\ , \\ a(1-x) & \text{for } (1+b)/2 \le x \le 1\ , \end{cases}$$

(see Fig. 1). As a is increased from 1 with fixed b, successive period doubling bifurcations take place. We assign the symbols L, C and R to an orbit x_i according

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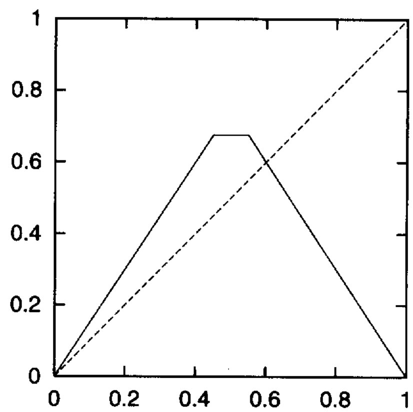


Fig. 1. Trapezoid map $T_{a,b}(x)$.

to the following rule. L corresponds to the case $x_i \in [0, (1-b)/2] \equiv I_L$, C to the case $x_i \in ((1-b)/2, (1+b)/2) \equiv I_C$, and R to the case $x_i \in [(1+b)/2, 1] \equiv I_R$. We denote this correspondence by $H(x_i)$. Further, we define the conjugate of R and L as $\bar{R} = L$ and $\bar{L} = R$. Then, the 2^m-cycle has the Metropolis-Stein-Stein symbolic sequence R^{*m}, as usual.[5] Defining a_m as the onset point of a 2^m-cycle, we calculate $\delta_m \equiv \frac{a_{m+1}-a_m}{a_{m+2}-a_{m+1}}$. Further, we investigate the asymmetric trapezoid map $x_{n+1} = A_{a,b,\gamma}(x_n)$,

$$A_{a,b,\gamma}(x) = \begin{cases} ax & \text{for } 0 \le x \le \alpha, \\ \alpha a & \text{for } \alpha \le x \le \beta, \\ \gamma a(1-x) & \text{for } \beta \le x \le 1, \end{cases}$$

where γ is the ratio of the two slopes of the trapezoid, $\alpha = \frac{\gamma}{1+\gamma}(1-b)$ and $\beta = \alpha + b$. When $\gamma = 1$, $A_{a,b,\gamma}(x)$ is reduced to $T_{a,b}(x)$.

In the following section, we treat the symmetric trapezoid map, and then in §3, we treat the asymmetric map. We give summary and discussion in the final section.

§2. The symmetric case

In Fig. 2, we show an example of a bifurcation diagram. The characteristic feature is that for $a < a_M = 2/(1-b)$, the orbit starting from $x_1 = a(1-b)/2$ is always stable, and its basin contains $B = \cup_{n=0}^{\infty} T_{a,b}^{-n}(I_C)$. For $a = a_M$, the map $T_{a,b}$ becomes surjective. Then the Lebesgue measure of B is 1, and its compliment $B^c = [0,1] - B$ is the Cantor set. All orbits in B^c are unstable, and B is the basin of the fixed point $x = 0$. Next, we go into the detailed calculations. First, we prove that at the onset point of the 2^{m+1} solution, $a = a_{m+1}$, the 2^m-cycle has the symbolic sequence R^{*m}.*)

Let us consider $T_{a,b}(x)$ for $1 < a \le a_1$. The graph of this map is shown in Fig. 3. At $a = a_1$, the fixed point x^* becomes $x^* = a(1-b)/2 = (1+b)/2$. That is, $a_1 = (1+b)/(1-b)$, which is in I_R. Thus, the symbolic sequence of the fixed point is R. For $a > a_1$, the fixed point becomes unstable and the period 2 points appear. For $a_1 \le a < a_2$, let us consider $T_{a,b}^2(x)$, whose graph is depicted in Fig. 4(a) for $a = a_1$ and in (b) for $a_1 < a < a_2$, respectively. Restricting $T_{a,b}^2(x)$ to the interval $[1-x^*, x^*]$, we obtain the trapezoid map $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$ for $a > a_1$ by the transformation

*) Strictly speaking, the last symbol of the sequence in our definition is R or L, and it is different from that in the MSS sequence, C.

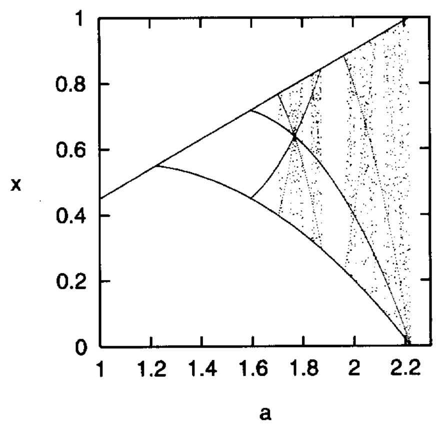


Fig. 2. Bifurcation diagram for the trapezoid map with $b = 0.1$. The abscissa is a, and the ordinate represents the coordinates of the attracting set.

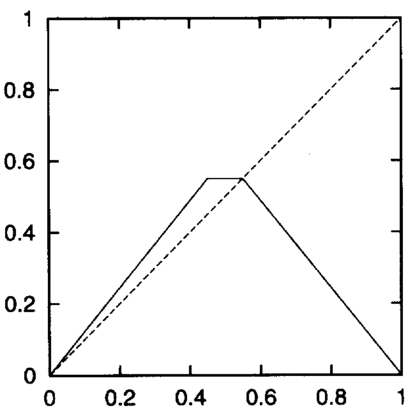
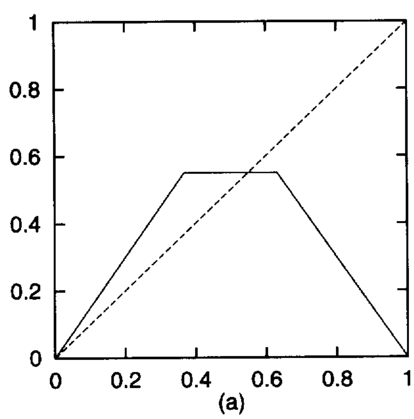


Fig. 3. Trapezoid map $T_{a,b}(x)$ for $a = a_1$ with $b = 0.1$.



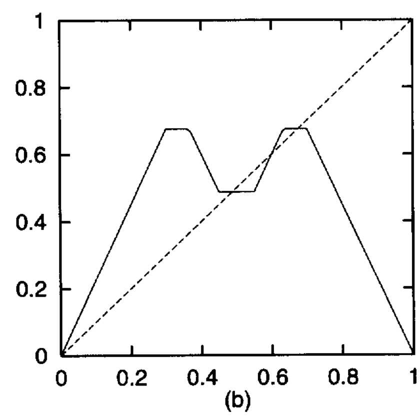


Fig. 4. Trapezoid map $T^2_{a,b}(x)$ for (a) $a = a_1$ with $b = 0.1$, and (b) $a_1 < a < a_2$ with $b = 0.1$.

$$x^{(1)} = \frac{x^* - x}{2x^* - 1}, \quad x^* = \frac{a}{a+1}, \quad \boldsymbol{a} \equiv (a, b),$$
$$\boldsymbol{\varphi}(\boldsymbol{a}) \equiv (a^{(1)}, b^{(1)}), \ a^{(1)} = a^2, \quad b^{(1)} = u(a)b, \quad u(a) \equiv \frac{a+1}{a-1},$$

where x^* is the fixed point. Note that $b^{(1)}$ is no more constant but decreases as a increases. As a increases, $a^{(1)}$ increases and $b^{(1)}$ decreases, and then the maximum value of $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$, $a^{(1)}(1 - b^{(1)})/2$, increases. Thus, the fixed point of $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$ becomes unstable at $a^{(1)} = a_1^{(1)} \equiv (1 + b^{(1)})/(1 - b^{(1)})$. Since $a_1^{(1)} = a_2^2$, this relation is nothing but the equation for a_2, $a_2 = \frac{1+\sqrt{b(2-b)}}{1-b}$. For $a_2 < a$, the 2-cycle of $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$ appears. We can continue the same procedure and obtain the renormalized map $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a > a_m(b)$,

$$\boldsymbol{\varphi}^m(\boldsymbol{a}) \equiv (a^{(m)}, b^{(m)}) \equiv (a^{2^m}, u_m(a)b),$$
$$u_m(a) \equiv \prod_{l=0}^{m-1} u(a^{2^l}) = \prod_{l=0}^{m-1} \frac{a^{2^l}+1}{a^{2^l}-1}.$$

As a increases, $a^{(m)}$ increases and $b^{(m)}$ decreases. Thus, the fixed point of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ becomes unstable, and the 2-cycle of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ appears. Therefore, the period doubling bifurcation cascade takes place.

Now, let us consider the symbolic sequence. We define the onset of the 2^n-cycle for $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ as $a_n^{(m)}$ and the i-th orbit of the 2^n-cycle as $x_{n,i}^{(m)}$. We set $x_{n,1}^{(0)} = a(1-b)/2$. For $m = 0$, we often omit the superscript (0). For the fixed point $x^* = x_{0,1}$ of $T_{a,b}(x)$, the sequence is

$$H(x_{0,1}) = \begin{cases} C & \text{for } 1 < a < a_1, \\ R & \text{for } a_1 \leq a. \end{cases} \tag{2·1}$$

For the 2-cycle,

$$H(x_{1,1})H(x_{1,2}) = \begin{cases} RC & \text{for } a_1 \leq a < a_2, \\ RL & \text{for } a_2 \leq a. \end{cases} \tag{2·2}$$

For the map $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$,

$$H(x_{0,1}^{(1)}) = \begin{cases} C & \text{for } 1 < a^{(1)} < a_1^{(1)}, \\ R & \text{for } a_1^{(1)} \leq a^{(1)}. \end{cases} \tag{2·3}$$

Since $x_{0,1}^{(1)}$ corresponds to $x_{1,2}^{(0)} = x_{1,2}$ and the scaling factor $\frac{-1}{2x^*-1}$ is negative in the coordinate transformation, $H(x_{1,2}^{(0)}) = \overline{H(x_{0,1}^{(1)})} = L$ for $a_1^{(1)} \leq a^{(1)}$. Thus, we obtain RL, and it agrees with Eq. (2·3).

Now, let us consider the relation between the 2^m-cycle and the 2^{m+1}-cycle. For $a_m < a < a_{m+1}$, the 2^m-cycle is stable and $x_{m,2^m}$ is in I_C because $x_{m,2^m}$ is mapped to $x_{m,1} = a(1-b)/2$. At $a = a_{m+1}$, the 2^m-cycle becomes unstable and the 2^{m+1}-cycle bifurcates from it. That is, $x_{m+1,i}$ and $x_{m+1,i+2^m}$ emerge from $x_{m,i}$. For all i, at $a = a_{m+1}$, $H(x_{m,i}) = H(x_{m+1,i}) = H(x_{m+1,i+2^m})$, and these are all L or R. For $a_{m+1} < a < a_{m+2}$, $H(x_{m+1,2^{m+1}})$ becomes C, but other symbols are unchanged. Then, $H(x_{m+1,2^{m+1}}(a_{m+2})) = \overline{H(x_{m,2^m}(a_{m+1}))}$ holds. We show this in the following. For the map $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$, the symbol of the fixed point $x_{0,i}^{(m)}$ which corresponds to $x_{m,2^m}$ is R at $a^{(m)} = a_1^{(m)} = (a_{m+1})^{2^m}$ for any m. Since the coordinate transformation from $x^{(m)}$ to $x^{(m+1)}$ inverts the ordering of the coordinates, $H(x_{1,0}^{(m)}(a_1^{(m)})) = H(x_{1,0}^{(m+1)}(a_1^{(m+1)})) = R$ implies $H(x_{m+1,2^{m+1}}(a_{m+2})) = \overline{H(x_{m,2^m}(a_{m+1}))}$. Thus, the rule to obtain the symbolic sequence for the 2^{m+1}-cycle at $a = a_{m+2}$ from that for the 2^m-cycle at $a = a_{m+1}$ is to duplicate the sequence for the 2^m-cycle and change the 2^{m+1}-th symbol to its conjugate symbol. Apparently, the sequence thus obtained is R^{*m}.

Next, we estimate δ_m. We assign 0 or 1 to x_i with the symbol L or R, respectively. We denote this correspondence as $s_i = s(x_i)$. Further, we define the function $f_s(x)$ for $x \in I_L \cup I_R$ as

$$f_s(x) = sa + (1-2s)ax,$$

where $s = s(x)$. Starting from the maximum value of $T_{a,b}(x)$, $x_1 = a(1-b)/2$, if $x_1, x_2, \cdots, x_{n-1} \in I_L \cup I_R$, x_n is expressed as

$$x_n = f_{s_{n-1}}(x_{n-1}) = f_{s_{n-1}} \circ f_{s_{n-2}} \circ \cdots \circ f_{s_1}(x_1) = \sum_{l=1}^{n} \xi_l a^l,$$

$$\xi_l = s_{n-l} \prod_{j=n-l+1}^{n-1} (1-2s_j) \quad \text{for} \quad 2 \le l \le n-1,$$

$$\xi_1 = s_{n-1}, \quad \xi_n = \frac{1-b}{2} \prod_{j=1}^{n-1} (1-2s_j). \tag{2·4}$$

Let us consider the period 2^m solution for $a \ge a_m$:

$$x_{2^m} = f_{s_{2^m-1}}(x_{2^m-1}) \equiv \sum_{l=0}^{2^m-1} c_l^{(m)} a^{2^m-l} \equiv F_m(a),$$

$$c_l^{(m)} = s_l \prod_{j=l+1}^{2^m-1} (1-2s_j) = \xi_{2^m-l}, \quad \text{for} \quad 1 \le l \le 2^m-2,$$

$$c_0^{(m)} = \frac{1-b}{2} \prod_{j=1}^{2^m-1} (1-2s_j),$$

$$c_{2^m-1}^{(m)} = s_{2^m-1}. \tag{2·5}$$

For example, $x_2 = F_1(a) = c_0^{(1)}a^2 + c_1^{(1)}a = -\frac{1-b}{2}a^2 + a$ (see Fig. 2). Note that $F_m(a)$ is determined by the sequence $(s_1, \cdots, s_{2^m-1})$ and is independent of s_{2^m}. Then, the condition for the onset point a_m of a 2^m-cycle is

$$F_m(a_m) = \frac{1+(-1)^{m-1}b}{2}, \tag{2·6}$$

or

$$F_{m-1}(a_m) = \frac{1+(-1)^{m-1}b}{2}. \tag{2·7}$$

Since the symbolic sequence is the Metropolis-Stein-Stein sequence R^{*m}, the relations $\Pi_{j=1}^{2^m-1}(1-2s_j) = (-1)^m$ and $s_{2^m} = \frac{1+(-1)^m}{2}$ follow. From these relations, $r_l \equiv (-1)^m c_l^{(m)}$ is shown to be m-independent as long as it is defined. Further, we can show that $r_0 = (1-b)/2$, $r_{2^m} = -(1+(-1)^m)/2$, and for $l > 0$ the successive values (r_l, r_{l+1}) take only the four sets of values $(0,1)$, $(1,0)$, $(1,-1)$ and $(-1,1)$. Next, we define

$$G_m(a) \equiv (-1)^m a^{-2^m} F_m(a) = \sum_{l=0}^{2^m-1} r_l a^{-l}.$$

$G_\infty(z) \equiv \lim_{m\to\infty} G_m(z)$ is the analytic function for $|z| > 1$. Equation (2·6) becomes

$$G_m(a_m) = a_m^{-2^m} \frac{(-1)^m - b}{2}, \tag{2·8}$$

and the accumulation point a_c satisfies the equation

$$G_\infty(a_c) = 0. \tag{2·9}$$

Let us estimate a_m for large m. Putting $a_m = a_c - \epsilon_m$ and neglecting the terms higher than ϵ_m, we obtain from Eq. (2·8),

$$\epsilon_m \simeq \left\{ G_m(a_c) + \frac{b-(-1)^m}{2} a_c^{-2^m} \right\} / G'_\infty(a_c). \tag{2·10}$$

Equation (2·9) is rewritten as

$$G_\infty(a_c) = G_m(a_c) + \sum_{l=0}^{\infty} r_{2^m+l} a_c^{-l-2^m} = 0. \tag{2·11}$$

Since $r_{2^m+l} = -r_l$ for $1 \le l \le 2^m - 1$, Eq. (2·11) is further changed to the following:

$$G_\infty(a_c) = G_m(a_c)(1 - a_c^{-2^m}) + a_c^{-2^m} \left(r_0 + r_{2^m} + a_c^{-2^m} \sum_{l=0}^{\infty} r_{2^m+l} a_c^{-l} \right) = 0. \tag{2·12}$$

Thus, we obtain

$$G_m(a_c) = \frac{b+(-1)^m}{2} a_c^{-2^m} + \mathcal{O}((a_c^{-2^m})^2). \tag{2·13}$$

Substituting Eq. (2·13) into Eq. (2·10), we obtain

$$\epsilon_m \simeq b a_c^{-2^m} / G'_\infty(a_c) + \mathcal{O}((a_c^{-2^m})^2). \tag{2·14}$$

Thus,

$$\delta_m = \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_{m+1} - \epsilon_{m+2}} \simeq \frac{\epsilon_m}{\epsilon_{m+1}} \simeq a_c^{2^m}.$$

Now, let us investigate the b dependence of a_c. For $b \sim 1$, a_c is very large. Thus, in Eq. (2·9) we retain only terms up to a_c^{-3} and obtain

$$a_c \simeq a_M - \frac{1}{a_M}. \tag{2·15}$$

For the case that b is not near 1, we consider $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ at $a = a_{m+1}$. For the map $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$, this value is the onset of the 2-cycle, that is,

$$\overline{a^{(m)}} \equiv a_1^{(m)} = \frac{1 + b^{(m)}(a_{m+1}, b)}{1 - b^{(m)}(a_{m+1}, b)}, \tag{2·16}$$

$$\overline{b^{(m)}} \equiv b^{(m)}(a_{m+1}, b). \tag{2·17}$$

We define the transformation $\mathcal{R}$ by

$$\mathcal{R}(\overline{a^{(m)}}, \overline{b^{(m)}}) = (\overline{a^{(m+1)}}, \overline{b^{(m+1)}}). \tag{2·18}$$

The operation of $\mathcal{R}$ is to increase the parameter a from a_{m+1} to a_{m+2}, iterate $T_{\boldsymbol{\varphi}^m(a_{m+2},b)}$ twice and then rescale. As is easily seen,

$$\overline{a^{(m+1)}} = a_1^{(m+1)} = (a_2^{(m)})^2 = (a_{m+2})^{2^{m+1}}, \tag{2·19}$$

$$\overline{b^{(m+1)}} = \sqrt{\widehat{b^{(m)}}(2 - \widehat{b^{(m)}})}, \tag{2·20}$$

$$\widehat{b^{(m)}} \equiv b^{(m)}(a_{m+2}, b).$$

Now, we prove that $\widehat{b^{(m)}}$ in Eq. (2·20) can be approximated by $\overline{b^{(m)}}$. First, we assume $a_{m+1} \gg 1$. Since $b^{(m)} = u_m(a)b$, we estimate $u_m(a)$ for $a \gg 1$,

$$\begin{aligned}\ln u_m(a) &= \sum_{l=0}^{m-1} \ln\left(1 + \frac{2a^{-2^l}}{1 - a^{-2^l}}\right) < \sum_{l=0}^{m-1} \frac{2a^{-2^l}}{1 - a^{-2^l}} \\ &< \sum_{n=1}^{\infty} \frac{2a^{-n}}{1 - a^{-n}} \sim \sum_{n=1}^{\infty} 2a^{-n} = \frac{2}{a-1}.\end{aligned} \tag{2·21}$$

Therefore, for $a \gg 1$, $u_m(a) \sim e^{2/a} \sim \mathcal{O}(1)$. Similarly, we estimate the derivative of $u_m(a)$ as $|u'_m(a)| \sim 2u_m(a)/a^2 \sim 0$ for $a \gg 1$. Thus,

$$\begin{aligned}\overline{b^{(m)}} - \widehat{b^{(m)}} &= b(u_m(a_{m+1}) - u_m(a_{m+2})) \simeq b|u'_m(a_{m+1})|(a_{m+2} - a_{m+1}) \\ &\sim 2b(a_{m+2} - a_{m+1})/a_{m+1}^2 \sim 2b^2 \frac{a_c^{-2^{m+1}}}{a_{m+1}^2 G'_\infty(a_c)} \ll 1.\end{aligned} \tag{2·22}$$

Next, let us consider the case that a_{m+1} is not small. This occurs when b and m are small. If $\overline{b^{(m)}} \ll 1$, since $u'_m(a) < 0$, $\widehat{b^{(m)}} = u_m(a_{m+2})b < u_m(a_{m+1})b = \overline{b^{(m)}}$. That is, $\widehat{b^{(m)}} \ll 1$. Thus, we can replace $\widehat{b^{(m)}}$ by $\overline{b^{(m)}}$ as long as $\overline{b^{(m)}}$ is small. Further, when $\overline{b^{(k)}}$ becomes rather large, for the map $T_{\boldsymbol{\varphi}^k(\boldsymbol{a})}(x^{(k)})$, the onset point of the 2^l-cycle, $a_l^{(k)}$, is rather large for $l \geq 1$. It is easily shown that k is not a very large number. Thus, since $a_{k+l} = (a_l^{(k)})^{\frac{1}{2^k}}$, this case is now reduced to the case $a_{k+1} \gg 1$. Therefore, we may always replace $\widehat{b^{(m)}}$ by $\overline{b^{(m)}}$. That is,

$$\overline{b^{(m+1)}} \simeq \phi(\overline{b^{(m)}}), \tag{2·23}$$

$$\phi(b) \equiv \sqrt{b(2-b)}. \tag{2·24}$$

Suppose that after a sufficiently large number of operations of $\mathcal{R}$, $\overline{b^{(n)}}$ becomes nearly 1:

$$\overline{b^{(n)}} \simeq \phi^n(b) \simeq 1. \tag{2·25}$$

Since $a_{n+1} \gg 1$, $\frac{\partial}{\partial a} b^{(n)} \sim 0$ for $a > a_{n+1}$. Thus, as a increases, $b^{(n)}$ in $T_{\boldsymbol{\varphi}^n(\boldsymbol{a})}(x^{(n)})$ is considered to be constant for $a > a_{n+1}$. Then, from Eq. (2·15) we obtain

$$a_c^{(n)} \simeq \frac{2}{1 - \overline{b^{(n)}}} - \frac{1 - \overline{b^{(n)}}}{2} \sim \frac{2}{1 - \phi^n(b)} - \frac{1 - \phi^n(b)}{2}. \tag{2·26}$$

Since $a_m^{(n)} = a_{n+m}^{2^n}$, we obtain

$$a_c^{(n)} = a_c^{2^n}, \tag{2·27}$$

and then

$$a_c \simeq \left(\frac{2}{1-\phi^n(b)}\right)^{1/2^n}. \tag{2·28}$$

It is easily shown that the right-hand side of Eq. (2·28) is independent of n as long as n satisfies $\phi^n(b) \sim 1$. To obtain the explicit b dependence of a_c for small b, we consider the iterations of ϕ in two steps. First, let k be the number of iterations such that $\overline{b^{(k)}}$ becomes rather large,

$$\phi^k(b) \simeq c,$$

where c is some constant, say $c = 0.2$. Since $\phi(b) \simeq \sqrt{2b}$ for $b \ll 1$, we obtain

$$c = \overline{b^{(k)}} = 2\left(\frac{b}{2}\right)^{1/2^k}. \tag{2·29}$$

Next, suppose that after l iterations, $\phi^l(c)$ becomes nearly 1, $\phi^l(c) \sim 1$. Thus,

$$a_c \simeq \left(\frac{2}{1-\phi^l(c)}\right)^{1/2^{k+l}}. \tag{2·30}$$

From Eqs. (2·29) and (2·30), we obtain

$$a_c \sim \exp\left[\frac{h(c,l)}{\ln 2 - \ln b}\right] \propto \exp\left[-\frac{h(c,l)}{\ln b}\right], \tag{2·31}$$
$$h(c,l) = \left(\ln\frac{2}{c}\right)\left(\ln\frac{2}{1-\phi^l(c)}\right)\frac{1}{2^l},$$

where $h(c,l)$ is a slowly varying function of c and does not depend on l as long as $\phi^l(c) \sim 1$. a_c tends to 1 as $b \to 0$.

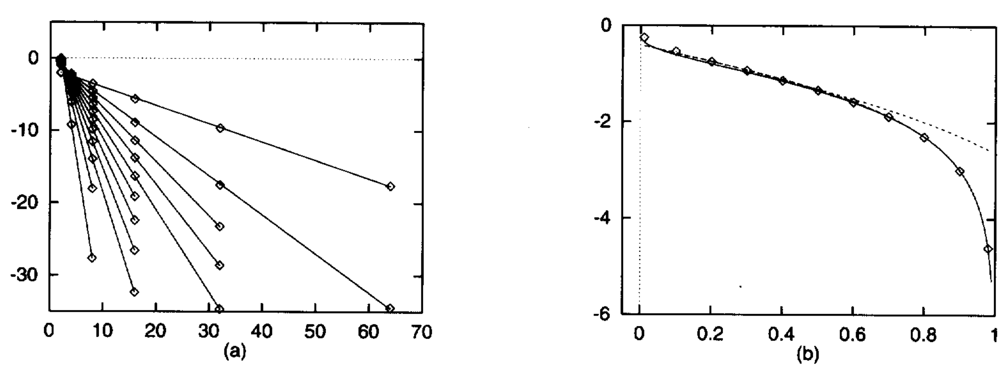


Fig. 5. The symmetric case. (a) The abscissa is 2^m, and the ordinate is $\ln \Delta\epsilon_m$. The diamonds are the numerical data, and the solid lines were obtained by a least square method, $\ln \Delta\epsilon_m = B + A2^m$. As b increases, the gradient $|A|$ increases. (b) b dependence of A. The diamonds are the numerical data, and the curves are theoretical results, $A_{th} = -\ln a_c$. The solid curve, dashed curve and dotted curve were calculated using Eqs. (2·28), (2·15) and (2·31), respectively.

We performed a numerical calculation for $b = 0.01,\ 0.1,\ 0.2, \cdots,\ 0.9,\ 0.98$ and estimated $\Delta\epsilon_m = a_{m+1} - a_m$ for $m = 1 - 6$. For any value of b, the data fit the relation $\ln \Delta\epsilon_m = B + A2^m$ fairly well(see Fig. 5(a)). From Eq. (2·14), theoretical results for A and B are $A_{th} = -\ln a_c$ and $B_{th} = \ln(b/G_\infty(a_c))$, respectively. We show the b dependence of A thus obtained in Fig. 5(b). In this figure, we also depict A_{th} calculated from Eqs. (2·15), (2·28) and (2·31). The agreement between the theoretical result, Eq. (2·28), and the numerical result is excellent in the full range of b. Further, the approximate formulas (2·15) and (2·31) agree well with the numerical data in the regions $b \sim 1$ and $b \ll 1$, respectively.

§3. The asymmetric case

In this case, we can make a similar argument as in the symmetric case. As a bifurcation parameter we take a, with b and γ fixed. Depending on the value of β, there are two different bifurcation processes for small a. For $\beta > 1/2$, the same process as in the symmetric case occurs, and $a_1 = \beta/\alpha$. For $\beta < 1/2$, since the fixed point continues to be stable for $\beta/\alpha < a < (1-\beta)/\alpha$, $a_1 = (1-\beta)/\alpha$. For $a > a_1$, the bifurcation process is the same as in the symmetric case.

Here, we prove that for $a > a_1$, there occurs a period doubling bifurcation cascade.

Let us define $a_L = a$ and $a_R = \gamma a$. As before, we define the renormalized map $A_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$ for $a > a_1$ as follows:

$$\begin{aligned}
x^{(1)} &= \frac{x^* - x}{\Delta x}, \quad x^* = \frac{\gamma a}{\gamma a + 1}, \quad \Delta x = \gamma(a-1)/(1+\gamma a), \\
\boldsymbol{\varphi} &: (a, b, \gamma) \to (a^{(1)}, b^{(1)}, \gamma^{(1)}), \\
& \quad a^{(1)} = a_R^2 = (\gamma a)^2, \ a_R^{(1)} = a_L a_R = \gamma a^2, \\
b^{(1)} &= u(a,\gamma) b, \ u(a,\gamma) \equiv \frac{\gamma a + 1}{\gamma(a-1)}, \ \gamma^{(1)} = 1/\gamma, \ \alpha^{(1)} = \frac{\alpha a - \beta}{\gamma(a-1)}.
\end{aligned}$$

As a is increased, the fixed point of $A_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$ becomes unstable and the 2-cycle appears. Thus, repeating the above procedure, we obtain the renormalized map $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a > a_m(b,\gamma)$:

$$\begin{aligned}
&\boldsymbol{\varphi}^m(\boldsymbol{a}) \equiv (a^{(m)}, b^{(m)}, \gamma^{(m)}), \ \ \boldsymbol{a} \equiv (a, b, \gamma), \\
&\quad a^{(m)} = \gamma^{2(2^m - (-1)^m)/3} a^{2^m}, \quad a_R^{(m)} = \gamma^{(2^{m+1} + (-1)^m)/3} a^{2^m}, \\
&\quad b^{(m)} = u_m(a,\gamma) b, \ \ u_m(a,\gamma) \equiv \prod_{l=0}^{m-1} u(a^{(l)}, \gamma^{(l)}), \quad \gamma^{(m)} = \gamma^{(-1)^m}.
\end{aligned}$$

As a increases, $a^{(m)}$ and $a_R^{(m)}$ increase and $b^{(m)}$ decreases. Thus, the maximum of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$, $a^{(m)}\alpha^{(m)}$, increases, and the period doubling bifurcation cascade appears.

As for the symbolic sequence, we define $H(x) = L$ for $x \in I_L \equiv [0, \alpha]$, $H(x) = C$ for $x \in I_C \equiv (\alpha, \beta)$, and $H(x) = R$ for $x \in I_R \equiv [\beta, 1]$. Then, it is easily shown that the symbolic sequence for the 2^m-cycle is R^{*m} by the same argument as that in the symmetric case.

Now, let us estimate δ_m. Defining $s(x)$ as before, $f_s(x)$ for $x \in I_L \cup I_R$ is defined as

$$f_s(x) = a(\eta + \nu x), \quad \eta = \gamma s(x), \quad \nu = 1 - (1+\gamma)s(x).$$

Starting from $x_1 = \alpha a$, we obtain x_{2^m} for $a \geq a_m$:

$$\begin{aligned}
x_{2^m} &= f_{s_{2^m-1}}(x_{2^m-1}) \equiv \sum_{l=0}^{2^m-1} c_l^{(m)} a^{2^m-l} \equiv F_m(a), \\
c_l^{(m)} &= \eta_l \prod_{j=l+1}^{2^m-1} \nu_j \quad \text{for } 1 \leq l \leq 2^m - 2, \\
c_0^{(m)} &= \alpha \prod_{j=1}^{2^m-1} \nu_j, \\
c_{2^m-1}^{(m)} &= \eta_{2^m-1}.
\end{aligned} \tag{3·1}$$

Thus,

$$F_m(a_m) = \frac{\alpha + \beta + (-1)^{m-1}b}{2}, \tag{3·2}$$

$$F_{m-1}(a_m) = \frac{\alpha + \beta + (-1)^{m-1}b}{2}. \tag{3·3}$$

Defining ζ_m and $\hat{\zeta}_m$ as the numbers of 1 in $s_1, s_2, \cdots, s_{2^m-1}$ and in $s_1, s_2, \cdots, s_{2^m}$, respectively, we obtain

$$\begin{aligned}
\hat{\zeta}_m &= (2^{m+1} + (-1)^m)/3 \quad \text{for } m \geq 0, \\
\zeta_m &= \hat{\zeta}_m \quad \text{for odd } m \ (m \geq 1), \\
\zeta_m &= \hat{\zeta}_m - 1 \quad \text{for even } m \ (m \geq 0), \\
\zeta_m &= \zeta_{m-1} + \hat{\zeta}_{m-1} \quad \text{for } m \geq 1,
\end{aligned}$$

where $\zeta_0 \equiv 0$. Then, $r_l \equiv (-\gamma)^{-\zeta_m} c_l^{(m)}$ is m-independent as long as it is defined, and

$$r_0 = \alpha, \quad r_{2^m} = -s_{2^m} \gamma^{1-\hat{\zeta}_m}. \quad (m \geq 0)$$

Thus, we define

$$G_m(a) \equiv (-\gamma)^{\zeta_m} a^{-2^m} F_m(a) = \sum_{l=0}^{2^m-1} r_l a^{-l}.$$

$G_\infty(z) \equiv \lim_{m\to\infty} G_m(z)$ is an analytic function for $|z| > \max(1, \gamma^{-1})$. Then, Eq. (3·2) becomes

$$G_m(a_m) = (-\gamma)^{\zeta_m} a^{-2^m} \frac{\alpha + \beta + (-1)^{m-1}b}{2}, \tag{3·4}$$

and

$$G_\infty(a_c) = 0. \tag{3·5}$$

Since $r_l = -\gamma^{\hat{\zeta}_m} r_{2^m+l}$ for $1 \le l \le 2^m - 1$, we obtain

$$G_m(a_c) = -(\alpha - \gamma s_{2^m})\gamma^{-\hat{\zeta}_m} a_c^{-2^m} + \mathcal{O}((a_c^{-2^m}\gamma^{-\hat{\zeta}_m})^2), \tag{3·6}$$

$$\epsilon_m = a_c - a_m \simeq b a_c^{-2^m}\gamma^{-\zeta_m}/G'_\infty(a_c) + \mathcal{O}((a_c^{-2^m}\gamma^{-\hat{\zeta}_m})^2). \tag{3·7}$$

Thus,

$$\delta_m \simeq \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}.$$

As for the b dependence of a_c, we estimate a_c for $b \sim 1$:

$$a_c \simeq \frac{1+\gamma}{\gamma(1-b)} - \frac{1-b}{1+\gamma}. \tag{3·8}$$

For other cases, we consider $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ at $a = a_{m+1}$. Since $\beta^{(m)} > 1/2$ for $m \ge 1$, we define

$$\overline{a^{(m)}} \equiv a_1^{(m)} = \frac{\beta^{(m)}(a_{m+1}, b, \gamma)}{\alpha^{(m)}(a_{m+1}, b, \gamma)}, \tag{3·9}$$

$$\overline{b^{(m)}} \equiv b^{(m)}(a_{m+1}, b, \gamma). \tag{3·10}$$

The transformation $\mathcal{R}$ is defined by

$$\mathcal{R}(\overline{a^{(m)}}, \overline{b^{(m)}}) = (\overline{a^{(m+1)}}, \overline{b^{(m+1)}}). \tag{3·11}$$

Using $a_2 = \frac{1}{2\alpha}(1 + \sqrt{1 - \frac{4\alpha^2}{\gamma}})$, we derive

$$\begin{aligned}
\overline{a^{(m+1)}} &= a_1^{(m+1)} = (\gamma^{(m)} a_2^{(m)})^2 = \cdots = \gamma^{2(2^{m+1} - (-1)^{m+1})/3}(a_{m+2})^{2^{m+1}}, \\
\overline{b^{(m+1)}} &= \phi(\widehat{b^{(m)}}, \gamma^{(m)}), \\
\widehat{b^{(m)}} &\equiv b^{(m)}(a_{m+2}, b, \gamma), \\
\phi(b, \gamma) &= \frac{1 + \gamma + 2(1-b) + \sqrt{(1+\gamma)^2 - 4\gamma(1-b)^2}}{1 + \gamma - 2\gamma(1-b) + \sqrt{(1+\gamma)^2 - 4\gamma(1-b)^2}} b.
\end{aligned}$$

Since $\ln u_m \le \sum_{n=1}^{\infty}(1+\hat{\gamma})/(\tilde{a}^n\tilde{\gamma}^{2/3} - 1)$, $u_m \sim \exp[(1+\hat{\gamma})/(\tilde{\gamma}^{2/3}\tilde{a})] \sim \mathcal{O}(1)$ for $a \gg 1$, where $\tilde{a} = \gamma^{2/3}a, \hat{\gamma} = \max(\gamma, \gamma^{-1})$ and $\tilde{\gamma} = \min(\gamma, \gamma^{-1}) = \hat{\gamma}^{-1}$. Further, we obtain $|\partial u_m/\partial a| \sim (1+\hat{\gamma})u_m\hat{\gamma}^{7/3}/a^2 \sim 0$ for $a \gg 1$. Thus, for $a_{m+2} \gg 1$,

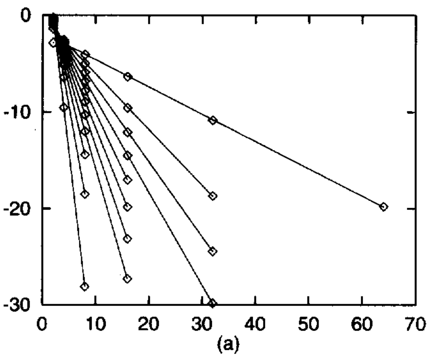
$$\overline{b^{(m)}} - \widehat{b^{(m)}} \sim (1+\hat{\gamma})b^2\hat{\gamma}^{7/3}a_c^{-2^{m+1}}\gamma^{-\zeta_{m+1}}/(a_{m+2}^2 G'_\infty(a_c)) \ll 1,$$

and we can replace $\widehat{b^{(m)}}$ by $\overline{b^{(m)}}$. In fact, except for the case of $\hat{\gamma} \sim 1$, a_{m+2} is large. If b is not small, this is obvious. For small b, estimating $\phi(b, \gamma)$, we obtain

$$\begin{aligned}
\overline{b^{(1)}} &\simeq (\hat{\gamma} - 1)/\hat{\gamma} \quad \text{for } \gamma > 1, \\
\overline{b^{(2)}} &\simeq (\hat{\gamma} - 1)/\hat{\gamma} \quad \text{for } \gamma < 1.
\end{aligned}$$

Then, except for the case of $\hat{\gamma} \sim 1$, $\overline{b^{(k)}} \simeq (\hat{\gamma} - 1)/\hat{\gamma}$ is not small for $k = 1$ or 2, and then $a_l^{(k)}$ is large for $l \ge 1$. Therefore, the following recursive relation is obtained:

$$\overline{b^{(m+1)}} \simeq \phi(\overline{b^{(m)}}, \gamma^{(m)}).$$



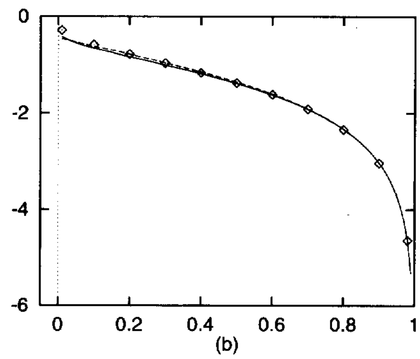


Fig. 6. The asymmetric case with $\gamma = 1.2$. (a) The abscissa is 2^m, and the ordinate is $\ln \Delta\epsilon_m - \ln \gamma_m$, where $\gamma_m = \gamma^{(3+(-1)^m)/6}$. The diamonds are the numerical data, and the solid lines were obtained by a least square method, $\ln \Delta\epsilon_m - \ln \gamma_m = B + A2^m$. As b increases, the gradient $|A|$ increases. (b) b dependence of A. The diamonds are the numerical data, and the curves are the theoretical results, $A_{th} = -\ln(\gamma^{2/3} a_c)$. The solid curve and the dashed curve were calculated using Eqs. (3·13) and (3·8), respectively.

Let us assume that $\overline{b^{(n)}}$ becomes nearly 1 for large n,

$$\overline{b^{(n)}} \simeq 1. \tag{3·12}$$

Since $a_{n+1} \gg 1$, $\frac{\partial}{\partial a} b^{(n)} \sim 0$ for $a > a_{n+1}$. Thus, as a increases, $b^{(n)}$ in $A_{\boldsymbol{\varphi}^n(\boldsymbol{a})}(x^{(n)})$ is considered to be constant for $a > a_{n+1}$. Thus, we obtain

$$a_c \simeq \left[\frac{1+\gamma^{(n)}}{1-\overline{b^{(n)}}} (\gamma^{(n)})^{-1/3} \right]^{1/2^n} \gamma^{-2/3}. \tag{3·13}$$

The right-hand side of Eq. (3·13) is independent of n as long as n satisfies $\overline{b^{(n)}} \sim 1$. For small b, a_c is b-independent. Now, let us see this. We put $c = \frac{\hat{\gamma}-1}{\hat{\gamma}}$. Then, $\overline{b^{(k)}} \simeq c$, where $k = 1$ for $\gamma > 1$ and $k = 2$ for $\gamma < 1$. Let us assume $\phi^l(c, \gamma^{(k)}) \sim 1$. (Here, we have used the following abbreviated notation,

$$\phi^2(\overline{b^{(k)}}, \gamma^{(k)}) \equiv \phi(\phi(\overline{b^{(k)}}, \gamma^{(k)}), \gamma^{(k+1)}),$$

etc.) Then,

$$a_c \simeq \left[\frac{1+\gamma^{(k+l)}}{1-\phi^l(c, \gamma^{(k)})} (\gamma^{(k+l)})^{-1/3} \right]^{1/2^{k+l}} \gamma^{-2/3}. \tag{3·14}$$

l is determined by the condition $\phi^l(c, \gamma^{(k)}) \sim 1$. Thus, a_c is b-independent.

In this case, we performed numerical calculations similar to those in the symmetric case and obtained excellent agreement between the theoretical results and numerical data (see Figs. 6(a) and (b)).

§4. Summary and discussion

The results of this paper are summarized as follows.

As a is increased from 1 with b and γ fixed, a cascade of period doubling bifurcations takes place for any b and γ. The symbolic sequence is R^{*m}, and for large m, δ_m is expressed as

$$\delta_m \simeq \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}.$$

Further, a_c is approximated by

$$a_c \simeq \left[\frac{1+\gamma^{(n)}}{1-\phi^n(b,\gamma)}(\gamma^{(n)})^{-1/3}\right]^{1/2^n}\gamma^{-2/3},$$

$$\phi(b,\gamma) = \frac{1+\gamma+2(1-b)+\sqrt{(1+\gamma)^2-4\gamma(1-b)^2}}{1+\gamma-2\gamma(1-b)+\sqrt{(1+\gamma)^2-4\gamma(1-b)^2}}b,$$

where $\gamma^{(n)} = \gamma^{(-1)^n}$, and n is an integer such that $\phi^n(b,\gamma) \sim 1$. In particular, for $\gamma = 1$, the above formula is reduced to

$$a_c \simeq \left(\frac{2}{1-\phi^n(b)}\right)^{1/2^n},$$

$$\phi(b) \equiv \phi(b,1) = \sqrt{b(2-b)}.$$

The explicit dependence of a_c on b is

$$a_c \simeq \frac{1+\gamma}{\gamma(1-b)} - \frac{1-b}{1+\gamma} \quad \text{for any } \gamma \text{ and } b \simeq 1,$$

$$a_c \simeq \text{constant} \quad \text{for } \gamma \neq 1 \text{ and } b \ll 1,$$

$$a_c \simeq \exp\left[\frac{h}{\ln 2 - \ln b}\right] \quad \text{for } \gamma = 1 \text{ and } b \ll 1,$$

where h is some positive constant.

The formula for δ_m implies that convergence of a_m to a_c is extremely rapid and the Feigenbaum constant δ is infinite. Let us call this behavior 'superconvergence'. This feature is attributed to the flatness of the summit. For example, in a nonlinear one-humped map with a flat summit, superconvergence will take place if the absolute value of the derivative is greater than some constant $\lambda > 1$ outside a region which contains the flat part of the map. This is because the onset point a_m for this map will be bounded from both the sides by the onset points a'_m and a''_m of an appropriate set of two trapezoid maps.

Our result implies $\delta(\infty) = \infty$. Since $\lim_{z\to\infty}\delta(z)$ is finite, this implies that $\lim_{z\to\infty}\delta(z)$ does not agree with $\delta(\infty)$, and superconvergence takes place only in the case $z = \infty$.

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Note added: After we finished this work, we were informed of the papers 6) $\sim$ 8). In the papers 6) and 7), the authors treat a symmetric trapezoid map and prove $\epsilon_m \propto a_c^{-2^m}$. In the paper 8), the authors treat an asymmetric trapezoid map and prove that a_m is quadratically convergent. This implies $\lim_{m\to\infty} \frac{\ln \delta_m}{\ln \delta_{m-1}} = 2$. Thus, their results are the same as ours. In this paper, we give the explicit expression for ϵ_m for the asymmetric case, and also, we give the b-dependence of a_c. Further, in 8), the authors mention the problem of convergence of the sequence $\Delta\epsilon_m/(\Delta\epsilon_{m-1})^2$. From our result, this sequence is $\epsilon_m/\epsilon_{m-1}^2 \simeq G'_\infty(a_c)\gamma^{(-1+(-1)^m)/2}/b$, and it does not converge for the asymmetric case.