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Superconvergence of Period-Doubling Cascade in Trapezoid Maps

—— *Its Rigorous Proof and Superconvergence of the Period-Doubling Cascade Starting from a Period p Solution* ——

Tatsuya Uezu

Graduate School of Human Culture, Nara Women's University
Nara 630-8506, Japan

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In the symmetric and the asymmetric trapezoid maps, as a slope a of the trapezoid is increased, a period-doubling cascade occurs. The symbolic sequence of periodic points in this cascade is the Metropolis-Stein-Stein sequence R^{*m}, and the convergence of the onset points a_m of the period 2^m solutions to the accumulation point a_c is exponentially fast. In a previous paper, we proved these results. In this paper, we give a detailed description of proofs regarding the results. Further, we study the period-doubling cascade starting from the period p (≥ 3) solution and show that the convergence of the onset points of the period $p \times 2^m$ solutions is also exponentially fast.

§1. Introduction

It is well known that in a class of one-dimensional maps, as a parameter is changed, there occurs a cascade of period-doubling bifurcations, and there exist several universal properties.[1] One of the universal quantities is the so-called Feigenbaum constant δ, which depends only on the exponent z characterizing the behavior of the map around the critical point.

About a decade ago, there was some controversy regarding the $z \to \infty$ limit of $\delta(z)$. As a typical example, the map $x_{n+1} = f(x_n) = 1 - a|x_n|^z$ has been extensively studied, and two different values were conjectured for this limit, — one finite and the other infinite.[2,3] This problem was solved by Eckmann and Epstein, and they proved that the limit is finite.[4]

In a previous paper,[5] instead of considering a map with finite z and taking $z \to \infty$, we investigated a map with $z = \infty$. As examples of such maps, we treated the symmetric and asymmetric trapezoid maps.

The symmetric trapezoid map $x_{n+1} = T_{(a,b)}(x_n)$ defined on the interval $[0,1]$ is given by

$$T_{(a,b)}(x) = \begin{cases} ax & \text{for } 0 \leq x \leq (1-b)/2, \\ a(1-b)/2 & \text{for } (1-b)/2 \leq x \leq (1+b)/2 \ , \\ a(1-x) & \text{for } (1+b)/2 \leq x \leq 1 \ , \end{cases} \tag{1·1}$$

where $0 < b < 1$ and $a > 0$ (see Fig. 1). As a is increased from 1 with fixed b, successive period-doubling bifurcations take place. We assign the symbols L, C and R to an orbit x_i according to the following rules: L for $x_i \in [0,(1-b)/2] \equiv I_L$, C for $x_i \in ((1-b)/2,(1+b)/2) \equiv I_C$, and R for $x_i \in [(1+b)/2,1] \equiv I_R$. We

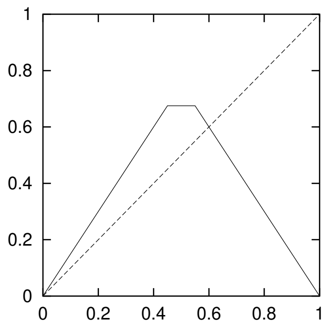

Fig. 1. The trapezoid map $T_{(a,b)}(x)$.

denote this correspondence by $H(x_i)$. Further, we define the conjugates of R and L as $\bar{R} = L$ and $\bar{L} = R$. In a previous paper [5)] we proved that as the slope a of the trapezoid map is increased, there occurs a cascade of period-doubling bifurcations, and the symbolic sequence is the Metropolis-Stein-Stein sequence R^{*m}, as usual. Also, defining a_m as the value of a at the appearance of a 2^m-cycle, we calculated $\delta_m \equiv \frac{a_{m+1}-a_m}{a_{m+2}-a_{m+1}}$ and proved that δ_m scales as

$$\delta_m \simeq a_c^{2^m},$$

where a_c is the accumulation point of the period-doubling cascade. We called this phenomenon the "superconvergent period-doubling cascade". Further, we investigated the asymmetric trapezoid map $x_{n+1} = A_{(a,b,\gamma)}(x_n)$, defined on the interval [0,1] as

$$A_{(a,b,\gamma)}(x) = \begin{cases} ax & \text{for } 0 \leq x \leq \alpha, \\ \alpha a & \text{for } \alpha \leq x \leq \beta \ , \\ \gamma a(1-x) & \text{for } \beta \leq x \leq 1 \ , \end{cases} \tag{1·2}$$

where γ is the ratio of the two slopes of the trapezoid, $\alpha = \frac{\gamma}{1+\gamma}(1-b)$ and $\beta = \alpha + b = \frac{b+\gamma}{1+\gamma}$. When $\gamma = 1$, $A_{(a,b,\gamma)}(x)$ is reduced to $T_{(a,b)}(x)$. In the asymmetric map, for the case that γ and b are fixed and a is increased, we proved results similar to those for the symmetric map. In particular, for the scaling of δ_m we obtained

$$\delta_m \simeq \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}.$$

Finally, we found approximate expressions for the accumulation point as a function of b, which is the length of the smaller side of the trapezoid in both symmetric and asymmetric cases.

In this paper, we give the detailed and complete description of the proofs given in the previous paper. The new results in this paper are the scaling relations for the period-doubling bifurcation starting with a period p (≥ 3) solution, which appears through a tangent bifurcation.

In the following section we treat the symmetric trapezoid map, and then in §3, we treat the asymmetric map. In §§4 and 5, we study the period-doubling bifurcation for a period p (≥ 3) solution in symmetric and asymmetric cases. We give a summary and discussion in the last section.

§2. The symmetric case

We consider $T_{\boldsymbol{a}}(x) \equiv T_{(a,b)}(x)$. For brevity we define $\boldsymbol{a} = (a, b)$. Let x_M be the maximum value of $T_{\boldsymbol{a}}(x)$; i.e., $x_M \equiv a\frac{1-b}{2}$. Let $x_{0,1}$ be the non-zero fixed point of $T_{\boldsymbol{a}}(x)$. Then the following lemma is easily proved.

Lemma 1

For $1 < a < a_1$, $T_{\boldsymbol{a}}(x)$ has the stable fixed point $x_{0,1} = x_M$, which satisfies $\frac{1-b}{2} < x_M < \frac{1+b}{2}$. Here a_1 is defined by the equation $x_M = \frac{1+b}{2}$, that is $a_1 = \frac{1+b}{1-b}$. For $a > a_1$, $T_{\boldsymbol{a}}(x)$ has the unstable fixed point $x_{0,1} = x^* \equiv \frac{a}{a+1}$ and $x^* > \frac{1+b}{2}$. At $a = a_1,\ x_M = x^*$ (see Fig. 2).

In the region $a > a_1$, by iterating $T_{\boldsymbol{a}}(x)$ twice and rescaling x such that the domain becomes $[0, 1]$, we obtain a trapezoid map with a different parameter value $\boldsymbol{a}^{(1)} = (a^{(1)}, b^{(1)})$ (see Figs. 3 and 4). In fact, we obtain the following lemma.

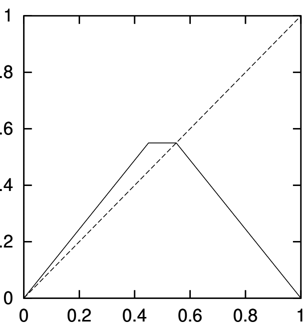

Fig. 2. The trapezoid map $T_{(a,b)}(x)$ for $a = a_1$ with $b = 0.1$.

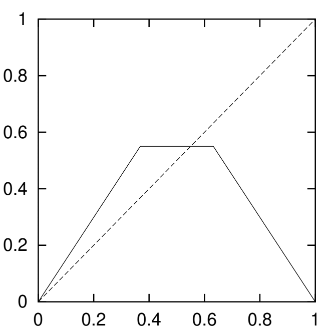

Fig. 3. The trapezoid map $T^2_{(a,b)}(x)$ for $a = a_1$ with $b = 0.1$.

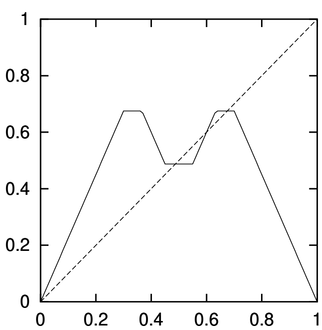

Fig. 4. The trapezoid map $T^2_{(a,b)}(x)$ for $a_1 < a < a_2$ with $b = 0.1$.

Lemma 2

When $T_{\boldsymbol{a}}(x)$ has a non-zero unstable fixed point $x^* = \frac{a}{a+1} (> \frac{1}{2})$ (that is, for $a > a_1$), by rescaling the coordinate x as $x^{(1)} = \frac{x^*-x}{2x^*-1}$ and restricting x to the interval $[1 - x^*, x^*]$, $T^2_{\boldsymbol{a}}(x)$ is transformed to $T_{\boldsymbol{a}^{(1)}}(x^{(1)})$, which is defined for $x^{(1)}$ on $[0, 1]$. Here $\boldsymbol{a}^{(1)} \equiv \boldsymbol{\varphi}(\boldsymbol{a}) \equiv (a^2, u(a)b)$ and $u(a) = \frac{a+1}{a-1}$.

The proof is straightforward. $\boldsymbol{\varphi}(\boldsymbol{a})$ is defined as long as $a \neq 1$. In this paper, we restrict ourselves to the case $a > 1$. Now, we define the m-th iteration of $\boldsymbol{\varphi}$, that is,

$$\boldsymbol{a}^{(m)} \equiv (a^{(m)}, b^{(m)}) \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}) = (a^{2^m}, u_m(a)b), \tag{2·1}$$

$$u_m(a) \equiv \prod_{l=0}^{m-1} u(a^{2^l}) = \prod_{l=0}^{m-1} \frac{a^{2^l}+1}{a^{2^l}-1}.$$

These are defined as long as $a \neq 1$. For $a > 1$, each $a^{(m)}$ is a strictly increasing, continuous function of a, and each $b^{(m)}$ is a strictly decreasing, continuous function of a. $\lim_{a\to\infty} b^{(m)} = b < 1$ for any $m > 0$. Let us prove the following lemma.

Lemma 3

For any positive integer m, there exists a unique value $a = a_m$ such that

$$b^{(m)}(a_m) = 1. \tag{2·2}$$

$\{a_m\}_{m=1}^{\infty}$ is an increasing sequence, i.e., $1 < a_1 < a_2 < \cdots$. Further, the relation $b < b^{(m)} < 1$ holds for $a > a_m$ with $m \geq 1$.

Proof

Let us consider the case $m = 1$. Then $b^{(1)}(a) = 1$ has the unique solution $a = \frac{1+b}{1-b} > 1$, and this is a_1 defined in Lemma 1. Next, let us assume that $b^{(m)}(a_m) = 1$ and $a_m > 1$ for $m \geq 1$. Since $b^{(m+1)}(a) = \frac{a^{(m)}+1}{a^{(m)}-1} b^{(m)}(a)$, $b^{(m+1)}(a_m) > 1$ follows. Thus, there exists the unique value of $a = a_{m+1} (> a_m)$ such that $b^{(m+1)}(a_{m+1}) = 1$. Therefore, by mathematical induction, $b^{(m)}(a) = 1$ has a unique solution a_m for any positive integer m and $\{a_m\}_{m=1}^{\infty}$ is an increasing sequence. The inequality $b < b^{(m)} < 1$ for $a_m < a$ immediately follows from the fact that the function $b^{(m)}(a)$ is strictly decreasing. Q.E.D.

For positive integer m, we define $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a > a_m$ from $T_{\boldsymbol{\varphi}^{m-1}(\boldsymbol{a})}(x^{(m-1)})$ successively by the same procedure as in Lemma 2. We define $x_M^{(m)} \equiv a^{(m)} \frac{1-b^{(m)}}{2}$, which is the maximum value of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. Further, we define $x^{(m)*} \equiv \frac{a^{(m)}}{a^{(m)}+1}$. $x_M^{(0)} = x_M$ and $x^{(0)*} = x^*$.

Lemma 4

For any non-negative integer m and for $a_m < a$ there exists a unique non-zero fixed point for $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ which is defined on $[0, 1]$. For $a_m < a < a_{m+1}$, $\frac{1-b^{(m)}}{2} < x_M^{(m)} < \frac{1+b^{(m)}}{2}$, and $x_M^{(m)}$ is the stable fixed point of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. At $a = a_{m+1}$, we have $x_M^{(m)} = x^{(m)*}$. For $a_{m+1} < a$, the non-zero fixed point of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ is $x^{(m)*}$, which is unstable and satisfies $x^{(m)*} > \frac{1+b^{(m)}}{2}$. Here, we define $x^{(0)} \equiv x$, $a_0 \equiv 1$, $a^{(0)} \equiv a$ and $b^{(0)} \equiv b$.

Proof

First of all, we note that for $m \geq 1$ $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ is really a trapezoid map, because from Lemma 3, $b < b^{(m)} < 1$ for $a_m < a$ and for $m \geq 1$. For the case $m = 0$, the statement follows from Lemma 1. Then, let us assume that the statement holds for $m \geq 0$. Since $x^{(m)*} = \frac{a^{(m)}}{a^{(m)}+1}$ is the unstable fixed point of $T_{\boldsymbol{a}^{(m)}}(x^{(m)})$ for $a_{m+1} < a$, from Lemma 2, taking $T^2_{\boldsymbol{a}^{(m)}}(x^{(m)})$ and rescaling $x^{(m)}$ as $x^{(m+1)} = \frac{x^{(m)*}-x^{(m)}}{2x^{(m)*}-1}$, we obtain $T_{\boldsymbol{a}^{(m+1)}}(x^{(m+1)})$, which is defined on $[0,1]$. The following equivalence relations are easily proved for $m \geq 0$:

$$b^{(m+1)}(a) = 1 \Longleftrightarrow x_M^{(m)} = \frac{1+b^{(m)}}{2} \Longleftrightarrow a^{(m)} = \frac{1+b^{(m)}}{1-b^{(m)}}. \tag{2·3}$$

Let us consider $x_M^{(m+1)} = a^{(m+1)}\frac{1-b^{(m+1)}}{2}$. From the relation (2·3), since $b^{(m+2)}(a_{m+2}) = 1$, it follows that $x_M^{(m+1)} = \frac{1+b^{(m+1)}}{2}$ at $a = a_{m+2}$. Since $x_M^{(m+1)}$ is a strictly increasing function w.r.t. a, we obtain $0 < \frac{1-b^{(m+1)}}{2} < x_M^{(m+1)} < \frac{1+b^{(m+1)}}{2}$ for $a_{m+1} < a < a_{m+2}$. This implies that $x_M^{(m+1)}$ is the unique non-zero fixed point for $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ and is stable. It is easily shown that for $a_{m+2} < a$, $x_M^{(m+1)}$ is no longer a fixed point, but $x^{(m+1)*} = \frac{a^{(m+1)}}{a^{(m+1)}+1}$ becomes an unstable fixed point and $x^{(m+1)*} > \frac{1+b^{(m+1)}}{2}$. At $a = a_{m+2}$, the equality $a_M^{(m+1)} = x^{(m+1)*}$ holds. Thus, by mathematical induction, the proof completes.

Lemma 5

The unique non-zero fixed point of $T_{\boldsymbol{\varphi}^{m}(\boldsymbol{a})}(x^{(m)})$ in Lemma 4 is a member of a periodic cycle with prime period 2^m of $T_{\boldsymbol{a}}(x)$.

Proof

The case $m = 0$ is trivial. Now, let us fix m (≥ 0) and assume that for $0 \leq k \leq m$, the unique non-zero fixed point of $T_{\boldsymbol{\varphi}^{k}(\boldsymbol{a})}(x^{(k)})$, which exists for $a_k < a$, is a member of the cycle with prime period 2^k. Thus, from Lemma 4 for $a_{m+1} < a$, the unique non-zero fixed point of $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ does not correspond to any periodic point with period less than 2^{m+1}, because it is stable for $a_{m+1} < a < a_{m+2}$. Thus, the non-zero fixed point of $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ is a member of the cycle with prime period 2^{m+1}. This completes the proof.

Here, let us investigate the upper bound of the sequence $\{a_m\}$. $x_M^{(m)}(a) = a^{(m)}\frac{1-b^{(m)}}{2}$ is defined for $a > 1$ and is a strictly increasing function w.r.t a. For any $m > 0$, $x_M^{(m)}(a_m) = 0$, and $\lim_{a\to\infty} x_M^{(m)}(a) = \infty$ follows. Thus, for $m > 0$ we define $a_M(m)$ as the unique solution of the equation

$$x_M^{(m)}(a) = 1. \tag{2·4}$$

For $m = 0$, we define $a_M(0) = \frac{2}{1-b}$. We abbreviate this as a_M. Then, we obtain the following lemma.

Lemma 6

$$\begin{aligned} &1 < a_1 < a_2 < \cdots < a_m < a_{m+1} < \cdots \\ &< a_M(m+1) < a_M(m) \cdots < a_M(2) < a_M(1) < a_M. \end{aligned} \tag{2·5}$$

Proof

For $m > 0$, since $x_M^{(m)}(a_m) = 0$, the inequality $a_m < a_M(m)$ follows. For $m \geq 0$, $x_M^{(m+1)}(a)$ can be rewritten as

$$x_M^{(m+1)}(a) = \{a^{(m)}\}^2 \frac{1}{a^{(m)} - 1} \left(x_M^{(m)}(a) - \frac{1 + b^{(m)}}{2} \right). \tag{2·6}$$

Then, $x_M^{(m+1)}(a_M(m)) = \frac{a^{(m)}}{a^{(m)}-1} > 1$. Therefore, $a_M(m+1) < a_M(m)$ for $m \geq 0$. Thus, we obtain $a_m < a_{m+1} < a_M(m+1) < a_M(m)$ for $m \geq 0$. Q.E.D.

From this,

$$a_c \equiv \lim_{m\to\infty} a_m \leq \lim_{m\to\infty} a_M(m) < a_M \tag{2·7}$$

follows. Therefore a_c is finite.

Using the above lemmas, we obtain the following theorem.

Theorem 1

$x = 0$ is a period 1 solution of $T_{\boldsymbol{a}}(x)$ for any $a > 0$. Except for this, for any non-negative integer m, for $a_m < a < a_{m+1}$ there is a unique cycle with prime period 2^k for $k = 0, 1, \cdots, m$ and no other periodic points exist. The periodic solution with prime period 2^m is stable for $a_m < a < a_{m+1}$ and unstable for $a_{m+1} < a$. As $a \to a_{m+1}+$, the 2^{m+1} cycle converges to the period 2^m cycle.

Proof

For the case $m = 0$, the statement is trivial. Let us assume that the statement holds for $0, 1, \cdots, m$. Then we only have to prove that for $a_{m+1} < a < a_{m+2}$, except for the period 2^k cycles for $k = 1 \sim m+1$, there is no other cycle.

Let us assume $a_{m+1} < a < a_{m+2}$. From Lemma 6, $a_l < a_c < a_M(n)$ follows for any $l \geq 0$ and $n \geq 0$. Thus, $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)}) \in [0, 1]$ for $x^{(1)} \in [0, 1]$. Let us consider any initial point x in (0, 1) which is not x^*. It is obvious from the graph of $T_{\boldsymbol{a}}^2(x)$ that the orbit $T_{\boldsymbol{a}}^i(x)$ finally enters the domain of the map $T_{\boldsymbol{\varphi}(\boldsymbol{a})}(x^{(1)})$ if i is appropriately chosen. Similarly, for $l = 1, 2, \cdots, m$, when $x^{(l)}$ is in $(0, 1)$ and is not $x^{(l)*}$, $T^j_{\boldsymbol{\varphi}^l(\boldsymbol{a})}(x^{(l)})$ enters the domain of the map $T_{\boldsymbol{\varphi}^{l+1}(\boldsymbol{a})}(x^{(l+1)})$ if j is appropriately chosen. Therefore we conclude that if x in (0, 1) is not a member of an unstable cycle of period $2^k (k = 0 \sim m)$ and is not a point which hits one of these cycles under a finite number of iterations, the orbit $T_{\boldsymbol{a}}^i(x)$ finally enters the domain of the map $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ if i is appropriately chosen. Thus, since any point other than $x^{(m+1)} = 0$ and 1 converges to the fixed point of $T_{\boldsymbol{\varphi}^{m+1}(\boldsymbol{a})}(x^{(m+1)})$ for $a_{m+1} < a < a_{m+2}$, the nonexistence of other periodic points for $a_{m+1} < a < a_{m+2}$ follows. The latter part of the theorem follows immediately from Lemma 4.

From the above arguments we conclude that as a is increased from 1, there occurs a period-doubling bifurcation cascade, and at $a = a_m$ the periodic solution with prime period 2^m appears for $m \geq 0$.

Now, let us investigate the symbolic sequence. The sequence of R and L for the period 2^m solution becomes the Metropolis-Stein-Stein (MSS) sequence. Let us prove this. For $m \geq 0$, we define the value of a at the onset of the 2^n-cycle for $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ as $a_n^{(m)}$, and the i-th orbit of the period 2^n cycle as $x_{n,i}^{(m)}$. We have $a_n^{(0)} = a_n$. (For $m = 0$, we often omit the superscript (0).) We set $x_{m,1}$ equal to the largest x value among the 2^m members of the periodic cycle. That is, $x_{m,1} = x_M = a(1-b)/2$ when the cycle is stable. Similarly, we define $x_{n,1}^{(m)} \equiv \frac{a^{(m)}(1-b^{(m)})}{2}$ when the period 2^n cycle is stable. Further, for $m \geq 0$ we divide the coordinate space $[0, 1]$ of $x^{(m)}$ into three intervals as $I_L^{(m)} \equiv [0, \frac{1-b^{(m)}}{2}]$, $I_C^{(m)} \equiv (\frac{1-b^{(m)}}{2}, \frac{1+b^{(m)}}{2})$ and $I_R^{(m)} \equiv [\frac{1+b^{(m)}}{2}, 1]$.

Lemma 7

For any positive integer m, when $a_m < a$, the following hold.

1. $I_C^{(m)} \ni x^{(m)}$ is equivalent to $I_C^{(m-1)} \ni x^{(m-1)}$.
2. If $I_L^{(m)} \ni x^{(m)}$ then $I_R^{(m-1)} \ni x^{(m-1)}$.
3. If $I_R^{(m)} \ni x^{(m)}$ then $I_L^{(m-1)} \ni x^{(m-1)}$.

Here $x^{(m)}$ and $x^{(m-1)}$ are related by the coordination transformation used to define $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ from $T_{\boldsymbol{\varphi}^{m-1}(\boldsymbol{a})}(x^{(m-1)})$.

Proof

For $a_m < a$, $x^{(m)} = 0$, $\frac{1-b^{(m)}}{2}$, $\frac{1+b^{(m)}}{2}$ and 1 correspond to $x^{(m-1)} = x^{(m-1)*}$, $\frac{1+b^{(m-1)}}{2}$, $\frac{1-b^{(m-1)}}{2}$ and $1 - x^{(m-1)*}$, respectively. Since this correspondence is linear, the statements hold.

Lemma 8

The fixed point $x_{0,1}^{(m)}$ of $T_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ corresponds to $x_{m,2^m}$.

Proof

Let $x_{m,i}$ be the orbit of the 2^m solution corresponding to $x_{0,1}^{(m)}$. From Lemma 4, for $a_m < a < a_{m+1}$, we have $x_{0,1}^{(m)} = x_M^{(m)}$ and $I_C^{(m)} \ni x_{0,1}^{(m)}$. Then, from Lemma 7, $I_C \ni x_{m,i}$ follows. Therefore, $x_{m,i}$ is mapped to x_M by $T_{\boldsymbol{a}}$, which is $x_{m,1}$. Thus $i = 2^m$.

Lemma 9

For any positive integer m, the symbols $H(x_{m,i})$ for the number of a 2^m cycle satisfy the following.

1. $H(x_{m,i})$ does not change for $a_m \leq a$ and for $i = 1, \cdots, 2^m - 1$.
2. For even m,

$$H(x_{m,2^m}) = \begin{cases} L & \text{for } a = a_m, \\ C & \text{for } a_m < a < a_{m+1}, \\ R & \text{for } a_{m+1} \leq a, \end{cases}$$

   and for odd m,

$$H(x_{m,2^m}) = \begin{cases} R & \text{for } a = a_m, \\ C & \text{for } a_m < a < a_{m+1}, \\ L & \text{for } a_{m+1} \leq a. \end{cases}$$

3. $\{H(x_{m,i})\}$ is the MSS sequence R^{*m} [*)] for $a_{m+1} \le a$.

Proof

Let us consider the case $m = 1$. At $a = a_1$, $x_{0,1} = \frac{1+b}{2} \in I_R$. Then, at $a = a_1$, we have $H(x_{0,1}) = H(x_{1,1}) = H(x_{1,2}) = R$. For $a_1 < a < a_2$, the relations $x_{0,1}^{(1)} = x_M^{(1)} = \frac{a^{(1)}(1-b^{(1)})}{2}$ hold, and for $a_2 \le a$, we have $x_{0,1}^{(1)} = x^{(1)*}$. From Lemma 8, $x_{0,1}^{(1)}$ corresponds to $x_{1,2}$. Thus, from Lemma 7, $H(x_{1,2})$ is C for $a_1 < a < a_2$ and is L for $a_2 \le a$. On the other hand, for $a_1 \le a$, since $x_{1,1} \ge x_{0,1} = \frac{a}{a+1}$, it follows that $H(x_{1,1}) = R$. Therefore, $H(x_{1,1})H(x_{1,2}) = RL$ for $a_2 \le a$. Thus, for $m = 1$ the lemma holds.

Next, we assume that the lemma holds for the case $m \ge 1$. At $a = a_{m+1}$, $x_{m+1,i}$ and $x_{m+1,i+2^m}$ emerge from $x_{m,i}$ ($i = 1 \sim 2^m$). Then, at $a = a_{m+1}$, $H(x_{m,i}) = H(x_{m+1,i}) = H(x_{m+1,i+2^m})$ for $i = 1, \cdots, 2^m$. These are L or R. Let us assume that at some value of $a > a_{m+1}$, $H(x_{m+1,i})$ becomes C. Then, $x_{m+1,i}$ is mapped to x_M by $T_{\boldsymbol{a}}$. Thus, in this case, $a < a_{m+2}$ and $x_{m+1,i+1} = x_M = x_{m+1,1}$ follow. Hence, i should be 2^{m+1}. Thus, the symbol of $x_{m+1,i}$ ($i = 1 \sim 2^{m+1} - 1$) does not change for $a_{m+1} \le a$. From Lemma 8, $x_{m+1,2^{m+1}}$ corresponds to $x_{0,1}^{(m+1)} = x_M^{(m+1)}$ for $a_{m+1} \le a \le a_{m+2}$. Thus, $H(x_{m+1,2^{m+1}}) = C$ for $a_{m+1} < a < a_{m+2}$. For $a_{m+2} \le a$, we have $x_{0,1}^{(m+1)} = x^{(m+1)*} \in I_R^{(m+1)}$. Thus, from Lemma 7, for $a_{m+2} \le a$,

$$H(x_{m+1,2^{m+1}}) = \begin{cases} L & \text{for odd } m+1, \\ R & \text{for even } m+1. \end{cases}$$

Therefore, the statements 1 and 2 hold for $m + 1$. At $a = a_{m+1}, H(x_{m+1,2^{m+1}}) = H(x_{m,2^m})$. Then, from the assumption,

$$H(x_{m+1,2^{m+1}}) = \begin{cases} R & \text{for odd } m+1, \\ L & \text{for even } m+1. \end{cases}$$

Let $H(x_{m,1})H(x_{m,2}) \cdots H(x_{m,2^m})$ be the MSS sequence for $a \ge a_{m+1}$. From the above argument, for $a \ge a_{m+2}$, the sequence for the 2^{m+1} cycle is

$$\begin{aligned} &H(x_{m,1})H(x_{m,2}) \cdots H(x_{m,2^m-1})H(x_{m,2^m}) \\ &\times H(x_{m,1})H(x_{m,2}) \cdots H(x_{m,2^m-1})\overline{H(x_{m,2^m})}. \end{aligned}$$

This is the MSS sequence. Thus, the statement 3 is proved. This completes the proof.

Now, we derive the equations that a_m and a_c should satisfy and obtain the asymptotic expression for δ_m. First, we assign 0 or 1 to any orbit x_i with the symbol L or R, respectively. We denote this correspondence as $s_i = s(x_i)$. Further, we define the function $f_s(x)$ for $x \in I_L \cup I_R$ as

$$f_s(x) = sa + (1 - 2s)ax, \tag{2·8}$$

[*)] Strictly speaking, the last symbol of the sequence in our definition is R or L and is different from that in the MSS sequence, C.

where $s = s(x)$. Starting from the maximum value of $T_{(a,b)}(x)$, $x_1 = x_M = a(1-b)/2$, if $x_1, x_2, \cdots, x_{n-1} \in I_L \cup I_R$, x_n is expressed as

$$x_n = f_{s_{n-1}}(x_{n-1}) = f_{s_{n-1}} \circ f_{s_{n-2}} \circ \cdots \circ f_{s_1}(x_1) = \sum_{l=1}^{n} \xi_l a^l, \tag{2·9}$$

$$\xi_l = s_{n-l} \prod_{j=n-l+1}^{n-1} (1-2s_j) \ \text{ for } \ 2 \le l \le n-1,$$

$$\xi_1 = s_{n-1}, \quad \xi_n = \frac{1-b}{2} \prod_{j=1}^{n-1} (1-2s_j).$$

For $m \ge 1$, let us consider the period 2^m solution $\{x_{m,i}\}$ for $a_m \le a \le a_{m+1}$:

$$x_{m,2^m} = f_{s_{2^m-1}}(x_{m,2^m-1}) \equiv \sum_{l=0}^{2^m-1} c_l^{(m)} a^{2^m-l} \equiv F_m(a), \tag{2·10}$$

$$c_l^{(m)} = s_l \prod_{j=l+1}^{2^m-1} (1-2s_j) = \xi_{2^m-l} \ \text{ for } \ 1 \le l \le 2^m-2,$$

$$c_0^{(m)} = \frac{1-b}{2} \prod_{j=1}^{2^m-1} (1-2s_j) = s_0 \prod_{j=1}^{2^m-1} (1-2s_j),$$

$$c_{2^m-1}^{(m)} = s_{2^m-1}.$$

Here $s_0 \equiv \frac{1-b}{2}$. Note that $F_m(a)$ is determined by the sequence $(s_1, \cdots, s_{2^m-1})$ and is independent of s_{2^m}. From Lemma 9, the symbols $H(x_{m,i})$ $(i = 1 \sim 2^m - 1)$ do not change for $a_m \le a$. Further, for any positive integer j these symbols are equal to the first $2^m - 1$ symbols of the period 2^{m+j} solution when $a_{m+j} \le a$. Therefore, $F_m(a)$ expresses $x_{2^{m+j},2^m}$ in the region $[a_{m+j}, a_{m+j+1}]$. From statement 2 in Lemma 9, the following relations hold:

$$x_{m,2^m}(a_m) = \frac{1+(-1)^{m-1}b}{2}, \tag{2·11}$$

$$x_{m,2^m}(a_{m+1}) = \frac{1+(-1)^{m}b}{2}. \tag{2·12}$$

Thus, we obtain the following conditions for a_m:

$$F_m(a_m) = \frac{1+(-1)^{m-1}b}{2}, \tag{2·13}$$

and

$$F_{m-1}(a_m) = \frac{1+(-1)^{m-1}b}{2}. \tag{2·14}$$

Since the symbolic sequence is the Metropolis-Stein-Stein sequence R^{*m}, the equalities

$$\prod_{j=1}^{2^m-1} (1-2s_j) = (-1)^m, \tag{2·15}$$

$$s_{2^m} = \frac{1+(-1)^m}{2} \tag{2·16}$$

follow. From these relations, for $l \geq 0$ we obtain

$$r_l \equiv \left\{ \prod_{j=1}^{2^m-1} (1-2s_j) \right\}^{-1} c_l^{(m)} = (-1)^m c_l^{(m)} = s_l \prod_{j=1}^{l} (1-2s_j).$$

That is, r_l is independent of m as long as it is defined. Thus, we obtain

$$r_l = s_l \prod_{j=1}^{l} (1-2s_j) \text{ for any } l \geq 0, \tag{2·17}$$

$$r_0 = (1-b)/2, \;\; r_{2^m} = -(1+(-1)^m)/2 \text{ for any } m \geq 0, \tag{2·18}$$

and for $l > 0$, the successive values (r_l, r_{l+1}) take the following six sets of values: $(0, \pm 1), (\pm 1, 0), (1, -1), (-1, 1)$.[*)] Let us define

$$G_m(a) \equiv (-1)^m a^{-2^m} F_m(a) = \sum_{l=0}^{2^m-1} r_l a^{-l}, \tag{2·19}$$

$$G_\infty(z) \equiv \lim_{m\to\infty} G_m(z). \tag{2·20}$$

Then, $G_\infty(z)$ is an analytic function for $|z| > 1$. Equation (2·13) becomes

$$G_m(a_m) = a_m^{-2^m} \frac{(-1)^m - b}{2}, \tag{2·21}$$

and the accumulation point a_c satisfies the equation

$$G_\infty(a_c) = 0. \tag{2·22}$$

Let us estimate a_m. Writing $a_m = a_c - \epsilon_m$ and using the mean value theorem, the right-hand side of Eq. (2·21) can be rewritten as

$$a_m^{-2^m} \frac{(-1)^m - b}{2} = (a_c^{-2^m} + 2^m \hat{a}_m^{-2^m-1} \epsilon_m) \frac{(-1)^m - b}{2},$$

where $a_m < \hat{a}_m < a_c$. On the other hand, the left-hand side of Eq. (2·21) can be expressed as

$$G_m(a_m) = G_m(a_c) - G'_m(\bar{a}_m)\epsilon_m,$$

where $a_m < \bar{a}_m < a_c$. Thus, we obtain from Eq. (2·22),

$$\epsilon_m = \left\{ G_m(a_c) + \frac{b-(-1)^m}{2} a_c^{-2^m} \right\} (1+\bar{h}_m)/G'_\infty(a_c), \tag{2·23}$$

where $\bar{h}_m = G'_\infty(a_c)/(G'_m(\bar{a}_m) + 2^m(\hat{a}_m)^{-2^m-1}\frac{(-1)^m-b}{2}) - 1$ and $\lim_{m\to\infty} \bar{h}_m = 0$. Here, we assume $G'_\infty(a_c) \neq 0$, which is proved below. Equation (2·22) can be rewritten as

$$G_\infty(a_c) = G_m(a_c) + \sum_{l=0}^{\infty} r_{2^m+l} a_c^{-l-2^m} = 0. \tag{2·24}$$

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*) In Ref. 5), the two cases $(0, -1)$ and $(-1, 0)$ are missing.

Using the relation

$$r_{2^m+l} = -r_l \quad \text{for } 1 \le l \le 2^m - 1, \tag{2·25}$$

Eq. (2·24) is further changed to the following:

$$G_\infty(a_c) = G_m(a_c)(1 - a_c^{-2^m}) + a_c^{-2^m}\left(r_0 + r_{2^m} + a_c^{-2^m}\sum_{l=0}^{\infty} r_{2^{m+1}+l}\, a_c^{-l}\right) = 0. \tag{2·26}$$

Thus, we get

$$G_m(a_c) = \frac{b + (-1)^m}{2} a_c^{-2^m} + (a_c^{-2^m})^2 q_m, \tag{2·27}$$

where $q_m = \frac{1}{1-a_c^{-2^m}}\left(\frac{b+(-1)^m}{2} - \sum_{l=0}^{\infty} r_{2^{m+1}+l}\, a_c^{-l}\right)$ and $|q_m| < \frac{a_c(2a_c-1)}{(a_c-1)^2}$. Substituting Eq. (2·27) into Eq. (2·23) we obtain

$$\epsilon_m = \frac{b a_c^{-2^m}}{G'_\infty(a_c)}(1 + h_m), \tag{2·28}$$

where $h_m = \bar{h}_m(1 + \frac{q_m}{b} a_c^{-2^m}) + \frac{q_m}{b} a_c^{-2^m}$ and $\lim_{m\to\infty} h_m = 0$. Thus,

$$\begin{aligned} \delta_m &= \frac{\epsilon_m - \epsilon_{m+1}}{\epsilon_{m+1} - \epsilon_{m+2}} = a_c^{2^m}(1 + l_m), \\ l_m &= [h_m - h_{m+1} - a_c^{-2^m}\{1 + h_{m+1} - a_c^{-2^m}(1 + h_{m+2})\}] \\ &\quad /[1 + h_{m+1} - a_c^{-2^{m+1}}(1 + h_{m+2})], \end{aligned} \tag{2·29}$$

and $\lim_{m\to\infty} l_m = 0$.

Next, we give an alternative relation for the onset point a_m. As is shown in Lemma 3, at $a = a_m$ we have the following (which is Eq. (2·2)) for $m > 0$:

$$b^{(m)}(a_m) = u_m(a_m) b = 1.$$

Defining $v(a) \equiv \frac{a-1}{a+1} = 1/u(a)$ and $v_m(a) \equiv 1/u_m(a) = \prod_{l=0}^{m-1} v(a^{2^l})$, we obtain for $m > 0$,

$$v_m(a_m) = b. \tag{2·30}$$

For $m \ge 1$, $v_m(a)$ is rewritten as

$$\begin{aligned} v_m(a) &= \frac{1 - a^{-1}}{1 + a^{-2^{m-1}}} \prod_{l=0}^{m-2}(1 - a^{-2^l}), \\ &\prod_{l=0}^{-1}(1 - a^{-2^l}) \equiv 1, \quad v_1(a) = v(a). \end{aligned} \tag{2·31}$$

As is easily shown, for $m \ge 2$, $G_{m-1}(a)$ can be expressed in terms of $v_m(a)$ as

$$G_{m-1}(a) = \frac{1}{2}\{(1 + a^{-2^{m-1}})v_m(a) - b - (-1)^m a^{-2^{m-1}}\}. \tag{2·32}$$

(See Appendix A.) Then the equation obtained from Eq. (2·14),

$$G_{m-1}(a_m) = (-1)^{m-1} a_m^{-2^{m-1}} \frac{1 + (-1)^{m-1} b}{2}, \tag{2·33}$$

is equivalent to Eq. (2·30). From the relation (2·32),

$$G_\infty(a) = \frac{1}{2}(v_\infty(a) - b) \tag{2·34}$$

follows for $a > 1$. Then $G_\infty(a_c) = 0$ is equivalent to $v_\infty(a_c) = b$, which follows from Eq. (2·30) immediately. Since

$$v'_\infty(a) = 2v_\infty(a) \sum_{l=0}^{\infty} \frac{2^l a^{2^l - 1}}{a^{2^{l+1}} - 1}, \tag{2·35}$$

we obtain

$$v'_\infty(a) > 0, \quad G'_\infty(a) > 0 \ \text{ for } a > 1. \tag{2·36}$$

Setting $a = a_c$ in Eq. (2·35), we get

$$v'_\infty(a_c) = 2b\tau(a_c), \tag{2·37}$$

$$\tau(a_c) \equiv \sum_{l=0}^{\infty} \frac{2^l a_c^{2^l - 1}}{a_c^{2^{l+1}} - 1} > 0. \tag{2·38}$$

Thus,

$$G'_\infty(a_c) = v'_\infty(a_c)/2 = b\tau(a_c) > 0. \tag{2·39}$$

Therefore, substituting Eq. (2·39) into Eq. (2·28), we obtain

$$\epsilon_m = \frac{a_c^{-2^m}}{\tau(a_c)}(1 + h_m). \tag{2·40}$$

§3. The asymmetric case

As in the symmetric case, we can discuss the period-doubling cascade when a is increased with fixed b and γ in the asymmetric case. For brevity we define $\boldsymbol{a} = (a, b, \gamma)$. Then, α and β are expressed as

$$\alpha = \frac{\gamma(1-b)}{1+\gamma}, \quad \beta = \alpha + b = \frac{b+\gamma}{1+\gamma}.$$

α, β and γ are related by $\alpha = \gamma(1-\beta)$. As long as $0 < b < 1$, it holds that $0 < \alpha < 1$ and $0 < \beta < 1$. Let us define $a_L \equiv a$ and $a_R \equiv \gamma a$. Let x_M be the maximum value of $A_{\boldsymbol{a}}(x)$, i.e., $x_M \equiv a\alpha$. We define $x^* \equiv \frac{\gamma a}{1+\gamma a}$. Then, let $x_{0,1}$ be the non-zero fixed point of $A_{\boldsymbol{a}}(x)$. The following lemma is easily proved.

Lemma 1'

1. The case $\beta \geq 1/2$. For $1 < a < a_1$, $A_{\boldsymbol{a}}(x)$ has the stable fixed point $x_{0,1} = x_M$, which satisfies $\alpha < x_M < \beta$. The value a_1 is defined by the equation $x_M = \beta$, that is $a_1 = \frac{\beta}{\alpha}$.
2. The case $\beta < 1/2$. For $1 < a < \frac{\beta}{\alpha}$, $A_{\boldsymbol{a}}(x)$ has the stable fixed point $x_{0,1} = x_M$, which satisfies $\alpha < x_M < \beta$. For $a \geq \frac{\beta}{\alpha}$, the non-zero fixed point becomes x^* and continues to be stable until $\frac{1-\beta}{\alpha}$. Thus, in this case we define $a_1 = \frac{1-\beta}{\alpha} = \frac{1}{\gamma}$.

In both cases, for $a > a_1 > 1$, $A_{\boldsymbol{a}}(x)$ has the unstable fixed point $x_{0,1} = x^* > \beta$.

In the region $a > a_1$, by iterating $A_{\boldsymbol{a}}(x)$ twice and rescaling x, we obtain a trapezoid map with the different parameter, $\boldsymbol{a}^{(1)} = (a^{(1)}, b^{(1)}, \gamma^{(1)})$. We obtain the following lemma.

Lemma 2'

When $A_{\boldsymbol{a}}(x)$ has a non-zero unstable fixed point $x^* = \frac{\gamma a}{\gamma a+1} (> \beta)$, by rescaling the coordinate x as $x^{(1)} = \frac{x^*-x}{\varDelta x}$, $A^2_{\boldsymbol{a}}(x)$ is transformed into $A_{\boldsymbol{a}^{(1)}}(x^{(1)})$, which is defined for $x^{(1)}$ on $[0, 1]$, where several parameters and variables are defined as

$$\begin{aligned}
&\varDelta x = \gamma(a-1)/(1+\gamma a), \quad \boldsymbol{a}^{(1)} = \boldsymbol{\varphi}(\boldsymbol{a}),\\
&\boldsymbol{\varphi} \,:\, (a, b, \gamma) \to (a^{(1)}, b^{(1)}, \gamma^{(1)}),\\
&\quad a^{(1)} = a_R^2 = (\gamma a)^2, \;\; a_R^{(1)} = a_L a_R = \gamma a^2,\\
&b^{(1)} = u(a,\gamma) b, \;\; u(a,\gamma) \equiv \frac{\gamma a + 1}{\gamma(a-1)}, \;\; \gamma^{(1)} = 1/\gamma, \;\; \alpha^{(1)} = \frac{\alpha a - \beta}{\gamma(a-1)}.
\end{aligned} \tag{3·1}$$

The proof is straightforward. Regardless of the value of β, $\beta^{(1)} > 1/2$ follows. Now, we define the m-th iteration of $\boldsymbol{\varphi}$ for $m \geq 0$. That is,

$$\begin{aligned}
&\boldsymbol{a}^{(m)} \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}) \equiv (a^{(m)}, b^{(m)}, \gamma^{(m)}),\\
&a^{(m)} = \gamma^{2(2^m-(-1)^m)/3} a^{2^m}, \quad a_R^{(m)} = \gamma^{(2^{m+1}+(-1)^m)/3} a^{2^m},\\
&b^{(m)} = u_m(a,\gamma) b, \;\; u_m(a,\gamma) \equiv \prod_{l=0}^{m-1} u(a^{(l)}, \gamma^{(l)}), \quad \gamma^{(m)} = \gamma^{(-1)^m},\\
&\alpha^{(m)} = \frac{\gamma^{(m)}(1-b^{(m)})}{1+\gamma^{(m)}}, \;\; \beta^{(m)} = \frac{b^{(m)} + \gamma^{(m)}}{1+\gamma^{(m)}},
\end{aligned} \tag{3·2}$$

where $u_0(a,\gamma) \equiv 1$, $a^{(0)} \equiv a$, $b^{(0)} \equiv b$ and $\gamma^{(0)} \equiv \gamma$.

Since $a^{(m)} = (\gamma a)^{2^m} \gamma^{-2[2^{m-1}+(-1)^m]/3}$ and $\gamma a_1 \geq 1$, it is easily shown that $a^{(m)} > 1$ holds for $a > a_1$ and $m \geq 0$. Thus, for $m \geq 1$, $u_m(a,\gamma)$ and $b^{(m)}(a,\gamma)$ are defined for $a > a_1$. In the following, we assume $a > 1$. For $m \geq 1$, $b^{(m)}$ is a strictly decreasing, continuous function w.r.t. a for $a > a_1$, and $\lim_{a\to\infty} b^{(m)}(a,\gamma) = b$.

Let us prove the following lemma.

Lemma 3'

For integer $m \geq 2$, there exists a unique value of $a = a_m$ greater than 1 such that

$$a^{(m-1)} = \frac{\beta^{(m-1)}}{\alpha^{(m-1)}}, \tag{3·3}$$

which is equivalent to

$$b^{(m)}(a_m) = 1. \tag{3·4}$$

$\{a_m\}_{m=1}^{\infty}$ is an increasing sequence $(1 < a_1 < a_2 < \cdots)$, and for $a_m < a$, $b < b^{(m)} < 1$ for $m \geq 1$.

Proof

The following equivalence relations are easily proved if these quantities are defined.

$$b^{(m)}(a) = 1 \Longleftrightarrow x_M^{(m-1)} = \beta^{(m-1)} \Longleftrightarrow a^{(m-1)} = \frac{\beta^{(m-1)}}{\alpha^{(m-1)}} > 1. \tag{3·5}$$

Let us consider the $m = 2$ case. For $a > 1$, $b^{(1)}$ is defined and a_2 is the solution of the equation $a^{(1)} = \frac{\beta^{(1)}}{\alpha^{(1)}}$. There is a unique solution of this equation that is greater than 1, $a_2 = \frac{1}{2\alpha}(1 + \sqrt{1 - \frac{4\alpha^2}{\gamma}})$. The relation $a_2 > a_1$ is easily proved. From the relation (3·4), we obtain $b^{(2)} = 1$ at $a = a_2$. Next, let us assume that $b^{(m)}(a_m) = 1$ for $m(\geq 2)$ and $a = a_m > 1$. Then for $a > a_m$, it follows that $b < b^{(m)} < 1$ and the function $\beta^{(m)}/\alpha^{(m)} = 1 + \frac{1+\gamma^{(m)}}{\gamma^{(m)}}\frac{b^{(m)}}{1-b^{(m)}}$ is continuous and strictly decreasing w.r.t. a. On the other hand, $a^{(m)}$ is a strictly increasing, continuous function w.r.t. a. Further, we obtain following limits:

$$\lim_{a \to a_m+0} \frac{\beta^{(m)}}{\alpha^{(m)}} = \infty, \quad \lim_{a \to \infty} \frac{\beta^{(m)}}{\alpha^{(m)}} = 1 + \frac{1+\gamma^{(m)}}{\gamma^{(m)}}\frac{b}{1-b} = \text{ (finite)},$$
$$\lim_{a \to a_m+0} a^{(m)} = \text{ (finite)}, \quad \lim_{a \to \infty} a^{(m)} = \infty.$$

Thus, there exists a unique value of $a_{m+1} > a_m$ such that $a^{(m)} = \beta^{(m)}/\alpha^{(m)} > 1$. Thus, $b^{(m+1)}(a_{m+1}) = 1$. Therefore, the first half of the lemma is proved. The second half immediately follows from the decreasing property of $b^{(m)}$ and the facts $b^{(1)}(a = \beta/\alpha) = 1, a_1 \geq \beta/\alpha$ and $1 < a_1 < a_2$. Q.E.D.

For positive integer m, we define $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ for $a > a_m$ from $A_{\boldsymbol{\varphi}^{m-1}(\boldsymbol{a})}(x^{(m-1)})$ successively by the same procedure as in the Lemma 2'. We define $x_M^{(m)} \equiv a^{(m)}\alpha^{(m)}$, which is the maximum value of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. Further, we define $x^{(m)*} \equiv \frac{\gamma^{(m)}a^{(m)}}{\gamma^{(m)}a^{(m)}+1}$. We have $x_M^{(0)} = x_M$ and $x^{(0)*} = x^*$.

Lemma 4'

For non-negative integer m and for $a_m < a$, there exists a unique non-zero fixed point of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$. For $a_m < a < a_{m+1}$, this fixed point is stable. For $m \geq 1$, the stable fixed point is $x_M^{(m)}$, and $\alpha^{(m)} < x_M^{(m)} < \beta^{(m)}$,[*)] and at $a = a_{m+1}$, the equalities

*) For $m = 0$, see Lemma 1'.

$x_M^{(m)} = x^{(m)*} = \beta^{(m)}$ hold. For $m \geq 0$, when $a_{m+1} < a$, the non-zero fixed point is $x^{(m)*}$, which is unstable and satisfies $x^{(m)*} > \beta^{(m)}$. Here, $a_0 \equiv 1,\ \alpha^{(0)} \equiv \alpha,\ \beta^{(0)} \equiv \beta$.

Proof

The statement regarding the case $m = 0$ follows from Lemma 1'. Then, let us consider the case $m \geq 1$. First, we note that $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ is really a trapezoid map for $m \geq 1$, because from Lemma 3', $b < b^{(m)} < 1$ for $a_m < a$ and $m \geq 1$. Further, from the relation (3·3) it follows that $x_M^{(m)} = \beta^{(m)} = x^{(m)*}$ at $a = a_{m+1}$. Since $x_M^{(m)}$ and $x^{(m)*}$ are strictly increasing, continuous functions w.r.t. a, and $\beta^{(m)}$ is a strictly decreasing, continuous function w.r.t. a, we obtain $\alpha^{(m)} < x_M^{(m)} < \beta^{(m)}$ for $a_m < a < a_{m+1}$, and $x^{(m)*} > \beta^{(m)}$ for $a > a_{m+1}$. The former relation implies that $x_M^{(m)}$ is the unique non-zero fixed point of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ and is stable. It is easily shown that for $a_{m+1} < a$, $x_M^{(m)}$ is no longer a fixed point, but $x^{(m)*}$ becomes an unstable fixed point. Q.E.D.

Lemma 5'

The unique non-zero fixed point of $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ in Lemma 4' is a member of a periodic cycle with prime period 2^m.

The proof is the same as that in the symmetric case.

Here, let us investigate the upper bound of the sequence $\{a_m\}$. For $m \geq 1$, $x_M^{(m)}(a)$ is defined at least for $a_1 < a$. These are strictly increasing functions w.r.t a. For any $m \geq 1$, $x_M^{(m)}(a_m) = 0$ and $\lim_{a\to\infty} x_M^{(m)}(a) = \infty$ hold. Therefore, for $m \geq 1$, we define $a_M(m)$ as the unique solution of the equation

$$x_M^{(m)}(a) = 1. \tag{3·6}$$

For $m = 0$, we define $a_M(0) = \frac{1}{\alpha}$ and $a_M \equiv a_M(0)$. Then, we obtain the following lemma.

Lemma 6'

$$\begin{aligned} 1 < a_1 < a_2 < \cdots < a_m < \cdots \\ < a_M(m) < \cdots < a_M(2) < a_M(1) < a_M. \end{aligned} \tag{3·7}$$

Proof

For $m = 1$, $x_M^{(1)}(a_1) = 0$ when $\beta \geq 1/2$, and $x_M^{(1)}(a_1) = \frac{1-2\beta}{1-2\beta+b} < 1$ when $\beta < 1/2$. Thus, $a_1 < a_M(1)$. For $m \geq 2$, since $x_M^{(m)}(a_m) = 0$, the inequality $a_m < a_M(m)$ follows. For $m \geq 0$, $x_M^{(m+1)}$ can be rewritten as

$$x_M^{(m+1)} = \gamma^{(m)}\{a^{(m)}\}^2 \frac{1}{a^{(m)}-1}(x_M^{(m)} - \beta^{(m)}). \tag{3·8}$$

Then, we obtain $x_M^{(m+1)}(a_M(m)) = \frac{a^{(m)}}{a^{(m)}-1} > 1$ for $m \geq 0$. Therefore, $a_M(m+1) < a_M(m)$ for $m \geq 0$. Thus, we obtain $a_m < a_{m+1} < a_M(m+1) < a_M(m)$. Q.E.D.

From this,

$$a_c \equiv \lim_{m\to\infty} a_m \le \lim_{m\to\infty} a_M(m) < a_M \qquad (3\cdot9)$$

follows. Therefore a_c is finite.

Using the above lemmas, we obtain the following theorem.

Theorem 2

$x = 0$ is period 1 solution of $A_{\boldsymbol{a}}(x)$ for any $a > 0$. Except for this, for any non-negative integer m, for $a_m < a < a_{m+1}$ there is a unique cycle with prime period 2^k for $k = 0, 1, \cdots, m$ and no other periodic points exist. The periodic solution with prime period 2^m is stable for $a_m < a < a_{m+1}$ and unstable for $a_{m+1} < a$. As $a \to a_{m+1}+$, the 2^{m+1} cycle converges to the period 2^m cycle.

The uniqueness of the cycle with prime period 2^k, $k = 0, 1, 2, \cdots, m$, for $a_m < a < a_{m+1}$ is proved by the same argument as that in the symmetric case. Thus, as in the symmetric case, we conclude that as a is increased, there occurs a period-doubling bifurcation cascade, and at $a = a_m$ the periodic solution with prime period 2^m appears.

Now, let us investigate the symbolic sequence. The sequence of R and L for the period 2^m solution becomes the Metropolis-Stein-Stein (MSS) sequence. Let us prove this.

As in the case of the symmetric map, for $m \ge 0$ we define the onset of the 2^n-cycle for $A_{\boldsymbol{\varphi}^m(\boldsymbol{a})}(x^{(m)})$ as $a_n^{(m)}$ and the i-th orbit of the period 2^n cycle as $x_{n,i}^{(m)}$. Here, $a_n^{(0)}$ is equal to the previously defined a_n. (For $m = 0$, we often omit the superscript (0).) We set $x_{m,1}$ equal to the largest x value among the 2^m members of the periodic cycle. Then, $x_{m,1} = x_M = a\alpha$ when the cycle is stable. Further, we divide the coordinate space $[0, 1]$ of $x^{(m)}$ into three intervals as $I_L^{(m)} \equiv [0, \alpha^{(m)}]$, $I_C^{(m)} \equiv (\alpha^{(m)}, \beta^{(m)})$ and $I_R^{(m)} \equiv [\beta^{(m)}, 1]$.

Then, we obtain Lemmas 7', 8' and 9' for $A_{\boldsymbol{a}}(x)$ corresponding to Lemmas 7, 8 and 9 for $T_{\boldsymbol{a}}(x)$, respectively. We omit the statements and proofs of these lemmas, since they are almost the same as those for $T_{\boldsymbol{a}}(x)$. Thus, we obtain the MSS sequence for the period 2^m solution.

Now, as in the symmetric case, we derive the equations for a_m and a_c, and we obtain an asymptotic expression for δ_m. Defining $s(x)$ as above, $f_s(x)$ for $x \in I_L \cup I_R$ is defined as

$$f_s(x) = a(\eta + \nu x), \quad \eta = \gamma s(x), \quad \nu = 1 - (1+\gamma)s(x). \qquad (3\cdot10)$$

In the following, s_i, η_i and ν_i are values evaluated at $x = x_{m,i}$. Starting from $x_{m,1} = \alpha a$: we obtain $x_{m,2^m}$ for $a_m \le a \le a_{m+1}$:

$$x_{m,2^m} = f_{s_{2^m-1}}(x_{m,2^m-1}) \equiv \sum_{l=0}^{2^m-1} c_l^{(m)} a^{2^m-l} \equiv F_m(a), \qquad (3\cdot11)$$

$$c_l^{(m)} = \eta_l \prod_{j=l+1}^{2^m-1} \nu_j \quad \text{for} \quad 1 \le l \le 2^m - 2,$$
$$c_0^{(m)} = \alpha \prod_{j=1}^{2^m-1} \nu_j = \eta_0 \prod_{j=1}^{2^m-1} \nu_j,$$
$$c_{2^m-1}^{(m)} = \eta_{2^m-1},$$

where $\eta_0 \equiv \alpha$. Thus,

$$F_m(a_m) = \frac{\alpha + \beta + (-1)^{m-1} b}{2}, \tag{3·12}$$
$$F_{m-1}(a_m) = \frac{\alpha + \beta + (-1)^{m-1} b}{2}. \tag{3·13}$$

We define ζ_l and $\hat{\zeta}_l$ as the number of appearances of 1 in $s_1, s_2, \cdots, s_{2^l-1}$ and in $s_1, s_2, \cdots, s_{2^l}$, respectively.[5] Then, their expressions are

$$\hat{\zeta}_l = (2^{l+1} + (-1)^l)/3 \quad \text{for } l \ge 0, \tag{3·14}$$
$$\zeta_l = \hat{\zeta}_l - \frac{1 + (-1)^l}{2} \quad \text{for } l \ge 0,$$
$$\zeta_l = \zeta_{l-1} + \hat{\zeta}_{l-1} \quad \text{for } l \ge 1,$$

where $\zeta_0 \equiv 0$. The following relations hold:

$$\sum_{l=0}^{m-2} \hat{\zeta}_l = \zeta_{m-1}, \tag{3·15}$$
$$\zeta_{2n+1} = \frac{4^{n+1} - 1}{3} = \hat{\zeta}_{2n+1} \ (n \ge 0) \ : \text{odd number},$$
$$\zeta_{2n} = \frac{2(4^n - 1)}{3} = \hat{\zeta}_{2n} - 1 \ (n \ge 1) \ : \text{even number},$$
$$\prod_{j=1}^{2^m-1} \nu_j = (-\gamma)^{\zeta_m} \quad \text{for } m \ge 1.$$

From these, it follows that $\hat{\zeta}_l$ is an odd number for $l \ge 1$. Then, from the last relation in Eq. (3·15), we obtain for $l \ge 0$,

$$r_l \equiv (-\gamma)^{-\zeta_m} c_l^{(m)} = \left\{ \prod_{j=1}^{2^m-1} \nu_j \right\}^{-1} \eta_l \prod_{j=l+1}^{2^m-1} \nu_j = \eta_l \left\{ \prod_{j=1}^{l} \nu_j \right\}^{-1}.$$

That is, r_l is independent of m as long as it is defined. Thus, we get

$$r_l = \eta_l \left\{ \prod_{j=1}^{l} \nu_j \right\}^{-1} \quad \text{for any } l \ge 0, \tag{3·16}$$
$$r_0 = \alpha, \quad r_{2^m} = -s_{2^m} \gamma^{1-\hat{\zeta}_m} \quad \text{for } m \ge 0. \tag{3·17}$$

Thus, we define

$$G_m(a) \equiv (-\gamma)^{-\zeta_m} a^{-2^m} F_m(a) = \sum_{l=0}^{2^m-1} r_l a^{-l}. \tag{3·18}$$

Let us consider the condition for the convergence of $G_m(a)$. Using the relations (3·17) and

$$r_l = -\gamma^{\hat{\zeta}_m} r_{2^m+l} \text{ for } 1 \le l \le 2^m - 1 \ , \ \ (m \ge 1) \tag{3·19}$$

for $m \ge 1$ we obtain the recursive relation

$$G_{m+1}(a) = (1 - \tilde{a}_m) G_m(a) + \tilde{a}_m(\alpha - \gamma s_{2^m}), \tag{3·20}$$

and then we obtain for $m \ge 1$

$$G_m(a) = \alpha \prod_{l=0}^{m-1} (1 - \tilde{a}_l) + \sum_{k=0}^{m-1} \left[\prod_{l=k+1}^{m-1} (1 - \tilde{a}_l) \right] \tilde{a}_k (\alpha - \gamma s_{2^k}), \tag{3·21}$$

where $\tilde{a}_l = a^{-2^l} \gamma^{-\hat{\zeta}_l} = \frac{1}{\sqrt{a^{(l-1)}}}$. Since $\tilde{a}_l = (a\gamma^{2/3})^{-2^l} \gamma^{-(-1)^l/3}$, setting $\hat{\gamma} = \max(\gamma, \gamma^{-1})$, we obtain

$$\tilde{a}_l \le (a\gamma^{2/3})^{-2^l} \hat{\gamma}^{1/3}. \tag{3·22}$$

Thus, if $a\gamma^{2/3} > 1$, $\lim_{m\to\infty} G_m(a)$ converges. That is, $G_\infty(z) \equiv \lim_{m\to\infty} G_m(z)$ is an analytic function for $|z| > \gamma^{-2/3}$.*) From Eq. (3·12) we obtain

$$G_m(a_m) = (-\gamma)^{-\zeta_m} a_m^{-2^m} \frac{\alpha + \beta + (-1)^{m-1} b}{2}, ^{**)} \tag{3·23}$$

and thus

$$G_\infty(a_c) = 0.^{***)} \tag{3·24}$$

Similarly to the symmetric case, we obtain from Eq. (3·23),

$$\begin{aligned} \epsilon_m &= \{G_m(a_c) - \mathit{\Gamma} a_c^{-2^m}\}(1 + \bar{h}_m)/G'_\infty(a_c), \\ \mathit{\Gamma} &= (-1)^{-\zeta_m} \frac{\alpha + \beta + (-1)^{m-1} b}{2}, \ \ \bar{h}_m = \frac{G'_\infty(a_c)}{G'_m(\bar{a}_m) + \mathit{\Gamma} 2^m (\hat{a}_m)^{-2^m - 1}} - 1, \end{aligned} \tag{3·25}$$

where $a_m < \bar{a}_m < a_c$, $a_m < \hat{a}_m < a_c$ and $\lim_{m\to\infty} \bar{h}_m = 0$. Further, using the relation (3·17) we obtain

$$\begin{aligned} G_m(a_c) &= -a_c^{-2^m} \gamma^{-\hat{\zeta}_m} (\alpha - \gamma s_{2^m}) + (a_c^{-2^m} \gamma^{-\hat{\zeta}_m})^2 q_m, \\ q_m &= -\frac{1}{1 - a_c^{-2^m} \gamma^{-\hat{\zeta}_m}} \left(\alpha - \gamma s_{2^m} + \gamma^{2\hat{\zeta}_m} \sum_{l=0}^{\infty} r_{2^{m+1}+l} a_c^{-l} \right). \end{aligned} \tag{3·26}$$

*) In a previous paper,[5)] as the condition for the convergence of $G_\infty(z)$, we found $|z| > \max(1, \gamma^{-1})$. This is a sufficient condition.

**) This is Eq. (3·4) in Ref. 5). There are several misprints in that paper. In particular, ζ_m in the definition of $G_m(a)$ and Eq. (3·4) should be changed to $-\zeta_m$.

***) Note that $a_c\gamma^{2/3} > 1$ follows from $a_1\gamma > 1$. Since $a_c^{-2^m}\gamma^{-\hat{\zeta}_m} = \tilde{a}_m(a = a_c) = (a_c\gamma^{2/3})^{-2^m}\gamma^{-(-1)^m/3}$ and $a_c^{-2^m}\gamma^{-\zeta_m} = a_c^{-2^m}\gamma^{-\hat{\zeta}_m}\gamma^{\frac{1+(-1)^m}{2}}$, these tend to 0 as $m \to \infty$.

Substituting (3·26) into (3·25), we obtain

$$\epsilon_m = a_c - a_m = \frac{ba_c^{-2^m}\gamma^{-\zeta_m}}{G'_\infty(a_c)}(1+h_m), \tag{3·27}$$

$$h_m = \bar{h}_m(1 + a_c^{-2^m}\gamma^{\zeta_m - 2\hat{\zeta}_m}\frac{q_m}{b}) + a_c^{-2^m}\gamma^{\zeta_m-2\hat{\zeta}_m}\frac{q_m}{b}. \tag{3·28}$$

The relation $\lim_{m\to\infty} h_m = 0$ is proved in Appendix B. Thus,

$$\delta_m = \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}(1+l_m), \tag{3·29}$$

$$l_m = [h_m - h_{m+1} - a_c^{-2^m}\gamma^{-\hat{\zeta}_m}(1+h_{m+1}) + a_c^{-2^{m+1}}\gamma^{-\hat{\zeta}_{m+1}}(1+h_{m+2})]$$
$$/[1+h_{m+1} - a_c^{-2^{m+1}}\gamma^{-\hat{\zeta}_{m+1}}(1+h_{m+2})],$$

and $\lim_{m\to\infty} l_m = 0$.

To obtain a relation similar to relation (2·32) in the case of a symmetric map, we define $v_m(a,\gamma)$ as

$$v_m(a,\gamma) = 1/u_m(a,\gamma) = \prod_{l=0}^{m-1} v(a^{(l)},\gamma^{(l)}), \quad v(a,\gamma) = \frac{1}{u(a,\gamma)} = \frac{\gamma(a-1)}{\gamma a + 1}. \tag{3·30}$$

For $m \geq 1$, $v_m(a)$ can be rewritten as

$$v_m(a) = \frac{1-a^{-1}}{g_{m-1}(a)}\prod_{l=0}^{m-2} h_l(a), \tag{3·31}$$

$$g_l(a) \equiv 1 + \gamma^{-\hat{\zeta}_l}a^{-2^l}, \quad h_l(a) \equiv 1 - \gamma^{-\hat{\zeta}_l}a^{-2^l} \text{ for } l \geq 0,$$

$$\prod_{l=0}^{-1} h_l(a) \equiv 1, \quad v_1(a) = v(a).$$

Then, for $m \geq 2$, the equation

$$G_{m-1}(a) = \alpha + \frac{1}{1+\gamma^{-1}}\{g_{m-1}(a)v_m(a) - 1 - (-1)^m a^{-2^{m-1}}\gamma^{-\zeta_{m-1}}\} \tag{3·32}$$

can be proved by using the above relations (see Appendix A). Thus, the condition $v_m(a_m) = b$ for a_m is equivalent to the following equation derived from Eq. (3·13):

$$G_{m-1}(a_m) = (-\gamma)^{-\zeta_{m-1}} a_m^{-2^{m-1}}\frac{\alpha+\beta+(-1)^{m-1}b}{2}. \tag{3·33}$$

From relation (3·32), we get for $a > \gamma^{-2/3}$,

$$G_\infty(a) = \alpha + \frac{1}{1+\gamma^{-1}}(v_\infty(a) - 1). \tag{3·34}$$

Thus,

$$G'_\infty(a) = \frac{1}{1+\gamma^{-1}}v'_\infty(a) = \frac{2b}{1+\gamma^{-1}}\tau(a), \tag{3·35}$$

$$\tau(a) \equiv \sum_{l=0}^{\infty}\frac{1+\gamma^{(l)}}{2(a^{(l)}-1)(\gamma^{(l)}a^{(l)}+1)}\gamma^{2(2^l-(-1)^l)/3}2^l a^{2^l-1} > 0.$$

Therefore, we obtain

$$\epsilon_m = \frac{1}{2\tau(a_c)} a_c^{-2^m} \gamma^{-\zeta_m} (1+\gamma^{-1})(1+h_m). \qquad (3\cdot36)$$

§4. Period-doubling bifurcation of the period *p* solution
— *Symmetric case* —

In this section, we consider the period-doubling bifurcation of the periodic solution with prime period p (≥ 3) that emerges through a tangent bifurcation.

Let us consider the mapping $T^p_{\boldsymbol{a}}(x)$. At $a = a_M = \frac{2}{1-b}$, the map looks like that in Fig. 5.

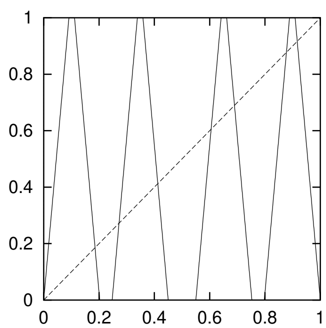

Fig. 5. Trapezoid map $T^3_{(a,b)}(x)$ for $a = a_M$ with $b = 0.1$.

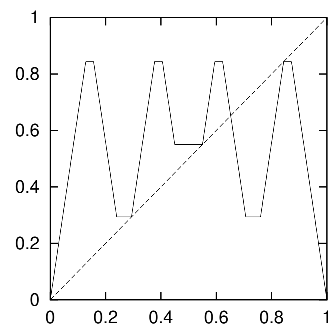

Fig. 6. Trapezoid map $T^3_{(a,b)}(x)$ for $a = a_{p,0}$ with $b = 0.1$.

As a is decreased from $a = a_M$, at some value of $a = a_{p,0}$, the map becomes tangent to the line $y = x$, as is illustrated in Fig. 6. If we set $x_1 = a\frac{1-b}{2}$, at this point the symbols for $x_1, x_2, \cdots, x_{p-1}$ are $RL \cdots L$. Then,

$$x_p = a^{p-1}\left(1 - \frac{a(1-b)}{2}\right). \qquad (4\cdot1)$$

As is shown in Appendix C, $x_p(a)$ takes the maximum value at $a = a_{p,\max} \equiv \frac{p-1}{p} a_M$ and $\frac{1+b}{2} < x_p(a_{p,\max})$ for $p \geq 3$. Thus, the point $a_{p,0}$, where the tangent bifurcation takes place, satisfies the condition

$$x_p(a_{p,0}) = \frac{1+b}{2}. \qquad (4\cdot2)$$

This equation has two solutions; one is $a = 1$, and the other corresponds to $a_{p,0}$. Thus, we obtain

$$1 < \frac{p-1}{p} a_M < a_{p,0} < a_{p,1},$$

where $a_{p,1}$ is defined by

$$x_p(a_{p,1}) = \frac{1-b}{2} \qquad (4\cdot3)$$

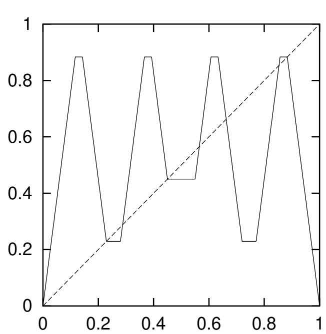

Fig. 7. Trapezoid map $T^3_{(a,b)}(x)$ for $a = a_{p,1}$ with $b = 0.1$.

and $a_{p,1} > 1$ (see Fig. 7).

$a_{p,1}$ is the onset point of the period $2p$ point. The symbolic sequence for $x_1, x_2, \cdots, x_{p-1}$ is $RL\cdots L$ for $a_{p,0} < a$ (see Appendix D). For $a_{p,0} < a < a_{p,1}$, let us consider the map $T^p_{\boldsymbol{a}}(x)$. The unstable periodic point with period p of $T_{\boldsymbol{a}}(x)$ is

$$x^* = \frac{a^{p-1}(a-1)}{a^p - 1}. \tag{4·4}$$

Rescaling the map $T^p_{\boldsymbol{a}}(x)$ around $x = 1/2$ for $a > a_{p,0}$, we obtain the trapezoid map $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$, where

$$\begin{aligned} &\boldsymbol{a}^{(0)} = (a^{(0)}, b^{(0)}), \;\; a^{(0)} = a^p, \;\; b^{(0)} = \frac{b}{\Delta x} = r(a)b, \;\; \Delta x = 2x^* - 1, \\ &r(a) = \frac{1}{\Delta x} = \frac{a^p - 1}{a^p - 2a^{p-1} + 1}, \;\; x^{(0)} = \frac{x^* - x}{2x^* - 1}. \end{aligned} \tag{4·5}$$

For $a > 1$, we obtain $r'(a) < 0$. As before, we define $x^{(0)}_M = a^{(0)}\frac{1-b^{(0)}}{2}$. For $a_{p,0} < a < a_{p,1}$, $x^{(0)}_M$ corresponds to x_p; that is $x^{(0)}_M = \frac{x^*-x_p}{2x^*-1}$. Therefore, we have $x^{(0)}_M = 0$ at $a = a_{p,0}$ and $x^{(0)}_M = \frac{1+b^{(0)}}{2}$ at $a = a_{p,1}$. We note that $b^{(0)}(a_{p,0}) = 1$. Since $r(a)$ is a strictly decreasing, continuous function w.r.t. a for $a > 1$, $b^{(0)}(a)$ is a strictly decreasing, continuous function w.r.t. a, and $x^{(0)}_M(a)$ is a strictly increasing, continuous function w.r.t. a. Therefore, for $a_{p,0} < a < a_{p,1}$, the inequalities $\frac{1-b^{(0)}}{2} < x^{(0)}_M < \frac{1+b^{(0)}}{2}$ hold, and for $a_{p,1} < a$, we have $\frac{1+b^{(0)}}{2} < x^{(0)}_M$. That is, $x^{(0)}_M$ is a stable fixed point of $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$ for $a_{p,0} < a < a_{p,1}$, and for $a_{p,1} < a$ the fixed point is $x^*_u \equiv \frac{a^{(0)}}{a^{(0)}+1}$ and is unstable. For $a_{p,1} < a$, rescaling the map $T^2_{\boldsymbol{a}^{(0)}}(x^{(0)})$, we obtain the trapezoid map $T_{\boldsymbol{a}^{(1)}}(x^{(1)})$, where,

$$\begin{aligned} &\boldsymbol{a}^{(1)} \equiv \boldsymbol{\varphi}(\boldsymbol{a}^{(0)}) = (a^{(1)}, b^{(1)}), \;\; a^{(1)} = \{a^{(0)}\}^2 = a^{2p}, \\ &b^{(1)} = u(a^{(0)})b^{(0)} = u(a^p)r(a)b, \;\; u(a) = \frac{a+1}{a-1}. \end{aligned} \tag{4·6}$$

As before, we define $\boldsymbol{a}^{(m)}$ as

$$\begin{aligned} &\boldsymbol{a}^{(m)} \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}^{(0)}) = (a^{(m)}, b^{(m)}), \;\; a^{(m)} = \{a^{(m-1)}\}^2 = a^{2^m p}, \\ &b^{(m)} = u(a^{(m-1)})b^{(m-1)} = u_m(a)r(a)b, \;\; u_m(a) = \prod_{l=0}^{m-1} u(a^{(l)}). \end{aligned} \tag{4·7}$$

Since $b^{(m)}(a)$ is a decreasing function w.r.t. a, we can repeat the same argument as in §2. At $a = a_{p,m}$, the period 2^m solution of $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$ appears, and $a_{p,m}$ satisfies the condition

$$b^{(m)}(a_{p,m}) = 1 \;\; \text{for } m \geq 0, \tag{4·8}$$

and

$$1 < a_{p,0} < a_{p,1} < a_{p,2} < \cdots < \frac{2}{1-b} = a_M. \tag{4·9}$$

The symbolic sequence for the period 2^m solution is the MSS sequence for the map $T_{\boldsymbol{a}^{(0)}}(x^{(0)})$. For $a_{p,m} \le a \le a_{p,m+1}$, starting from $x_1^{(0)} = x_M^{(0)}$, $x_{2^m}^{(0)}$ is expressed in terms of the function $F_m(a^{(0)}, b^{(0)})$. Here, $F_m(a,b)$ is defined by Eq. (2·10), and we include b as an independent variable of F_m explicitly. Thus, we obtain the other condition for $a_{p,m}$,

$$F_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m})) = \frac{1 + (-1)^{m-1} b^{(0)}(a_{p,m})}{2}. \tag{4·10}$$

In terms of $G_m(a,b)$, defined by Eq. (2·19), Eq. (2·21) becomes

$$G_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m})) = a^{(0)}(a_{p,m})^{-2^m} \frac{(-1)^m - b^{(0)}(a_{p,m})}{2}, \tag{4·11}$$

and we obtain

$$G_\infty(a^{(0)}(a_{p,c}), b^{(0)}(a_{p,c})) = 0, \tag{4·12}$$

where $a_{p,c} = \lim_{m\to\infty} a_{p,m}$. From these equations, we obtain for $\epsilon_{p,m} = a_{p,c} - a_{p,m}$,

$$\epsilon_{p,m} \simeq \frac{a_{p,c}^{-p2^m} r(a_{p,c}) b}{\frac{\partial G_\infty}{\partial a^{(0)}} p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})| b}.^{*)} \tag{4·13}$$

Therefore, we obtain

$$\delta_m \sim a_{p,c}^{p2^m}. \tag{4·14}$$

As for the relations among $a_{p,0}$, the following ordering holds:

$$a_c \le a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M. \tag{4·15}$$

(See Appendix E.)

§5. Period-doubling bifurcation of the period $p \ge 3$ solution —*Asymmetric case*—

As in the previous case, we investigate the period-doubling bifurcation of the period p (≥ 3) solution in the asymmetric case.

In this case also, we consider the period p solution starting with $x_1 = a\alpha$ with the sequence of symbols $RL\cdots L$. Then, we obtain

$$x_p = \gamma a^{p-1}(1 - a\alpha). \tag{5·1}$$

As shown in Appendix C, $x_p(a)$ takes the maximum value at $a = a_{p,\max} \equiv \frac{p-1}{p} a_M$ and $\beta < x_p(a_{p,\max})$ for $p \ge 3$, where $a_M = 1/\alpha$, as before. Then, the onset point $a_{p,0} > a_{p,\max}$ of the period p solution satisfies

$$x_p(a_{p,0}) = \beta. \tag{5·2}$$

Although this equation has two solutions, we should adopt the larger one, as in the symmetric case. The symbolic sequence for $x_1, x_2, \cdots, x_{p-1}$ is $RL\cdots L$ for $a_{p,0} \le a$

*) Since $\frac{\partial G_\infty(a,b)}{\partial a} > 0$, the denominator is positive.

(see Appendix D). On the other hand, the onset point of the period $2p$ solution, $a_{p,1}$, satisfies

$$x_p(a_{p,1}) = \alpha. \tag{5·3}$$

Then, for $a_{p,0} < a$, the unstable periodic point x^* with period p for $A_{\boldsymbol{a}}(x)$ is

$$x^* = \frac{(\gamma a - 1)\gamma a^{p-1}}{\gamma^2 a^p - 1}. \tag{5·4}$$

Similarly to the symmetric case, we obtain an asymmetric trapezoid map $A_{\boldsymbol{a}^{(0)}}(x^{(0)})$ by rescaling the map $A^p_{\boldsymbol{a}}(x)$ in the vicinity of I_C for $a > a_{p,0}$, where

$$\begin{aligned}
&\boldsymbol{a}^{(0)} = (a^{(0)}, b^{(0)}, \gamma^{(0)}),\ \ a^{(0)} = \gamma^2 a^p,\ \ \gamma^{(0)} = \gamma^{-1},\ \ b^{(0)} = \frac{b}{\Delta x} = r(a)b, \qquad (5·5)\\
&\Delta x = \frac{\gamma g(a)}{\gamma^2 a^p - 1},\ \ g(a) = \gamma a^p - (1+\gamma)a^{p-1} + 1,\\
&r(a) = 1/\Delta x = \frac{\gamma^2 a^p - 1}{\gamma g(a)},\ \ x^{(0)} = \frac{x^* - x}{\Delta x},\\
&\alpha^{(0)} \equiv \frac{x^* - \beta}{\Delta x} = \frac{\gamma^2 a^p(1-\beta) - \gamma a^{p-1} + \beta}{\gamma g(a)},\\
&\beta^{(0)} \equiv \frac{x^* - \alpha}{\Delta x} = \frac{\gamma^2 a^p(1-\alpha) - \gamma a^{p-1} + \alpha}{\gamma g(a)}.
\end{aligned}$$

Then, $r'(a)$ becomes

$$r'(a) = -\frac{(1+\gamma)a^{p-2}f(a)}{\gamma g(a)^2}, \tag{5·6}$$

where $f(a) = \gamma^2 a^p - p\gamma a + p - 1$. We can prove that, at least for $a_{p,0} \le a$, the inequality $f(a) > 0$ holds. Thus, for $a_{p,0} \le a$, we have $r'(a) < 0$. Further, $\alpha^{(0)\prime}(a) = \frac{ba^{p-2}f(a)}{\gamma g(a)^2} > 0$ and $\beta^{(0)\prime}(a) = -\gamma\alpha^{(0)\prime}(a) < 0$ for $a \ge a_{p,0}$. Thus, $b^{(0)}$ is a strictly decreasing, continuous function, and $x_M^{(0)} = a^{(0)}\alpha^{(0)}$ is a strictly increasing, continuous function w.r.t. a. At $a = a_{p,0}$, $\alpha^{(0)} = 0$ and $\beta^{(0)} = b^{(0)} = 1$. For $a_{p,0} \le a \le a_{p,1}$, $x_p(a)$ corresponds to $x_M^{(0)}$; that is, $x_M^{(0)} = \frac{x^* - x_p}{\Delta x}$. Then, at $a = a_{p,0}$, we have $x_M^{(0)} = 0$ and at $a = a_{p,1}$, $x_M^{(0)} = \beta^{(0)}$. Thus, $\alpha^{(0)} < x_M^{(0)} < \beta^{(0)}$ for $a_{p,0} < a < a_{p,1}$, and $x_M^{(0)}$ is a stable fixed point of $A_{\boldsymbol{a}^{(0)}}(x^{(0)})$. Further, for $a_{p,1} < a$, we have $x_M^{(0)} > \beta^{(0)}$, and the fixed point becomes $x_u^* \equiv \frac{\gamma^{(0)}a^{(0)}}{1+\gamma^{(0)}a^{(0)}}$ and is unstable. For $a_{p,1} < a$, rescaling the map $A^2_{\boldsymbol{a}^{(0)}}(x^{(0)})$, we obtain $A_{\boldsymbol{a}^{(1)}}(x^{(1)})$, where

$$\begin{aligned}
&\boldsymbol{a}^{(1)} \equiv \boldsymbol{\varphi}(\boldsymbol{a}^{(0)}) = (a^{(1)}, b^{(1)}, \gamma^{(1)}),\ \ a^{(1)} = \{\gamma^{(0)}a^{(0)}\}^2 = \gamma^2 a^{2p}, \qquad (5·7)\\
&b^{(1)} = u(a^{(0)}, \gamma^{(0)})b^{(0)} = u(a^p)r(a)b,\ \ \gamma^{(1)} = \{\gamma^{(0)}\}^{-1},\ \ u(a,\gamma) = \frac{\gamma a + 1}{\gamma(a-1)}.
\end{aligned}$$

Thus, $a^{(1)}$ is increasing, $b^{(1)}$ is decreasing, and $\gamma^{(1)}$ is constant w.r.t. a. Further, we define $\boldsymbol{a}^{(m)}$ as

$$\boldsymbol{a}^{(m)} \equiv \boldsymbol{\varphi}^m(\boldsymbol{a}^{(0)}) = (a^{(m)}, b^{(m)}, \gamma^{(m)}),\ \ a^{(m)} = \{\gamma^{(m-1)}a^{(m-1)}\}^2, \tag{5·8}$$

$$b^{(m)} = u(a^{(m-1)}, \gamma^{(m-1)})b^{(m-1)} = u_m(a,\gamma)r(a)b, \quad u_m(a,\gamma) = \prod_{l=0}^{m-1} u(a^{(l)}, \gamma^{(l)}),$$

$$\gamma^{(m)} = \{\gamma^{(m-1)}\}^{-1}.$$

Since $b^{(m)}$ is a decreasing function w.r.t. a, we can make an argument quite similar to that in §4. In particular, at $a = a_{p,m}$, a period-doubling bifurcation takes place, and

$$b^{(m)}(a_{p,m}) = 1 \ \text{ for } m \geq 0, \tag{5·9}$$

with

$$1 < a_{p,0} < a_{p,1} < a_{p,2} < \cdots < \frac{1}{\alpha} = a_M. \tag{5·10}$$

The symbolic sequence for the period 2^m solution is the MSS sequence for the map $A_{\boldsymbol{a}^{(0)}}(x^{(0)})$. For $a_{p,m} \leq a \leq a_{p,m+1}$, starting from $x_1^{(0)} = x_M^{(0)}$, $x_{2^m}^{(0)}$ is expressed in terms of the function $F_m(a^{(0)}, b^{(0)}, \gamma^{(0)})$, and we get

$$F_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m}), \gamma^{(0)}) = \frac{\alpha^{(0)}(a_{p,m}) + \beta^{(0)}(a_{p,m}) + (-1)^{m-1}b^{(0)}(a_{p,m})}{2}, \tag{5·11}$$

where $F_m(a,b,\gamma)$ is defined by Eq. (3·11). In terms of $G_m(a,b,\gamma)$, defined by Eq. (3·18), Eq. (5·11) becomes

$$\begin{aligned} &G_m(a^{(0)}(a_{p,m}), b^{(0)}(a_{p,m}), \gamma^{(0)}) \\ &\quad = (-\gamma^{(0)})^{-\zeta_m}\{a^{(0)}(a_{p,m})\}^{-2^m}\frac{\alpha^{(0)}(a_{p,m}) + \beta^{(0)}(a_{p,m}) + (-1)^{m-1}b^{(0)}(a_{p,m})}{2}. \end{aligned} \tag{5·12}$$

Let us define $a_{p,c} = \lim_{m\to\infty} a_{p,m}$. Thus, we obtain

$$G_\infty(a^{(0)}(a_{p,c}), b^{(0)}(a_{p,c}), \gamma^{(0)}) = 0, \tag{5·13}$$

and $\epsilon_{p,m} = a_{p,c} - a_{p,m}$ is given by

$$\epsilon_{p,m} \simeq \frac{(\gamma^2 a_{p,c}^p)^{-2^m}\gamma^{\zeta_m} r(a_{p,c})b}{\frac{\partial G_\infty}{\partial a^{(0)}}\gamma^2 p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})|b}. \tag{5·14}$$

Thus,

$$\delta_m \simeq \gamma^{(-1)^{m-1}/3}(a_{p,c}^p \gamma^{4/3})^{2^m}. \tag{5·15}$$

As for the relations among $a_{p,0}$, the following ordering holds:

$$a_c \leq a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M. \tag{5·16}$$

(See Appendix E.)

§6. Summary and discussion

In this paper, we have studied the symmetric and the asymmetric trapezoid maps rigorously. We gave proofs of several results regarding the period-doubling bifurcation which occurs as the parameter a is increased. We obtained the following scaling results for the period-doubling bifurcation starting from a period one solution.

1. Symmetric case

$$\epsilon_m = \frac{b a_c^{-2^m}}{G'_\infty(a_c)}(1+h_m),$$
$$\delta_m = a_c^{2^m}(1+l_m),$$

where $\lim_{m\to\infty} h_m = 0$ and $\lim_{m\to\infty} l_m = 0$.

2. Asymmetric case

$$\epsilon_m = \frac{b a_c^{-2^m}\gamma^{-\zeta_m}}{G'_\infty(a_c)}(1+h_m),$$
$$\delta_m = \gamma^{(-1)^m/3}(a_c\gamma^{2/3})^{2^m}(1+l_m),$$

where $\lim_{m\to\infty} h_m = 0$ and $\lim_{m\to\infty} l_m = 0$.

The new results in this paper regard the period-doubling bifurcation that starts from a periodic solution with any prime period p (≥ 3). Here, we have the following.

1. Symmetric case

$$\epsilon_{p,m} \simeq \frac{a_{p,c}^{-p2^m} r(a_{p,c}) b}{\frac{\partial G_\infty}{\partial a^{(0)}} p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})|b},$$
$$\delta_m \simeq a_{p,c}^{p2^m}.$$

2. Asymmetric case

$$\epsilon_m \simeq \frac{(\gamma^2 a_{p,c}^p)^{-2^m}\gamma^{\zeta_m} r(a_{p,c}) b}{\frac{\partial G_\infty}{\partial a^{(0)}}\gamma^2 p a_{p,c}^{p-1} + \frac{1}{2}|r'(a_{p,c})|b},$$
$$\delta_m \simeq \gamma^{(-1)^{m-1}/3}(a_{p,c}^p\gamma^{4/3})^{2^m}.$$

These results imply that for any period-doubling cascade, the accumulation rate to the accumulation point is extremely fast, and in fact, exponential.

In a one-dimensional map with one hump, the Feigenbaum constant depends on the power z which characterizes the behavior of the map in the vicinity of the critical point. It has been proved that $\lim_{z\to\infty}\delta(z)$ is finite.[4] The result in this paper shows that $\delta(\infty)$ is infinite. That is, superconvergence takes place only in the case $z=\infty$. Therefore, this feature is attributed to the flatness of the summit. For example, in a one humped map with a flat summit, superconvergence will take place if it satisfies some conditions, e.g. the absolute value of the derivative is greater than some constant $\lambda > 1$ outside a region which contains the flat part of the map.

In terms of the renormalization group, we can interpret the usual convergence and the superconvergence of the period-doubling cascades as follows. Under the renormalization transformation, the behavior of a map with finite z and that of a map with $z = \infty$ are different. As the number of iterations of the renormalization transformation becomes large, renormalized maps with finite z tend to a fixed-point function and are almost identical, but those maps with $z = \infty$ keep changing, and their slopes tend to infinity. This difference in behavior results in the finite and infinite values of δ.

In previous papers,[7)-9)] similar results were obtained with a different method. In those studies, the authors estimated quantities in question using inequalities and found that a_m is quadratically convergent in the case of period doubling of the period one solution. This implies that $\lim_{m\to\infty} \frac{\ln \delta_m}{\ln \delta_{m-1}} = 2$. On the other hand, our method is constructive. In fact, we gave a precise equation for the onset point a_m of the periodic point with period 2^m and also gave an equation for the accumulation point a_c. From these equations, we obtained the above scaling relations. We also extended the argument not only to the asymmetric trapezoid map but also to period doubling of the prime period p (≥ 3) solution which emerges from a tangent bifurcation.

In Ref. 8), the authors mention the problem of convergence of the sequence $\Delta\epsilon_m/(\Delta\epsilon_{m-1})^2$, and from our result, this is $\epsilon_m/\epsilon_{m-1}^2 \simeq G'_\infty(a_c)\gamma^{(-1+(-1)^m)/2}/b$ and does not converge for the asymmetric case.

The trapezoid maps studied here are exactly solvable models of the renormalization group. It is interesting to investigate further, detailed bifurcation structures in these models. This is a future problem.

In this paper, we gave an elementary proof of the existence of the period-doubling cascade in trapezoid maps. It will help us to gain an intuitive understanding of the mechanism of the period-doubling cascade in one-dimensional maps. On the other hand, in the case of maps of greater than one dimensional, it seems that the mechanism of the period-doubling cascade has not yet been fully understood. By studying two-dimensional maps, which are extensions of trapezoid maps, we may clarify this mechanism. This is another future problem.

Appendix A
—— *Proof of Relations (2·32) and (3·32)* ——

In this appendix, we prove the relation (3·32). The relation (2·32) is then obtained by setting $\gamma = 1$.

For $m \geq 2$, the relation (3·32) is

$$G_{m-1}(a) = \alpha + \frac{1}{1+\gamma^{-1}}\{g_{m-1}(a)v_m(a) - 1 - (-1)^m a^{-2^{m-1}}\gamma^{-\zeta_{m-1}}\}, \quad \text{(A·1)}$$

$$v_m(a) = \frac{1-a^{-1}}{g_{m-1}(a)}\prod_{l=0}^{m-2} h_l(a),$$

$$g_l(a) \equiv 1 + \gamma^{-\hat{\zeta}_l}a^{-2^l}, \;\; h_l(a) \equiv 1 - \gamma^{-\hat{\zeta}_l}a^{-2^l} \text{ for } l \geq 0.$$

Let us define ϕ_m for $m \geq 1$ as

$$\phi_m(a) \equiv g_m(a)v_{m+1}(a) = (1-a^{-1})\prod_{l=0}^{m-1} h_l(a). \quad \text{(A·2)}$$

We expand $\phi_m(a)$ as

$$\phi_m(a) \equiv \sum_{l=0}^{2^m} w_l^{(m)} a^{-l}, \tag{A·3}$$

and then using the relations in (3·15), we obtain for $m \geq 1$,

$$w_0^{(m)} = 1, \;\; w_{2^m}^{(m)} = (-1)^{m-1}\gamma^{-\zeta_m}. \tag{A·4}$$

In particular,

$$w_1^{(1)} = -(1+\gamma^{-1}). \tag{A·5}$$

Let us derive the recursive relations for $w_l^{(m)}$. For $m \geq 1$, we have

$$\phi_{m+1}(a) = g_m(a)v_{m+1}(a) = h_m(a)\phi_m(a) = \phi_m(a) - \gamma^{-\hat{\zeta}_m} a^{-2^m}\phi_m(a). \tag{A·6}$$

Comparing coefficients, we obtain for $m \geq 1$,

$$\mathcal{O}(a^{-2^{m+1}}): \;\; w_{2^{m+1}}^{(m+1)} = -\gamma^{-\hat{\zeta}_m} w_{2^m}^{(m)}, \tag{A·7}$$

$$\mathcal{O}(a^{-2^m}): \;\; w_{2^m}^{(m+1)} = -w_{2^m}^{(m)} - \gamma^{-\hat{\zeta}_m} w_0^{(m)}, \tag{A·8}$$

$$\mathcal{O}(a^{-(2^m+l)}): \;\; w_{2^m+l}^{(m+1)} = -\gamma^{-\hat{\zeta}_m} w_l^{(m)} \;\; \text{for } 0 < l < 2^m, \tag{A·9}$$

$$\mathcal{O}(a^{-l}): \;\; w_l^{(m+1)} = w_l^{(m)} \;\; \text{for } 0 \leq l < 2^m. \tag{A·10}$$

Using Eq. (A·4), Eq. (A·7) is automatically satisfied. Then, from Eq. (A·8), we get

$$w_{2^m}^{(m+1)} = -(1+\gamma^{-1})s_{2^m}\gamma^{1-\hat{\zeta}_m}. \tag{A·11}$$

Now, let us return to the quantity $G_{m-1}(a)$. We refer to the r.h.s. of relation (A·1) as $\hat{G}_{m-1}(a)$ and expand it in a^{-1} for $m \geq 2$:

$$\hat{G}_{m-1}(a) = \sum_{l=0}^{2^{m-1}-1} \bar{r}_l^{(m-1)} a^{-l}. \tag{A·12}$$

Thus, we obtain the following relations for $m \geq 2$:

$$\bar{r}_0^{(m-1)} = \alpha, \tag{A·13}$$

$$\bar{r}_l^{(m-1)} = \frac{w_l^{(m-1)}}{1+\gamma^{-1}} \quad \text{for } 0 < l \leq 2^{m-1}-1. \tag{A·14}$$

Thus, from (A·11), (A·9) and (A·10), we obtain for $m \geq 1$,

$$\bar{r}_{2^m}^{(m+1)} = -s_{2^m}\gamma^{1-\hat{\zeta}_m}, \tag{A·15}$$

$$\bar{r}_{l+2^m}^{(m+1)} = -\gamma^{-\hat{\zeta}_m}\bar{r}_l^{(m)} \quad \text{for } 0 < l \leq 2^m - 1, \tag{A·16}$$

$$\bar{r}_l^{(m+1)} = \bar{r}_l^{(m)} \quad \text{for } 0 \leq l \leq 2^m - 1. \tag{A·17}$$

The relation (A·17) implies that $\bar{r}_l^{(m)}$ is independent of m as long as it is defined, and then we omit the superscript (m). From Eq. (A·16), it turns out that $\bar{r}_l$ for

$l = 2^m + 1, \cdots, 2^{m+1} - 1$ are determined by $\bar{r}_l$ for $l = 1, \cdots, 2^m - 1$. Thus, $\bar{r}_1$ and $\bar{r}_{2^m}$ for $m \geq 1$ determine $\bar{r}_l$ for $l > 0$. From Eqs. (A·5) and (A·14), $\bar{r}_1 = -1 = r_1$ follows. Also, from Eqs. (3·17) and (3·19), we note that r_l satisfies the same relations as (A·15) and (A·16). Then, $\bar{r}_{2^m} = r_{2^m}$ for $m \geq 1$. Therefore, $r_l = \bar{r}_l$ follows for $l \geq 1$. Finally, from Eq. (A·13), we have $\bar{r}_0 = \alpha = r_0$. Therefore, we obtain $\hat{G}_{m-1}(a) = G_{m-1}(a)$. That is, the relation (3·32) holds.

Appendix B
—— *Proof of* $\lim_{m\to\infty} h_m = 0$ ——

In this appendix, we prove

$$\lim_{m\to\infty} h_m = 0,$$

where h_m is

$$h_m = \bar{h}_m \left(1 + a_c^{-2^m} \gamma^{\zeta_m - 2\hat{\zeta}_m} \frac{q_m}{b}\right) + a_c^{-2^m} \gamma^{\zeta_m - 2\hat{\zeta}_m} \frac{q_m}{b},$$

$$q_m = -\frac{1}{1 - a_c^{-2^m} \gamma^{-\hat{\zeta}_m}} \left(\alpha - \gamma s_{2^m} + \gamma^{2\hat{\zeta}_m} \sum_{l=0}^{\infty} r_{2^{m+1}+l} a_c^{-l}\right).$$

To show this, we only have to prove that

$$\lim_{m\to\infty} a_c^{-2^m} \gamma^{\zeta_m - 2\hat{\zeta}_m} q_m = 0.$$

First we estimate the series $S \equiv \sum_{l=0}^{\infty} r_{2^{m+1}+l} a_c^{-l}$. Let τ_l be the number of appearances of 1 in $s_1, \ s_2, \ \cdots, \ s_l$. Then, r_l is

$$r_l = \gamma s_l \left[\prod_{j=1}^{l} \{1 - (1+\gamma) s_j\}\right]^{-1} = \gamma s_l (-\gamma)^{-\tau_l}. \qquad \text{(B·1)}$$

Thus,

$$|r_l| \leq \gamma^{1-\tau_l}.$$

Let $\tilde{\tau}_l$ be the number of appearances of 1 in $s_{2^{m+1}+1}, \ s_{2^{m+1}+2}, \ \cdots, \ s_{2^{m+1}+l}$. Then, for $l \geq 0$, we obtain

$$\tau_{2^{m+1}+l} = \hat{\zeta}_{m+1} + \tilde{\tau}_l, \ \ \tilde{\tau}_l \leq l, \ \ \tilde{\tau}_0 \equiv 0.$$

Then,

$$|S| \leq \sum_{l=0}^{\infty} \gamma^{1-\hat{\zeta}_{m+1}-\tilde{\tau}_l} a_c^{-l}.$$

Since $a_c\gamma > 1$, we obtain

$$|S| \leq \frac{a_c}{a_c - 1} \gamma^{1-\hat{\zeta}_{m+1}} \quad \text{for } \gamma \geq 1,$$

$$|S| \leq \frac{a_c\gamma}{a_c\gamma - 1} \gamma^{1-\hat{\zeta}_{m+1}} \quad \text{for } \gamma < 1.$$

Thus for any γ,

$$|S| \leq \text{const} \times \ \gamma^{-\hat{\zeta}_{m+1}}.$$

From the relation

$$\zeta_m - \hat{\zeta}_{m+1} = -\hat{\zeta}_m - \frac{1+(-1)^{m-1}}{2},$$

we get

$$a_c^{-2^m}\gamma^{\zeta_m}|S| \leq \text{const} \times \ a_c^{-2^m}\gamma^{-\hat{\zeta}_m}\gamma^{-\frac{1+(-1)^{m-1}}{2}}.$$

Further,

$$\zeta_m - 2\hat{\zeta}_m = -\hat{\zeta}_m - \frac{1+(-1)^m}{2},$$

and therefore

$$a_c^{-2^m}\gamma^{\zeta_m - 2\hat{\zeta}_m} = a_c^{-2^m}\gamma^{-\hat{\zeta}_m}\gamma^{-\frac{1+(-1)^m}{2}}.$$

Thus, since $a_c^{-2^m}\gamma^{-\hat{\zeta}_m}$ tends to 0 as $m \to \infty$, we obtain

$$\lim_{m\to\infty} a_c^{-2^m}\gamma^{\zeta_m - 2\hat{\zeta}_m} q_m = 0.$$

Appendix C

—— *Proofs of* $a_{p,\max} = \frac{p-1}{p}a_M$ *and* $x_p(a_{p,\max}) > \beta$ *for* $p \geq 3$ ——

In this appendix, we prove that for the asymmetric case, the maximum of $x_p = \gamma a^{p-1}(1-a\alpha)$ appears at $a = a_{p,\max} \equiv \frac{p-1}{p}a_M$, and $x_p(a_{p,\max}) > \beta$ for $p \geq 3$. Similar results for the symmetric case are obtained by setting $\gamma = 1$.

Differentiating $x_p(a)$ w.r.t. a, we get

$$x_p'(a) = \gamma a^{p-2}(p-1)\left(1 - \frac{p}{p-1}a\alpha\right). \tag{C·1}$$

Then, at $a = a_{p,\max} \equiv \frac{p-1}{p}a_M$, $x_p(a)$ has a maximum value

$$x_p(a_{p,\max}) = \frac{\gamma}{p}\left(\frac{p-1}{p}\right)^{p-1} a_M^{p-1} = \frac{1}{p(1-\beta)}\left(\frac{p-1}{p}\right)^{p-1}\alpha^{2-p}. \tag{C·2}$$

Since $0 < \alpha < \beta < 1$ and $p \geq 3$, we have

$$x_p(a_{p,\max}) > \frac{1}{p(1-\beta)}\left(\frac{p-1}{p}\right)^{p-1}\beta^{2-p} = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}\frac{\beta}{g(\beta)}, \tag{C·3}$$

where $g(\beta) \equiv \beta^{p-1}(1-\beta) > 0$. The function $g(\beta)$ is maximal at $\beta = \frac{p-1}{p}$, where

$$g\left(\frac{p-1}{p}\right) = \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}. \tag{C·4}$$

Thus,

$$x_p(a_{p,\max}) > \frac{1}{p}\left(\frac{p-1}{p}\right)^{p-1}\frac{\beta}{g\left(\frac{p-1}{p}\right)} = \beta. \tag{C·5}$$

Appendix D

—— Proof of $H(x_1)H(x_2)\cdots H(x_{p-1}) = RL\cdots L$ for $a_{p,0} \le a$ ——

In this appendix, we prove that for $a_{p,0} \le a$, the symbolic sequence for $x_1, x_2, \cdots, x_{p-1}$ is $RL\cdots L$. We prove this in the asymmetric case. We obtain the results for the symmetric case by setting $\gamma = 1$.

First, we prove the relation

$$a_{p,0} > \frac{\beta}{\alpha}. \tag{D·1}$$

For $p \ge 3$, the relation

$$x_p\left(\frac{\beta}{\alpha}\right) = \left(\frac{\beta}{\alpha}\right)^{p-2}\beta > \beta = x_p(a_{p,0}) \tag{D·2}$$

holds. Since $x_p(a) = \gamma a^{p-1}(1 - a\alpha)$ is a strictly decreasing continuous function for $a_{p,\max} \le a$, and $a_{p,\max} < a_{p,0}$ holds, we obtain $a_{p,0} > \frac{\beta}{\alpha}$. Now, let us fix $a \ge a_{p,0}$. Then, from Eq. (D·1),

$$x_1 = a\alpha \ge a_{p,0}\alpha > \beta.$$

Thus, $H(x_1) = R$. Then,

$$x_2 = \gamma a(1 - x_1) = \frac{\gamma a^{p-1}(1 - a\alpha)}{a^{p-2}} = \frac{x_p}{a^{p-2}} \le \frac{\beta}{a^{p-2}} \le \frac{\beta}{a_{p,0}^{p-2}} < \frac{\beta}{a_{p,0}} < \alpha.$$

Thus, $x_2 \in I_L$. Now, let us assume $x_n < \beta$ for $2 \le n \le p-2$. Then,

$$x_{n+1} = ax_n = a^n\gamma(1 - a\alpha) = \frac{a^{p-1}\gamma(1 - a\alpha)}{a^{p-1-n}} < \frac{\beta}{a^{p-1-n}} < \frac{\beta}{a} \le \frac{\beta}{a_{p,0}} < \alpha.$$

Therefore, $x_{n+1} \in I_L$. Thus, we obtain

$$x_2 < x_3 < \cdots < x_{p-1} < \alpha.$$

Q.E.D.

Appendix E

—— Proof of $a_c \le a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M$ ——

In this appendix, we prove the relations

$$a_c \le a_{3,0} < a_{4,0} < \cdots < a_{p,0} < a_{p+1,0} < \cdots < a_M.$$

We only have to carry out the proof for the asymmetric case, because for $\gamma = 1$ this reduces to the symmetric case.

Let us estimate $x_{p+1}(a_{p,0})$ for $p \ge 3$:

$$x_{p+1}(a_{p,0}) = a_{p,0}x_p(a_{p,0}) = a_{p,0}\beta > \beta.$$

Then, since $x_{p+1}(a)$ is decreasing for $a_{p+1,\max} \le a$ and $a_{p+1,\max} < a_{p+1,0}$ holds, the relation $a_{p,0} < a_{p+1,0}$ follows. Now, let us prove the first relation $a_c \le a_{3,0}$. For $a_{3,0} < a < a_{3,1}$, there is a stable period three solution. If $a_c > a$, there is a stable periodic solution with period 2^m for some integer m, and no other stable solution. Thus, $a_c \le a_{3,0}$. Q.E.D.

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